The volume of Gaussian states by information geometry

Stefano Mancini

School of Science and Technology, University of Camerino, Italy
& INFN Sezione di Perugia, Italy

July 11, 2016

Motivations

Computation of the volume of different classes of states

- Distinguishing classical from quantum states
- Distinguishing separable from entangled sates
- Typicality of a set of states

Natural metrics

- Fisher-Rao metric for classical states
- Fubini-Study metric for pure quantum states

Phase space & Information geometry

- Phase space can be a common playground for classical and quantum states
- Employ information geometry there ?

Outline

- Gaussian states: classical & quantum
- Information geometry and Gaussian states
- Volume measure for Gaussian states and its properties
- Regularized volume for Gaussian states
- Application to two-mode systems

Gaussian states

The phase space Γ of N modes is the $2N$-dimensional space of the canonical position and momentum variables $\xi = (q_1, p_1, \ldots, q_N, p_N)^T$ of such modes.

- Classical state in Γ

$$\rho(\xi) = \frac{1}{(2\pi)^{2N}} \int d\tau \; e^{-i\xi^T\tau} \chi_\rho(\tau).$$

- Quantum state $\hat{\rho}$ on $\mathcal{H} = L^2(\mathbb{R})^{\otimes N}$.
- Phase space representation of $\hat{\rho}$

$$W(\xi) = \frac{1}{(2\pi)^{2N}} \int d\tau \; e^{-i\xi^T\tau} \; \chi_{\hat{\rho}}(\tau),$$

$$\chi_{\hat{\rho}}(\xi) := \mathrm{Tr}\left[\hat{\rho}\hat{D}(\xi)\right], \;\; \hat{D}(\xi) := \exp\left[i \sum_k \left(q_k \hat{q}_k + p_k \hat{p}_k\right)\right].$$

Gaussian States

Gaussian states are those for which the characteristic function is a Gaussian function of the phase space coordinates ξ, namely

$$\chi_\rho(\xi) = e^{-\frac{1}{2}\xi^T V \xi + i x^T \xi}, \quad \chi_{\hat{\rho}}(\xi) = e^{-\frac{1}{2}\xi^T V \xi + i x^T \xi},$$

where V is the $2N \times 2N$ covariance matrix and $x \in \mathbb{R}^{2N}$ the first moment vector.

- Classical states $\Rightarrow$ $V > 0$
- Quantum states $\Rightarrow$ $V + i\Omega \geq 0$ where

$$\Omega = \bigoplus_{j=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Gaussian States

The Gaussian form of characteristic functions reflects on the corresponding phase space representations $\rho(\xi)$ and $W(\xi)$ which we commonly write as

$$P(\xi) = \frac{e^{-\frac{1}{2}(\xi-x)^T V^{-1}(\xi-x)}}{(2\pi)^N \sqrt{\det V}}.$$

Among quantum states we can also distinguish between:

- A composite Gaussian state with two subsystems A and B is separable if and only if there exist covariance matrices V_A and V_B such that

$$V \geq V_A \oplus V_B.$$

- A two-mode Gaussian system is separable if and only if

$$\widetilde{V} + i\Omega \geq 0,$$

where $\widetilde{V} = \Lambda_B V \Lambda_B$, with $\Lambda_B(q_1, p_1, q_2, p_2)^T = (q_1, p_1, q_2, -p_2)^T$.

Information Geometry

A Gaussian pdf with zero mean in the $2N$-dimensional phase space Γ may be parametrized using $m \leq N(2N+1)$ *real*-valued variables $\theta^1, \ldots, \theta^m$, so that

$$\mathcal{S} := \left\{ P(\xi) \equiv P(\xi;\theta) = \frac{e^{-\frac{1}{2}\xi^T V^{-1}(\theta)\xi}}{(2\pi)^N \sqrt{\det V(\theta)}}, \;\Big|\; \theta \in \Theta \right\},$$

turns out to be an m-dimensional statistical model.

Given $\theta \in \Theta$, the *Fisher information matrix* of $\mathcal{S}$ at θ is the $m \times m$ matrix $g(\theta)$ whose entries are given by

$$g_{\mu\nu}(\theta) := \int_{\mathbb{R}^{2N}} dx\, P(\xi;\theta)\, \partial_\mu \ln P(\xi;\theta) \partial_\nu \ln P(\xi;\theta),$$

with $\partial_\mu = \frac{\partial}{\partial \theta^\mu}$. With this metric, the manifold $\mathcal{M} := (\Theta, g(\theta))$ becomes a Riemannian manifold.

Information Geometry

Definition

Set of classical states

$$\Theta_{\text{classic}} := \{\theta \in \mathbb{R}^m | V(\theta) > 0\}.$$

Set of quantum states

$$\Theta_{\text{quantum}} := \{\theta \in \mathbb{R}^m | V(\theta) + i\Omega \geq 0\}.$$

Set of separable quantum states

$$\Theta_{\text{separable}} := \{\theta \in \mathbb{R}^m | V(\theta) \geq V_A \oplus V_B\}.$$

Set of entangled states

$$\Theta_{\text{entangled}} := \Theta_{\text{quantum}} - \Theta_{\text{separable}}.$$

The volume measure

Definition

Let Θ be the parameter space and $\mathcal{M} = (\Theta, g(\theta))$ be the Riemannian manifold associated to the class of Gaussian states Θ, with $g(\theta)$ being the Fisher-Rao metric. Then the volume of the physical states represented by Θ is

$$\mathcal{V}(V) := \int_{\Theta} d\theta \sqrt{\det g(\theta)}.$$

Proposition

The entries of the Fisher-Rao metric are related to V by

$$g_{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left[V^{-1}\ (\partial_\mu V)\ V^{-1}\ (\partial_\nu V)\right],$$

for every $\mu, \nu \in \{1, \ldots, m\}$.

The volume measure

- A labeling permutation of the system's modes acts on $P(\xi;\theta)$ by a permutation congruence of the covariance matrix.
- The uncertainty relation $V(\theta) + i\Omega \geq 0$ has a symplectic invariant form.

Proposition

If there exists a permutation matrix Π (resp. a symplectic matrix S) such that $V' = \Pi^T\ V\ \Pi$ (resp. $V' = S^T\ V\ S$), then

$$\mathcal{V}(V') = \mathcal{V}(V).$$

The volume measure

Given the volume form

$$\nu_g = \sqrt{\det g}\; d\theta^1 \wedge \ldots \wedge d\theta^m$$

it results

$$\det g(\theta) = \frac{1}{(\det V(\theta))^{2m}} \widetilde{F}(V(\theta)),$$

where $\widetilde{F}(V(\theta))$ denotes a non-rational function of the coordinates $\theta^1, \ldots, \theta^m$.

Occurring divergences

- The set Θ is not compact because the variables θ^l are unbounded from above
- $\det g(\theta)$ diverges since $\det V$ approaches zero for some $\theta^l \in \Theta$

Regularized volume

In general, the trace of the covariance matrix is directly linked to the mean energy per mode, namely $\mathcal{E} = \frac{1}{2N}\mathrm{Tr}(V)$. Thereby, we define a regularizing function as

$$\Phi(V) := H(\mathbf{E} - \mathrm{Tr}(V)) \; \log\left[1 + (\det V)^m\right],$$

where $H(\cdot)$ denotes the Heaviside step function and $\mathbf{E}$ is a positive *real* constant (equal to $2N\mathcal{E}$).

Definition

Given a set of Gaussian states represented by a parameter space Θ, we define its volume, regularized by the functional Φ, to be

$$\widetilde{\mathcal{V}}_{\Phi}(V) := \int_{\Theta} \Phi(V) \; \nu_g .$$

Regularized volume

Theorem

Let E denote the constant $m \times m$ matrix defined by

$$E_{\mu\nu} = \frac{1}{2}\mathrm{Tr}[(\partial_\mu V)(\partial_\nu V)], \quad 1 \leq \mu, \nu \leq m.$$

The Fisher-Rao information matrix g satisfies

$$\det g \leq \left(\frac{\lambda_{\max}[\mathrm{adj}(V)]}{\det V}\right)^{2m} \det(E) = \left(\frac{1}{\lambda_{\min}(V)}\right)^{2m} \det(E),$$

where $\lambda_{\max}[\mathrm{adj}(V)]$ denotes the largest eigenvalue of $\mathrm{adj}(V)$ and $\lambda_{\min}(V)$ denotes the smallest eigenvalue of V.

Regularized volume

Corollary

The regularized volume element satisfies

$$\Phi(V)\sqrt{\det g} \leq \sqrt{\det E}\ H(\mathbf{E} - \mathrm{Tr}(V))\lambda^m_{\max}[\mathrm{adj}(V)]\frac{\log[1 + (\det V)^m]}{(\det V)^m}.$$

Consequently, the integral

$$\int_\Theta \Phi(V)\sqrt{\det g}\, d\theta,$$

is well-defined and bounded for any measurable subset $\Theta \subset \mathbb{R}^m$ over which V is positive definite.

Remark

The function $\Phi(V)$ is not invariant under symplectic transformations.

Regularized volume

Consider the function

$$\Upsilon(V) := e^{-\frac{1}{\kappa}\mathrm{Tr}[\mathrm{adj}(V)]} \log[1 + (\det V)^m],$$

with $\kappa \in \mathbb{R}_+$.

Proposition

Let V, V' be two covariance matrices and Π be a permutation matrix (resp., S be a symplectic matrix) such that $V' = \Pi^T\ V\ \Pi$ (resp. $V' = S^T\ V\ S$), then

$$\Upsilon(V') = \Upsilon(V).$$

Regularized volume

Definition

Given a set of Gaussian states represented by a parameter space Θ, we define its volume, regularized by the functional Υ, to be

$$\widetilde{\mathcal{V}}_{\Upsilon}(V) := \int_{\Theta} \Upsilon(V)\ \nu_g .$$

Corollary

The regularized volume element satisfies

$$\Upsilon(V)\sqrt{\det g} \leq \sqrt{\det E}\exp(-\mathrm{Tr}[\mathrm{adj}(V)])\lambda_{\max}^{m}[\mathrm{adj}(V)]\frac{\log[1+(\det V)^m]}{(\det V)^m}.$$

Consequently, the integral $\int_\Theta \Upsilon(V)\sqrt{\det g}\, d\theta$ *is well-defined and bounded for any measurable subset* $\Theta \subset \mathbb{R}^m$ *over which* V *is positive definite.*

Application to two-mode systems

The most general parametrization of a two-mode covariance matrix $V(\theta)$ is realized through its *canonical form* and it only employs four parameters,

$$V(\theta) = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix}.$$

Thus,

$$\begin{aligned} \Theta_{\text{classic}} &= \{(a,b,c,d) \in \mathbb{R}^4 |\ V(\theta) > 0\} \\ \Theta_{\text{quantum}} &= \{(a,b,c,d) \in \mathbb{R}^4 |\ V(\theta) + i\Omega \geq 0\} \\ \Theta_{\text{separable}} &= \{(a,b,c,d) \in \mathbb{R}^4 |\ V(\theta) + i\Omega \geq 0, V(\theta) + i\widetilde{\Omega} \geq 0\}, \end{aligned}$$

where $\theta^1 = \theta^5 = a \in \mathbb{R}$, $\theta^8 = \theta^{10} = b \in \mathbb{R}$, $\theta^3 = c \in \mathbb{R}$ and $\theta^7 = d \in \mathbb{R}$.

Application to two-mode systems

Finally,

$$\int_{\Theta_{\text{separable}}} \Phi(V)\ \nu_g \le \int_{\Theta_{\text{quantum}}} \Phi(V)\ \nu_g \le \int_{\Theta_{\text{classic}}} \Phi(V)\ \nu_g,$$

for every $\mathbf{E} \in \mathbb{R}_+$. Here,

$$\Phi(V) = H(\mathbf{E} - 2(a+b))\ \log\left[1 + \left((ab - c^2)(ab - d^2)\right)^4\right].$$

And

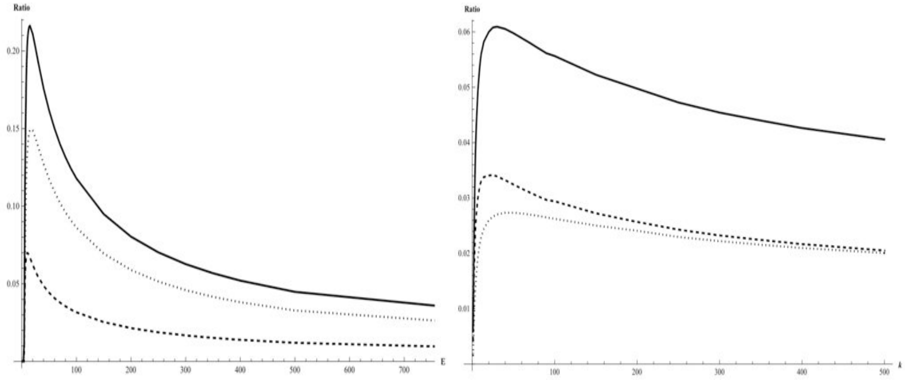
$$\int_{\Theta_{\text{separable}}} \Upsilon(V)\ \nu_g \le \int_{\Theta_{\text{quantum}}} \Upsilon(V)\ \nu_g \le \int_{\Theta_{\text{classic}}} \Upsilon(V)\ \nu_g,$$

with

$$\Upsilon(V) = e^{-\frac{1}{\kappa}\left(2a^2b + a(2b^2 - c^2 - d^2) - b(c^2+d^2)\right)}\ \log\left[1 + \left((ab - c^2)(ab - d^2)\right)^4\right]$$

and for all $\kappa \in \mathbb{R}_+$.

Application to two-mode systems



Solid: quantum over classical volume; Dashed: entangled over classical volume; dotted separable over classical volume.

Conclusion and outlook

- We have considered the phase space as the common playground for describing both classical and quantum states
- We have dealt with classical and quantum Gaussian states as pdfs
- By Information Geometry we have associated Riemannian manifolds to different sets of states
- Regularization for the volume measures is needed
- We have shown strict chains of inclusions for volume of sets of states depending on the regularization's symmetry
- Extension to other states by using Husimi-Q
- Possible comparison with volumes derived by the measure introduced in [C. Lupo et al. J. Math. Phys. (2012)]
- What's about quantum Fisher [P. Facchi et al. Phys. Lett. A (2010)]

D. Felice, M. Hà Quang, S. Mancini, *The volume of Gaussian states by information geometry*, arXiv:1509.01049 [math-ph] (2015).