Haantjes manifolds of KdV stationary flows

Giorgio Tondo, Università di Trieste, Italy

in the framework of a research program in collaboration with

Piergiulio Tempesta,
Universidad Complutense de Madrid and
ICMAT, Madrid, Spain.

to honour G. Marmo on the occasion of his 70's
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Motivations

- Theorem of Liouville–Haantjes.

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- Results about Bihamiltonian structures and Stäckel separability, by Ibort, Magri & Marmo, 2000 *J. of Geometry and Physics* **33**. 210–228

Theorem of Liouville–Haantjes

Theorem (Tempesta & T., 2015)

Let M be a $2n$-dimensional $\omega\mathcal{H}$ manifold and $(H_1, H_2, \ldots, H_n)$ be smooth independent functions forming a Lenard-Haantjes chain. Then, the foliation generated by these functions turns out to be Lagrangian. Consequently, each Hamiltonian system, with Hamiltonian functions H_j, $1 \leq j \leq n$ is integrable by quadratures. Conversely, If a Hamiltonian system in n dimensions is completely integrable in the Liouville-Arnold sense, and its Hamiltonian function $H = H_1$ is non degenerate, then it admits an associated $\omega\mathcal{H}$ structure in any tubular neighborhood of an Arnold torus, given by

$$K_\alpha = \sum_{i=1}^{n} \frac{\nu_i^{(\alpha+1)}}{\nu_i}(\boldsymbol{J})\left(\frac{\partial}{\partial J_i} \otimes \mathrm{d}J_i + \frac{\partial}{\partial \phi_i} \otimes \mathrm{d}\phi_i\right) \quad \alpha = 0, \ldots, n-1 \ , \quad (1)$$

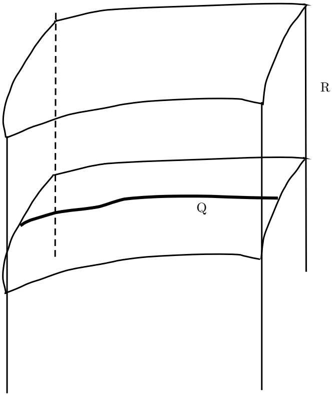
where $\nu_i^{\alpha+1} := \frac{\partial H_{\alpha+1}}{\partial J_i}$, are just the frequencies of the $(\alpha + 1 = j)$ linear flows on the torus.

Ibort, Magri & Marmo (2000)

$(T^*Q \times \mathbb{R}, P_0, P_1)$: bi–Hamiltonian manifold

$(T^*Q, \Omega = \check{P_0}^{-1}, N)$: ωN manifold

(Q, g, L) : Benenti manifold



Outline

Search for integrals of motion

Problem: Given an integrable Hamiltonian system, how to find enough integrals of motion in involution?

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- Lenard–Magri chains in *PN* manifolds;

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- Lenard–Magri chains in PN manifolds;
- Lenard–Haantjes chains in $P\mathcal{H}$ manifolds.

Integrability of Nijenhuis and Haantjes

Theorem (Haantjes, 1955)

Let $\boldsymbol{K} : TM \to TM$ a smooth field of operators and suppose that

- *its eigen-distributions have constant rank in each point of M;*

Then, the n.a.s condition in order that an integrable frame exists which is an eigen–frame of $\boldsymbol{K}$ is that the Haantjes tensor of $\boldsymbol{K}$ vanishes.

Remark

If $\boldsymbol{K}$ is not semisimple, the vanishing of the Haantjes tensor is only sufficient to the existence of an integrable frame which is also an eigen–frame of $\boldsymbol{K}$.

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Nijenhuis tensor

- Let M be a differentiable manifold and $\boldsymbol{K} : TM \to TM$ be a (1,1) tensor field. The *Nijenhuis torsion* of $\boldsymbol{K}$ is the skew-symmetric $(1,2)$ tensor field defined by

$$\mathcal{T}_{\boldsymbol{K}}(X,Y) := \boldsymbol{K}^2[X,Y] + [\boldsymbol{K}X, \boldsymbol{K}Y] - \boldsymbol{K}\Big([X, \boldsymbol{K}Y] + [\boldsymbol{K}X, Y]\Big),$$

where $X, Y \in TM$ and $[\ ,\ ]$ denotes the commutator of two vector fields.

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- The *Haantjes tensor* associated with $\boldsymbol{K}$ is the (1, 2) tensor field defined by

$$\mathcal{H}_{\boldsymbol{K}}(X,Y) := \boldsymbol{K}^2\mathcal{T}_{\boldsymbol{K}}(X,Y)+\mathcal{T}_{\boldsymbol{K}}(\boldsymbol{K}X,\boldsymbol{K}Y)-\boldsymbol{K}\Big(\mathcal{T}_{\boldsymbol{K}}(X,\boldsymbol{K}Y)+\mathcal{T}_{\boldsymbol{K}}(\boldsymbol{K}X,Y)\Big).$$

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- A Haantjes (Nijenhuis) field of operators is a field of operators whose associated Haantjes (Nijenhuis) tensor identically vanishes.

A remarkable property of Haantjes operators

Theorem (Bogoyavlenskij, 2004)

Let $\boldsymbol{K}$ be an operator with vanishing Haantjes tensor in M. Then for any polynomial in $\boldsymbol{K}$, with coefficients $a_j \in C^\infty(M)$, the associated Haantjes tensor also vanishes, i.e.

$$\mathcal{H}_{\boldsymbol{K}}(X,Y) = 0 \implies \mathcal{H}_{(\sum_j a_j(\boldsymbol{x})\boldsymbol{K}^j)}(X,Y) = 0. \tag{2}$$

This means that a single Haantjes operator generates a module over the ring of smooth functions of M.

Semisimple Haantjes and Nijenhuis operators

The class of Haantjes operators is more general than that of Nijenhuis operators.

Theorem

Let $\boldsymbol{K}$ *a smooth field of operators. If there exists a local coordinate chart* $\{(x_1, \dots, x_n)\}$, *where* $\boldsymbol{K}$ *takes a diagonal form, i.e.*

$$\boldsymbol{K} = \sum_{i=1}^{n} l_i(\boldsymbol{x}) \frac{\partial}{\partial x_i} \otimes \mathrm{d}x_i,$$

then the Haantjes tensor of $\boldsymbol{K}$ *vanishes. In particular, if* $l_i(x) = \lambda_i(x_i)$, $i = 1, \dots, n$, *the Nijenhuis tensor of* $\boldsymbol{K}$ *also vanishes.*

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- $(\{\ ,\ \}_0, \{\ ,\ \}_1)$ are *compatible* Poisson brackets, i.e. are such that $\{\ ,\ \}_1 + \lambda\{\ ,\ \}_0$ is a Poisson pencil $\forall\, \lambda \in \mathbb{C}$.

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- Equivalently, a Bi-Hamiltonian manifold can be defined by two compatible Poisson bi–vector fields (P_0, P_1), i.e., such that $P_0 + \lambda P_1$ is a Poisson pencil.

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- (P, N) are *compatible*, i.e. $(M, P_0 := P, P_1 := NP)$ must be a bi-Hamiltonian manifold.

Lenard-Magri Chains in PN manifolds

$$I_k := \frac{1}{2k} tr(N^k), \quad k = 1, \dots, n$$

Lenard chain generated by I

$$\begin{aligned} dI_1 &= dI \\ dI_2 &= N^T dI \\ &\vdots \\ dI_n &= (N^T)^{n-1} dI \end{aligned}$$

$$\begin{tikzcd} dI_k \arrow[d, "N^T"'] \arrow[dr, "P_1=NP_0"] & \\ dI_{k+1} \arrow[r, "P_0=P"'] & X_k \end{tikzcd}$$

BH formulation

$\mathcal{L}_{X_k}(N)=0 \quad \Rightarrow$ Recursion operator $\quad \{I_j, I_k\}_{0,1}=0$

Problem

Most of the classical integrable Hamiltonian systems do not admit a BH formulation (Brouzet, 1993).
Some generalizations of the BH theory:

- A BH formulation of X w.r.t. *alternative* (i.e. not including P_0) Poisson structures, still compatible each other (Marmo, Vilasi et al., 1984);

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- a BH formulation of X w.r.t. a *degenerate* bi–Hamiltonian structure in an *extended* phase space (BH manifold) (Ibort, Magri & Marmo, 2000);

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- a BH formulation of X w.r.t. a *degenerate* bi–Hamiltonian structure in an *extended* phase space (BH manifold) (Ibort, Magri & Marmo, 2000);
- a Quasi-BH formulation (or generalized Lenard-chains) of X w.r.t. a standard bi–Hamiltonian structure in its original phase space (ωN manifold) (Morosi & T., 1997, Falqui, Magri & T., 2000, Falqui & Pedroni, 2003, Tempesta & T., 2012).

$P\mathcal{H}$ manifolds

$(M, P, K_0, K_1, \dots, K_\alpha, \dots)$

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- The operators $K_\alpha$form a commutative ring of Haantjes operators: $K_\alpha K_\beta = K_\beta K_\alpha$;
- The operators K_α generate a module over the ring of smooth function on M:

$$\mathcal{H}_{\left(\sum_\alpha a_\alpha(\boldsymbol{x}) K_\alpha\right)}(X, Y) = 0 \ , \qquad \forall X, Y \in TM \ , \tag{3}$$

being $a_\alpha(\boldsymbol{x})$ arbitrary smooth functions on M.

A paradigmatic example

A $(2n+1)$-dimensional PN manifold, with N having its minimal polynomial of degree $n+1$, has a *standard* $P\mathcal{H}$ structure, given by

$$(M, P, I, N, \ldots, N^{n-1}, N^n) \ ,$$

as each Nijenhuis operator N generates a module of Haantjes operators.

Lenard–Haantjes Chains

Let us introduce a natural extension of Lenard-Magri chains

Lenard–Magri chains	*Lenard-Haantjes chains*
(M, P, N)	$(M, P, K_0, K_1, \ldots, K_n)$
$\mathrm{d}H_1 = \mathrm{d}H$ $\mathrm{d}H_2 = N^T \mathrm{d}H$ $\vdots$ $\mathrm{d}H_{n+1} = (N^T)^n \mathrm{d}H$	$\mathrm{d}H_1 = K_0^T \mathrm{d}H$ $\mathrm{d}H_2 = K_1^T \mathrm{d}H$ $\vdots$ $\mathrm{d}H_{n+1} = K_n^T \mathrm{d}H$
$\mathcal{L}_{X_H}(N) = 0$	$\mathcal{L}_{X_H}(K_\alpha) \neq 0$
$\{H_i, H_j\} = 0$	$\{H_i, H_j\} =< dH_i, P^{-1} dH_j >$ $=< {K_{i-1}}^T dH, P {K_{j-1}}^T dH >$ $=< dH, K_{i-1} P {K_{j-1}}^T dH >= 0$

KdV hierarchy: $\dot{u} = \hat{X}_k$

$u^{(n)} := \frac{\partial}{\partial x^n} u(x, t)$

$$
\begin{aligned}
u_{t_1} &= u^{(1)} \\
u_{t_2} &= u^{(3)} + 6uu^{(1)} \\
u_{t_3} &= u^{(5)} + 10u^{(3)}u^{(1)} + 20u^{(2)}u^{(1)} + 30u^{(1)}u^2 \\
u_{t_4} &= u^{(7)} + 4u^{(5)}u + 42u^{(4)}u^{(1)} + 70u^{(3)}(u^{(2)} + u^2) + \\
&\quad + 280u^{(2)}u^{(1)}u + 70u^{(1)}\left((u^{(1)})^2 + 2u^3\right) \\
. &= .... \\
. &= ... \\
. &= ...
\end{aligned}
$$

First stationary flow of seventh order: X_1

$$
\begin{aligned}
M_7 := \{0 = u^{(7)} + 4u^{(5)}u + 42u^{(4)}u^{(1)} + 70u^{(3)}(u^{(2)} + u^2) \\
+ 280u^{(2)}u^{(1)}u + 70u^{(1)}\big((u^{(1)})^2 + 2u^3\big)\}
\end{aligned}
$$

$$
\begin{aligned}
\dot{u} &= u^{(1)} \\
\dot{u}^{(1)} &= u^{(2)} \\
\dot{u}^{(2)} &= u^{(3)} \\
\dot{u}^{(3)} &= u^{(4)} \\
\dot{u}^{(4)} &= u^{(5)} \\
\dot{u}^{(5)} &= u^{(6)} \\
\dot{u}^{(6)} &= -\Big(4u^{(5)}u + 42u^{(4)}u^{(1)} + 70u^{(3)}(u^{(2)} + u^2) + \\
&\quad + 280u^{(2)}u^{(1)}u + 70u^{(1)}\big((u^{(1)})^2 + 2u^3\big)\Big)
\end{aligned}
$$

Second stationary flow of seventh order: X_2

$$M_7 := \{(u_1 := u, u_2 := u^{(1)}, u_3 := u^{(2)}, u_4 := u^{(3)}, u_5 := u^{(4)}, u_6 := u^{(5)}, u_7 := u^{(6)})\}$$

$$\begin{aligned}
\dot{u}_1 &= u_4 + 6u_2u_1 \\
\dot{u}_2 &= u_5 + 6u_2^2 + 6u_3u_1 \\
\dot{u}_3 &= u_6 + 6u_4u_1 + 18u_3u_2 \\
\dot{u}_4 &= u_7 + 6u_5u_1 + 24u_4u_2 + 18u_3^2 \\
\dot{u}_5 &= -8u_6u_1 - 12u_5u_2 - 10u_4(u_3 + 7u_1^2) - 70u_2^3 - 140u_1^2u_2 - 280u_3u_2u_1 \\
\dot{u}_6 &= -8u_1u_7 + \ldots \\
\dot{u}_7 &= -28u_7u_2 + \ldots
\end{aligned}$$

Third stationary flow of seventh order: X_3

$$
\begin{aligned}
\dot{u}_1 =& u_6 + 20 u_2 u_3 + 10 u_4 u_1 + 30 u_1^2 u_2 \\
\dot{u}_2 =& u_7 + 10 u_5 u + 30 u_4 u_2 + 10 u_3 (u_3 + 3 u_1^2) + 60 u_2^2 u_1 \\
\dot{u}_3 =& -4 u_6 u_1 + \dots \\
\dot{u}_4 =& -4 u_7 u_1 + \dots \\
\dot{u}_5 =& -10 u_7 u_2 + \dots \\
\dot{u}_6 =& 2 u_7 (-9 u_3 + 8 u_1^2) + \dots \\
\dot{u}_7 =& -28 u_7 u_4 + 112 u_7 u_2 u_1 + \dots
\end{aligned}
$$

Poisson tensors in adapted coordinates

$$M_7 := \{(v_1 = \partial_x^{-1}\hat{X}_1, v_2 = \partial_x^{-1}\hat{X}_2, v_3 = \partial_x^{-1}\hat{X}_3, w_1 = \hat{X}_1, w_2 = \hat{X}_2, w_3 = \hat{X}_3, E)\}$$

$$P_0 = \left[\begin{array}{cc|c} 0_3 & B & 0 \\ -B^T & 0_3 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \qquad B = 4\begin{bmatrix} 0 & 1 & 0 \\ 1 & v_1 & 0 \\ 0 & 0 & -v_3 \end{bmatrix},$$

$$P_1 = \left[\begin{array}{cc|c} 0_3 & C & * \\ -C^T & 0_3 & * \\ \hline * & * & 0 \end{array}\right], \qquad C = 4\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & v_1 \\ 1 & v_1 & v_2 \end{bmatrix},$$

Gelfand-Zakharevich systems of co-rank=1

$$H(\lambda) = H_1\lambda^3 + H_2\lambda^2 + H_3\lambda + H_4$$

$$\begin{aligned}
H_1 =& \frac{1}{4}w_1(w_2 - \frac{1}{2}w_1 v_1) - \frac{1}{8}\frac{w_3^2}{v_3} - \frac{1}{8}v_1^4 + \frac{3}{8}v_1^2 v_2 - \frac{1}{4}v_1 v_3 - \frac{1}{8}v_2^2 - \frac{1}{8}\frac{E}{v_3}\ , \\
H_2 =& \frac{1}{4}w_1(-\frac{1}{2}w_1 w_2 + w_3) + \frac{1}{8}w_2^2 - \frac{1}{8}\frac{w_3^2 v_1}{v_3} + \frac{1}{4}v_1 v_2(v_2 - \frac{1}{2}v_1^2) - \frac{1}{4}v_2 v_3\ , \\
& + \frac{1}{8}v_1^2 v_3 - \frac{1}{8}\frac{v_1}{v_3}E\ ,
\end{aligned}$$

$$H_3 = -\frac{1}{8}w_1^2 v_3 + \frac{1}{4}w_3(w_2 - \frac{1}{2}w_3\frac{v_2}{v_3}) - \frac{1}{8}v_1^3 v_3 - \frac{1}{8}v_3^2 + \frac{1}{4}v_1 v_2 v_3 - \frac{1}{8}\frac{v_2}{v_3}E\ , \quad H_4 = -\frac{1}{8}E$$

$$\begin{tikzcd}
 & dH_1 \arrow[dl, "P_1"'] \arrow[dr, "P_0"] & & dH_2 \arrow[dl, "P_1"'] \arrow[dr, "P_0"] & & dH_3 \arrow[dl, "P_1"'] \arrow[dr, "P_0"] & & dH_4 \arrow[dl, "P_1"'] \arrow[dr, "P_0"] & \\
0 & & X_1 & & X_2 & & X_3 & & 0
\end{tikzcd}$$

Nijenhuis operator for the GZ system

Deformed Poisson bivector

$$\tilde{P}_1 := P_1 - X_3 \wedge (-8\frac{\partial}{\partial E}) \qquad [\tilde{P}_1] = \left[\begin{array}{cc|c} 0_3 & C & 0 \\ -C^T & 0_3 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] ,$$

a Nijenhuis operator such that $\tilde{P}_1 = NP_0$

$$N =: \left[\begin{array}{cc|c} A & 0_3 & 0 \\ 0_3 & A & 0 \\ \hline 0 & 0 & 0 \end{array}\right] , \qquad A = \begin{bmatrix} 0 & 0 & -\frac{1}{v_3} \\ 1 & 0 & -\frac{v_1}{v_3} \\ 0 & 1 & -\frac{v_2}{v_3} \end{bmatrix} ,$$

Haantjes operators

$$K_\alpha = e_\alpha I_7 + f_\alpha N + g_\alpha N^2 + h_\alpha N^3 \qquad \alpha = 1,2,3$$

such that

$$K_1{}^T \mathrm{d}H_3 = \mathrm{d}H_2 \quad K_2{}^T \mathrm{d}H_3 = \mathrm{d}H_1 \quad K_3{}^T \mathrm{d}H_3 = \mathrm{d}H_4$$

$$K_1 =: \left[\begin{array}{cc|c} A_1 & 0_3 & 0 \\ 0_3 & A_1 & 0 \\ \hline 0 & 0 & v_1/v_2 \end{array}\right], \qquad A_1 = \begin{bmatrix} \frac{v_2}{v_3} & 0 & \frac{1}{v_3} \\ 1 & \frac{v_2}{v_3} & -\frac{v_1}{v_3} \\ 0 & 1 & 0 \end{bmatrix},$$

$$K_2 =: \left[\begin{array}{cc|c} A_2 & 0_3 & 0 \\ 0_3 & A_2 & 0 \\ \hline 0 & 0 & 1/v_2 \end{array}\right], \qquad A_2 = \begin{bmatrix} \frac{v_1}{v_3} & -\frac{1}{v_3} & 0 \\ \frac{v_2}{v_3} & 0 & -\frac{1}{v_3} \\ 1 & 0 & 0 \end{bmatrix},$$

$$K_3 =: \left[\begin{array}{cc|c} 0_3 & 0_3 & 0 \\ 0_3 & 0_3 & 0 \\ \hline 0 & 0 & v_3/v_2 \end{array}\right],$$

$$
\begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{c_2}{c_1} & 1+\frac{c_2^2}{c_1 c_3}+\frac{c_1 c_2}{c_3} & \frac{c_1^2}{c_3}+\frac{c_2}{c_3} & -(\frac{c_1}{c_3}+\frac{c_2}{c_1 c_3}) \\ \frac{c_3}{c_1} & -c_1+\frac{c_2}{c_1}+\frac{c_2^2}{c_3} & 2+\frac{c_1 c_2}{c_3} & -(\frac{c_2}{c_3}+\frac{1}{c_1}) \\ -\frac{1}{c_1} & -\frac{c_2}{c_1 c_3} & -\frac{1}{c_3} & \frac{1}{c_1 c_3} \end{bmatrix} = \begin{bmatrix} I \\ N \\ N^2 \\ N^3 \end{bmatrix} \quad (4)
$$

where

$$
m_N(\lambda) = \lambda^4 - c_1\lambda^3 - c_2\lambda^2 - c_3\lambda = \lambda^4 - \frac{v_2}{v_3}\lambda^3 - \frac{v_1}{v_3}\lambda^2 - \frac{1}{v_3}\lambda
$$

is the minimal polynomial of the Nijenhuis operator N.

Restriction to the symplectic leaves of P_0

$$S_6 := \{E = const\}$$

Theorem

The Poisson pencil $P_0 - \lambda \tilde{P}_1$ restricts to the symplectic leaves of P_0. Moreover, also N, K_1,K_2,K_3, the Hamiltonian functions and the Hamiltonian vector fields retrict to S_6. In particular, the relations (4) restricts to the Benenti relations

$$\begin{aligned}
\check{K}_1 &= -\, c_1 I + \check{N} \\
\check{K}_1 &= -\, c_2 I - c_1 \check{N} + \check{N}^2 \\
\check{K}_3 &= -\, c_3 I - c_2 \check{N} - c_1 \check{N}^2 + \check{N}^3 = 0 \ ,
\end{aligned}$$

with the minimal polynomial of $\check{N}(\lambda)$ being

$$m_{\check{N}}(\lambda) = \lambda^3 - c_1 \lambda^2 - c_2 \lambda - c_3 \ .$$

References

Falqui, G. & Pedroni, M. 2002 On a Poisson Reduction for Gelfand–Zakharevic Manifolds

Gerdjikov, V.S. Vilasi, G. & Yanovski A.B. 2008 Integrable Hamiltonian Hierarcchies, Berlin: Springer-Verlag.

Ibort, A. Magri, F. & Marmo, G. 2000 Bihamiltonian structures,and Stäckel separability. *J. of Geometry and Physics* **33**. 210–228

Magri, F. 2012 Recursion operators and Frobenius manifolds. *SIGMA* **8**, paper 076, 7 pp.

Magri, F. 2014 Haantjes manifolds. *Journal of Physics: Conference Series* **482**, paper 012028, 10 pp.

Tempesta, P. & Tondo, G. 2012 Generalized Lenard chains, separation of variables, and superintegrability. *Physical Review E* **85**, 046602, 11 pp.

Tempesta, P. & Tondo, G. 2016 Haantjes manifolds of classical integrable systems. Preprint ArXiv: 1405.5118v1.

Tempesta, P. & Tondo, G. 2016 Haantjes structure for the Jacobi-Calogero model and the Benenti systems, *SIGMA* **12**, paper 023, 18 pp.

Happy 70's,

Buon Compleanno,

Beppe!