From Calogero models to spin chains: entanglement and entropy of a su$(1|1)$ supersymmetric spin chain with long-range interactions

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Outline

Introduction

Calogero-Sutherland models:

$$H = -\sum_i \partial^2_{x_i} + \sum_{i<j} V(x_i - x_j)$$

$$\psi(x_1, \dots, x_N)$$

A_n rational model:

$$H_{\mathrm{sc}} = -\sum_i \partial^2_{x_i} + \omega^2 \sum_i x_i^2 + a(a-1) \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

Spin models

A_n rational model with $SU(m)$ spin:

$$H_S = -\sum_i \partial_{x_i}^2 + \omega^2 \sum_i x_i^2 + a \sum_{i<j} \frac{a + S_{ij}}{(x_i - x_j)^2}$$

$$\psi(x_1, \dots, x_N) |s_1, \dots, s_N\rangle$$

$$S_{ij} |s_1, \dots, s_i, \dots, s_j, \dots, s_N\rangle = |s_1, \dots, s_j, \dots, s_i, \dots, s_N\rangle$$

Spin models

Polychronakos freezing trick

Scalar potential ($\omega = a\Omega,\ \Omega = 1$):

$$U(\mathbf{x}) = \sum_{i<j} \frac{1}{(x_i - x_j)^2} + a^2 \sum_i x_i^2, \quad V(\mathbf{x}) = \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

Scalar and spin Hamiltonians

$$H_{\mathrm{sc}} = -\sum_i \partial_{x_i}^2 + a^2 U(\mathbf{x}) - a V(\mathbf{x})$$

$$H_S = -\sum_i \partial_{x_i}^2 + a^2 U(\mathbf{x}) - a\mathcal{H}(\mathbf{x}) = H_{\mathrm{sc}} + a(V(\mathbf{x}) - \mathcal{H}(\mathbf{x}))$$

$$\mathcal{H}(\mathbf{x}) = \sum_{i<j} \frac{1}{(x_i - x_j)^2} S_{ij}$$

Spin Hamiltonian

Equilibrium point of the scalar potential $U(\mathbf{x})$:

$$\sum_{j,j>i} \frac{1}{\xi_i - \xi_j} - \xi_i = 0, \quad i = 1, \ldots N$$

Spin chain Hamiltonian

$$\mathfrak{H} = \sum_{i<j} \frac{1}{(\xi_i - \xi_j)^2}(1 + S_{ij})$$

In the limit $a \to \infty$, the spin and dynamical degrees of freedom decouple, and the particles are "frozen" at the equilibrium point of the scalar potential. The spectrum of the spin model and the partition function can be obtained from the spectra and partition functions of the scalar and dynamical spin models:

$$Z(T) = \lim_{a\to\infty} \frac{Z_S(aT)}{Z_{\mathbf{sc}}(aT)}$$

Spin chains

- Translation-invariant closed spin chain.
- Sites: occupied by a boson or a (spinless) fermion
- Boson and fermion creation operators at the i-th site:

$$b_i^\dagger, \quad f_i^\dagger$$

- Hilbert space: 2^N-dimensional subspace of the Fock space

$$b_i^\dagger b_i + f_i^\dagger f_i = 1\,, \qquad 1 \leq i \leq N\,.$$

Hamiltonian

$$H = \sum_{i<j} h_N(j-i)(1-S_{ij}) - \lambda N_{\mathrm{f}}$$

$$S_{ij} = b_i^\dagger b_j^\dagger b_i b_j + f_i^\dagger f_j^\dagger f_i f_j + f_j^\dagger b_i^\dagger f_i b_j + b_j^\dagger f_i^\dagger b_i f_j$$

$|0\rangle \equiv$ boson, $|1\rangle \equiv$fermion

$$S_{ij}|\ldots, s_i, \ldots, s_j, \ldots\rangle = (-1)^n|\ldots, s_j, \ldots, s_i, \ldots\rangle\,,$$

$$\begin{cases} n = s_i = s_j, & s_i = s_j \\ \#\text{ fermions at sites} : i+1, \ldots, j-1, & s_i \neq s_j \end{cases}$$

Periodic chains: $h_N(x) = h_N(N-x)$:

$$h_N(x) = h_N(-x) = h_N(x+N) \geq 0\,, \qquad \forall x \in \mathbf{R}\,.$$

Spinless hoping fermions

Any chain of this form can be recast into a model of spinless hopping fermions:

identify $|0\rangle$ with the fermion vacuum

- Fermion creation operators:

$$a_i^\dagger = f_i^\dagger b_i, \quad 1 \le i \le N$$

- The chain sites are either empty ($|0\rangle$) or occupied by a fermion ($|1\rangle$)
- Hilbert space: whole 2^N-dimensional Fock space, operators $a_i^\dagger$ acting on the vacuum $|0, \ldots, 0\rangle$
- Operator S_{ij}:

$$S_{ij} = 1 - a_i^\dagger a_i - a_j^\dagger a_j + a_i^\dagger a_j + a_j^\dagger a_i \,.$$

Hamiltonian

$$H = -\sum_{i,j} h_N(i-j) a_i^\dagger a_j - \lambda \sum_i a_i^\dagger a_i ,$$

Diagonalization

Fourier-transform:

$$c_\ell = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \mathrm{e}^{-2\pi\mathrm{i}\, k\ell/N} a_k \,, \qquad 0 \le \ell \le N-1 \,.$$

c_ℓ is a new set of fermionic operators
$c_\ell^\dagger$ creates a fermion with momentum $p = 2\pi\ell/N \pmod{2\pi}$

Hamiltonian

$$H = \sum_{\ell=0}^{N-1} \Big(\varepsilon_N(\ell) - \lambda\Big) c_\ell^\dagger c_\ell$$

$$\varepsilon_N(\ell) = \sum_{j=1}^{N-1} \big[1 - \cos(2\pi j\ell/N)\big]\, h_N(j)$$

- $\varepsilon_N(\ell)$ depends on ℓ and N only through the corresponding momentum $2\pi\ell/N$,

$$\varepsilon_N(\ell) = \mathcal{E}(2\pi\ell/N)\,, \qquad 0 \leq \ell \leq N-1\,,$$

- $\mathcal{E}$: *dispersion relation* (a smooth function defined in the interval $[0, 2\pi]$).
- If $\mathcal{E}$ exists, it is necessarily unique, and $\mathcal{E}(p) = \mathcal{E}(2\pi - p)$.

Particular case:

$$h_N(x) = \left(\frac{\alpha}{\pi}\right)^2 \sinh^2\left(\frac{\pi}{\alpha}\right) \left(\wp_N(x) - \frac{2\hat{\eta}_1}{\alpha^2}\right)$$

$$\alpha > 0, \quad \wp_N(x) \equiv \wp(x; N/2, \mathrm{i}\,\alpha/2), \quad \hat{\eta}_1 = \zeta(1/2; 1/2, \mathrm{i}\,N/(2\alpha))$$

$\wp(x; \omega_1, \omega_3) \equiv$ Weierstrass elliptic function
$\zeta(x; \omega_1, \omega_3) \equiv$ Weierstrass zeta function

The model

$$H = \sum_{\ell=0}^{N-1} \Big(\varepsilon_N(\ell) - \lambda\Big) c_\ell^\dagger c_\ell$$

$$h_N(x) = \left(\frac{\alpha}{\pi}\right)^2 \sinh^2\left(\frac{\pi}{\alpha}\right) \left(\wp_N(x) - \frac{2\hat{\eta}_1}{\alpha^2}\right)$$

smoothly interpolates between

- the Heisenberg chain (for $\alpha = 0$)
- the Haldane–Shastry (for $\alpha = \infty$) $\mathrm{su}(1|1)$ chain (with a chemical potential)

$$\lim_{\alpha\to 0+} h_N(x) = \delta_{1,x} + \delta_{N-1,x}\,, \qquad \lim_{\alpha\to\infty} h_N(x) = \frac{(\pi/N)^2}{\sin^2\left(\frac{\pi x}{N}\right)}$$

The Heisenberg chain can be transformed into the spin 1/2 (closed) XX Heisenberg Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \left(1 - \frac{\lambda}{2}\right)\sum_{i=1}^{N}(1 + \sigma_i^z),$$

through the standard Wigner–Jordan transformation

$$a_k = \sigma_1^z \cdots \sigma_{k-1}^z \cdot \frac{1}{2}(\sigma_k^x - \mathrm{i}\,\sigma_k^y), \qquad 1 \le k \le N$$

The dispersion relation

for the elliptic interaction

$$\mathcal{E}(p) = 2\sinh^2(\pi/\alpha)\left[\wp(p) - \left(\zeta(p) - \frac{\eta_1 p}{\pi}\right)^2 - \frac{2\eta_1}{\pi}\right]$$

$$\wp(p) \equiv \wp(p; \pi, \mathrm{i}\,\pi/\alpha)\,, \quad \zeta(p) \equiv \zeta(p; \pi, \mathrm{i}\,\pi/\alpha)\,, \quad \eta_1 = \zeta(\pi)\,.$$

2π-periodic function, independent of the number of particles N.
The dispersion relations of the XX model and the $\mathrm{su}(1|1)$ are the limits when $\alpha \to 0+$ and $\alpha \to \infty$

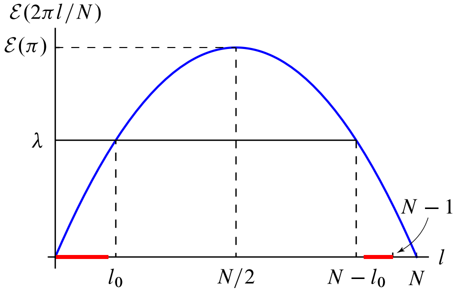
$$\mathcal{E}_{\mathrm{XX}}(p) = 2(1 - \cos p), \qquad \mathcal{E}_{\mathrm{HS}}(p) = \frac{1}{2}\,p(2\pi - p)$$

Critical behavior and central charges

$$H = \sum_{\ell=0}^{N-1} \Big(\varepsilon_N(\ell) - \lambda\Big) c_\ell^\dagger c_\ell$$

Ground state of the model: the modes excited in the ground state are those whose momenta $p = 2\pi\ell/N$ satisfy the condition $\lambda > \mathcal{E}(p)$

- the dispersion relation has a positive derivative in $(0, \pi)$ (monotonically increasing with maximum at $p = \pi$)
- this is the case for the elliptic interaction (in particular, for the XX model and the $\mathrm{su}(1|1)$ Haldane–Shastry chains)



Dispersion relation $\mathcal{E}(2\pi\ell/N)$ as a function of the mode number $\ell = 0, \ldots, N-1$ (modes excited in the ground state: thick red line)

The model is gapless for $\lambda \in [0, \mathcal{E}(\pi)]$.

- the system is gapped for $\lambda < 0$ or $\lambda > \mathcal{E}(\pi)$. If $\lambda < 0$ the gap between the first excited state $c_0^\dagger|0,\dots,0\rangle$ and the ground state is $\Delta E = |\lambda| > 0$, $(N \to \infty)$
- if $0 \le \lambda \le \mathcal{E}(\pi)$, the gap between the first excited state and the ground state

$$\Delta E = \min\big(\lambda - \mathcal{E}(2\pi\lfloor\ell_0\rfloor/N), \mathcal{E}(2\pi(\lfloor\ell_0\rfloor + 1)/N) - \lambda\big),$$

is $\mathrm{O}(1/N)$, since $\lambda = \mathcal{E}(2\pi\ell_0/N)$.

ΔE tends to zero as $N \to \infty$ and the system is gapless.

Criticality

- If $\lambda \in (0, \mathcal{E}(\pi))$ the $\mathrm{su}(1|1)$ chain is critical: at low energies its spectrum is that of a $(1+1)$-dimensional CFT with one free boson:

$$\Delta E \simeq \mathcal{E}'(p_0)\Delta p, \quad p_0 = 2\pi\ell_0/N \equiv \mathcal{E}^{-1}(\lambda) \in (0, \pi)$$

as in a $(1+1)$-dimensional CFT with $v = \mathcal{E}'(p_0)$.

Central charges

At low temperatures the free energy (per unit length) of a $(1+1)$-dimensional CFT

$$f(T) \simeq f_0 - \frac{\pi c T^2}{6v}$$

$c \equiv$ the central charge

Free energy of the spin chain given by

$$F(T) = -T \log Z = -T \sum_{\ell=0}^{N-1} \log Z_\ell$$

$Z_\ell = 1 + \mathrm{e}^{-\beta(\mathcal{E}(2\pi\ell/N)-\lambda)}$ the partition function of the ℓ-th normal mode.

$$f(T) = \lim_{N\to\infty} \frac{F(T)}{N} = -\frac{T}{\pi} \int_0^\pi \log\left[1 + \mathrm{e}^{-\beta(\mathcal{E}(p)-\lambda)}\right] \mathrm{d}\, p$$

low temperature behavior of $f(T)$

$$f(T) = f_0 - \frac{\pi T^2}{6v} + \mathrm{O}(T^3)$$

The spin chain is critical for $0 < \lambda < \mathcal{E}(\pi)$, with central charge $c = 1$.

Critical behavior of the $\mathrm{su}(1|1)$ chain at the endpoints $\lambda = 0, \mathcal{E}(\pi)$

- $\lambda = 0$, $\mathcal{E}'(0) \neq 0$ the chain is critical but has central charge $c = 1/2$; its low energy behavior is described by a CFT with one free *fermion*.

$$f(T) = f_0 - \frac{\pi T^2}{12v} + \mathrm{O}(T^3)$$

  - elliptic interaction
  - the $\mathrm{su}(1|1)$ Haldane–Shastry chain ($\alpha = \infty$ case) can be described at low energies by a $(1+1)$-dimensional CFT with one free fermion

- at the endpoint $\lambda = \mathcal{E}(\pi)$ the model is not critical

Entropy

Aim: construct the von Neumann entanglement entropy S of the ground state of the $\mathrm{su}(1|1)$ supersymmetric model

von Neumann entropy

of the reduced density matrix ρ_L of a block of L consecutive sites when the system is in its ground state.

$|\psi\rangle$: ground state of the chain

$$\rho_L = \mathrm{tr}_{N-L} |\psi\rangle\langle\psi|$$

tr_{N-L}: trace over the Hilbert space of the remaining $N-L$ sites
The von Neumann entanglement entropy is given by

$$S = -\,\mathrm{tr}(\rho_L \log \rho_L)$$

Rényi entropy

$$S_q = \frac{\log \operatorname{tr}(\rho_L^q)}{1-q}, \quad q > 0, \quad \lim_{q\to 1} S_q = S$$

The von Neumann and Rényi ground-state entanglement entropies of a $(1+1)$-dimensional CFT scale as $r_q \log L$ when $L \to \infty$

$$r_q = \frac{1}{12}(1+q^{-1})(c+\bar{c})$$

Since the $\mathrm{su}(1|1)$ supersymmetric chain is critical for $0 < \lambda < \mathcal{E}(\pi)$, with central charge $c = \bar{c} = 1$, it is to be expected that

$$S_q \simeq \frac{1}{6}(1+q^{-1}) \log L, \quad L \to \infty$$

.

The ground state is not entangled for λ outside the interval $[0, \mathcal{E}(\pi)]$

- If $\lambda < 0$ the ground state is the vacuum $|0, \dots, 0\rangle$ (all the modes have positive energy $\mathcal{E}(2\pi\ell/N) - \lambda$)
  The ground state is a product state ($|0\rangle^{\otimes N}$), and is not entangled.
- If $\lambda > \mathcal{E}(\pi)$, $\mathcal{E}(2\pi\ell/N) - \lambda < 0$ for all ℓ and all the modes are excited: $c_\ell^\dagger|\psi\rangle = 0$ for all $\ell = 0, \dots, N-1$

$$a_k^\dagger|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \mathrm{e}^{-2\pi\mathrm{i}\, k\ell/N} c_\ell^\dagger|\psi\rangle = 0\,, \qquad 1 \le k \le N$$

Hence $|\psi\rangle = |1, \dots, 1\rangle = |1\rangle^{\otimes N}$, again a product state and therefore not entangled.

$\lambda \in (0, \mathcal{E}(\pi))$

First step: evaluation of the ground-state correlation matrix A

$$A_{mn} = \langle\psi|a_m^\dagger a_n|\psi\rangle \equiv \langle a_m^\dagger a_n\rangle\,, \qquad 1 \le m, n \le N$$

For the ground state:

$$\begin{cases} c_j|\psi\rangle = 0\,, & \lfloor l_0\rfloor + 1 \le j \le N - \lfloor l_0\rfloor - 1 \\ c_j^\dagger|\psi\rangle = 0\,, & \text{otherwise} \end{cases}$$

Then

$$\langle c_j^\dagger c_k\rangle = \begin{cases} 0\,, & \lfloor \ell_0\rfloor + 1 \le j \le N - \lfloor \ell_0\rfloor - 1 \\ \delta_{jk}\,, & \text{otherwise,} \end{cases}$$

Using the inverse Fourier transform formula

$$a_k = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}\, k\ell/N} c_\ell$$

we get

$$\begin{aligned} A_{mn} &= \frac{1}{N}\left(\sum_{\ell=0}^{\lfloor \ell_0 \rfloor} + \sum_{\ell=N-\lfloor \ell_0 \rfloor}^{N-1}\right) \mathrm{e}^{-2\pi\mathrm{i}\,(m-n)\ell/N} \\ &= \frac{1}{N} + \frac{2}{N}\sum_{\ell=1}^{\lfloor \ell_0 \rfloor} \cos\bigl(2\pi(m-n)\ell/N\bigr) \\ &\overset{N\gg 1}{\simeq} \frac{1}{\pi}\int_0^{p_0} \cos\bigl(p(m-n)\bigr)\,\mathrm{d}\,p = \frac{\sin\bigl(p_0(m-n)\bigr)}{\pi(m-n)}\,. \end{aligned}$$

Second step: evaluation of the correlation matrix A_L for a block of L consecutive sites

Reduced density matrix ρ_L, $1 \leq m, n \leq L$

$$\begin{aligned}(A_L)_{mn} =& \langle a_m^\dagger a_n \rangle_L \equiv \mathrm{tr}_L(a_m^\dagger a_n \rho_L) \\ =& \,\mathrm{tr}(a_m^\dagger a_n |\psi\rangle\langle\psi|) = \langle\psi| a_m^\dagger a_n |\psi\rangle \equiv A_{mn}\end{aligned}$$

tr_L: the trace over the Hilbert space of the first L sites.
A_L is the submatrix of A consisting of its first L rows and columns.

Third step: Construction of an alternative basis of fermionic operators whose correlation matrix is diagonal

$$U = (u_{mn})_{1\le m,n\le L}, \quad UA_L U^\dagger = \mathrm{diag}(\mu_1, \dots, \mu_L)$$

$\mu_1, \dots, \mu_L \in [0,1]$: eigenvalues of A_L.

$$g_k = \sum_{m=1}^{L} u^*_{km} a_m, \quad 1 \le k \le L$$

acting on the Hilbert space of the first L sites.

Fourth step: Correlation matrix in the basis g_k

$$\langle g_k^\dagger g_l \rangle_L = \mu_k \delta_{kl}$$

This equation and Wick's theorem for Gaussian states imply that the correlation matrix factorizes as

$$\rho_L = \otimes_{k=1}^{L} \varrho_k, \quad \varrho_k = \mu_k g_k^\dagger g_k + (1-\mu_k) g_k g_k^\dagger$$

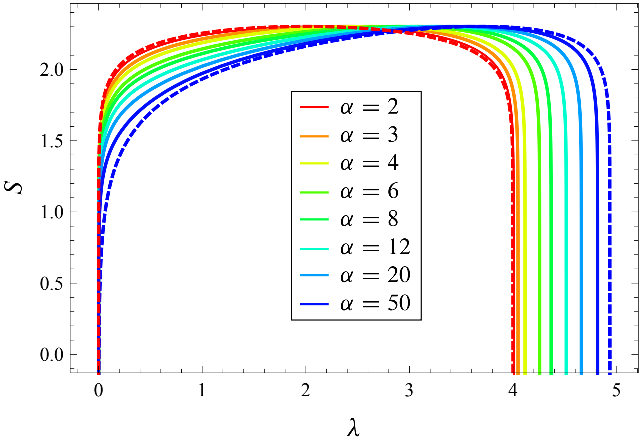
- The Hilbert space of the system is the tensor product of the two-dimensional spaces spanned by the vectors $|v\rangle_k, g_k^\dagger|v\rangle_k$ $(1 \leq k \leq N)$, where $g_k|v\rangle_k = 0$.
- ϱ_k is diagonal in the basis $\{|v\rangle_k, g_k^\dagger|v\rangle_k\}$, with respective eigenvalues $1-\mu_k$ and μ_k.
- The von Neumann and Rényi entropies of ϱ_k are respectively equal to $s(\mu_k)$ and $s_q(\mu_k)$, where

$$\begin{cases} s(x) = -x\log x - (1-x)\log(1-x)\,, \\ s_q(x) = (1-q)^{-1}\log\left[x^q + (1-x)^q\right]. \end{cases}$$

By the additivity property of these entropies

$$S = \sum_{k=1}^{L} s(\mu_k)\,, \qquad S_q = \sum_{k=1}^{L} s_q(\mu_k)\,.$$

The entropy of all of these models is a *universal* function of the Fermi momentum p_0, the difference between two models is given by the dependence of p_0 on the parameter λ.



Approximation to the von Neumann entanglement entropy ($q = 1$) of the elliptic $\mathrm{su}(1|1)$ chain for $L = 1000$ and several values of the parameter α between 2 and 50. The red and blue dashed curves correspond respectively to the XX Heisenberg model ($\alpha = 0$) and the $\mathrm{su}(1|1)$ Haldane–Shastry chain ($\alpha = \infty$)

Asymptotics: $\lambda \to 0, \mathcal{E}(\pi)$

Behavior of the entropy as λ approaches its extreme critical values 0 and $\mathcal{E}(\pi)$.

$\lambda \to 0$:

$$S_q \simeq s_q(Lp_0/\pi) \simeq \begin{cases} \dfrac{(Lp_0/\pi)^q}{1-q}, & 0 < q < 1\,; \\[2ex] \dfrac{q}{q-1}\dfrac{Lp_0}{\pi}, & q > 1\,. \end{cases}$$

If $Lp_0 \ll 1$ the von Neumann entropy can be approximated by

$$S \simeq s(Lp_0/\pi) \simeq -\frac{Lp_0}{\pi}\log\left(\frac{Lp_0}{\pi}\right).$$

- S_q and S are continuous at $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$.
- These entropies have a discontinuous first derivative (with respect to the chemical potential λ) at $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$.
- The analysis suggests that there is a quantum phase transition at $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$ between an ordered (non-entangled) and a disordered (entangled) ground state, with the entanglement entropy as the order parameter.

Asymptotics: $0 < \lambda < \mathcal{E}(\pi)$ fixed, and $L \gg 1$

- A_{mn} is a function of $m - n$ only: the correlation matrix A_L is a Toeplitz matrix.
- Using the Fisher–Hartwig conjecture

$$S_q = \frac{q+1}{6q} \log(L \sin p_0) + \gamma_1^{(q)} + o(1)\,,$$

- $\mathrm{su}(1|1)$ HS chain:

$$S_q = \frac{q+1}{6q} \log\left[L \sin\left(\sqrt{\pi^2 - 2\lambda}\right)\right] + \gamma_1^{(q)} + o(1)\,.$$

Conclusions

- Exactly solvable one-dimensional quantum models are very useful as a source of key ideas in many fields of Physics (as in condensed matter or the theory of critical phenomena)
- Thermodynamical properties of the supersymmetric $\mathrm{su}(1|1)$ spin chain can be explicitly computed and allow to relate it to CFT theories with specific values of the central charge
- The entanglement entropy can also be computed in analytic form and many interesting properties of its asymptotic behavior can be discussed in a precise way.

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