Nonlinear equations from linear ones

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Geometria è Fisica, a Geometrical Vision of Physics, July 11, 2016

Matrix equations

- G – a n-dimensional Lie group,
- $\mathfrak{g}$ – its Lie algebra.
- Equation
$$\frac{d}{dt}K(t) = M(t)K(t), \quad K(0) = I,$$
where
- $\mathbb{R} \ni t \mapsto M(t) \in \mathfrak{g}$ – a given curve in $\mathfrak{g}$,
- $K(t)$ – a curve in G, which is a solution to the equation.

Motivation

In quantum mechanical applications the Schrödinger equation for an n-level system governed by a time-dependent Hamiltonian reads:

$$i\frac{d\psi}{dt} = H(t)\psi. \tag{1}$$

Writing the solution of with an initial condition $\psi(0)$ as

$\psi(t) = U(t)\psi(0)$ and substituting $H(t) = iM(t)$, we obtain:

$$\frac{d}{dt}U(t) = M(t)U(t), \quad U(0) = I,$$

where $U(t) \in G = U(n)$.

This equations can be also treated as a classical control system on the Lie group G.

Wei-Norman method

- G – n-dimensional Lie group
- $\mathfrak{g}$ – its Lie algebra (simple, complex)
- $\mathbb{R} \ni t \mapsto M(t) \in \mathfrak{g}$ – a curve in $\mathfrak{g}$.
- $K(t)$ – a curve in G given by the differential equation:

$$\frac{d}{dt}K(t) = M(t)K(t), \quad K(0) = I.$$

- X_k, $k = 1, \ldots, n$ is some basis in $\mathfrak{g}$, then:

$$M(t) = \sum_{k=1}^{n} a_k(t)X_k.$$

- We look for the solution $K(t)$ in the form

$$K(t) = \prod_{k=1}^{n} \exp\left(u_k(t)X_k\right).$$

Wei-Norman method

- Differentiating ($' = d/dt$) and commuting...

$$K' = \sum_{l=1}^{n} u_l' \prod_{k<l} \mathrm{Ad}_{\exp(u_k X_k)} X_l \, K,$$

where Ad is the adjoint action of G on $\mathfrak{g}$,

$$\mathrm{Ad}_g X := g X g^{-1}, \quad g \in G, \quad X \in \mathfrak{g}.$$

- Using

$$\mathrm{Ad}_{\exp(f \cdot X)} = \exp(f \cdot \mathrm{ad}_X),$$

where $\mathrm{ad}_X = [X, \ \cdot \ ]$ is the adjoint action of $\mathfrak{g}$ on itself, we obtain:

- ... we get

$$K' = \sum_{l=1}^{n} u_l' \prod_{k<l} \exp(u_k \mathrm{ad}_{X_k}) X_l \, K.$$

Wei-Norman method

- Comparing

$$\frac{d}{dt}K(t) = \underbrace{\sum_{l=1}^{n} u_l' \prod_{k<l} \exp(u_k \mathrm{ad}_{X_k}) \cdot X_l}_{M(t) = \sum_{k=1}^{n} a_k(t) X_k} K(t)$$

- ... we obtain equations for the (unknown) coefficients u_j

$$\mathbf{a} = A\mathbf{u}', \quad \mathbf{u}' = A^{-1}\mathbf{a}.$$

where

$$A_{jl} = A_{jl}^{(l)}, \quad A^{(l)} = \prod_{k<l} \exp(u_k \mathrm{ad}_{X_k})$$

It can be shown, that A is invertible, at least locally

Choice of basis

- A depends on the choice of (an ordered) basis in $\mathfrak{g}$
- Example: $\mathfrak{sl}(2,\mathbb{C})$ (J. Cariñena, J. Grabowski, G. Marmo, *Lie-Scheffers Systems: A Geometric AApproach*) for

$$X_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

we get

$$u_1' = a_1 e^{-u_2} - a_3 u_1^2 e^{u_2}, \quad u_2' = a_2 + 2a_3 u_1 e^{u_2}, \quad u_3' = a_3 \mathrm{e}^{u_2}$$

whereas, for

$$X_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

a much 'nicer' Riccati system

$$u_1' = a_1 + a_2 u_1 + a_3 u_1^2, \quad u_2' = a_2 + 2a_3 u_1, \quad u_3' = a_3 \mathrm{e}^{u_2}$$

- How to find such a 'canonical' ordered basis for an arbitrary Lie group?

Example: spin in rotating magnetic field

- Time dependent Hamiltonian:

$$H(t) = -\frac{B\cos(\omega t)}{2}\sigma_x - \frac{B\sin(\omega t)}{2}\sigma_y = -\frac{B}{2}\begin{bmatrix} 0 & \exp(-i\omega t) \\ \exp(i\omega t) & 0 \end{bmatrix}, \quad U' = -iHU$$

- In the above 'nice' basis the equations read:

$$u_1'(t) = \frac{iB}{2}\left(e^{-i\omega t} + e^{i\omega t}(u_1(t))^2\right), \quad u_2'(t) = iBe^{i\omega t}u_1(t), \quad u_3'(t) = \frac{iB}{2}e^{i\omega t}e^{u_2(t)}$$

- ... and are easily solved

$$U = \begin{bmatrix} \left(\cos\left(\frac{\Omega t}{2}\right) + \frac{i\sin\left(\frac{\Omega t}{2}\right)\omega}{\Omega}\right)e^{-i\frac{\omega t}{2}} & -\frac{B}{2\Omega}\left(e^{-it\Omega} - 1\right)e^{i\frac{\Omega-\omega}{2}t} \\ \frac{B}{2\Omega}\left(e^{it\Omega} - 1\right)e^{-i\frac{\Omega-\omega}{2}t} & \left(\cos\left(\frac{\Omega t}{2}\right) - \frac{i\sin\left(\frac{\Omega t}{2}\right)\omega}{\Omega}\right)e^{i\frac{\omega t}{2}} \end{bmatrix}, \quad \Omega = \sqrt{B^2 + \omega^2}$$

Example: $\mathfrak{sl}(3,\mathbb{C})$

1. A system of two coupled Riccati equations:

$$
\begin{aligned}
u_1' &= a_1 + (2a_5 - a_4)u_1 + a_6 u_2 - a_8 u_1^2 - a_7 u_1 u_2,\\
u_2' &= a_2 + a_3 u_1 + (a_4 + a_5)u_2 - a_8 u_1 u_2 - a_7 u_2^2,
\end{aligned}
$$

2. A scalar Riccati equation for u_3:

$$
u_3' = (a_3 - a_8 u_2) + (2a_4 - a_5 + a_8 u_1 - a_7 u_2)\, u_3 + (a_7 u_1 - a_6)\, u_3^2,
$$

3. The rest

$$
\begin{aligned}
u_4' &= a_4 - a_6 u_3 + a_7\,(u_1 u_3 - u_2)\\
u_5' &= a_5 - a_8 u_1 - a_7 u_2\\
u_6' &= (a_6 - a_7 u_1)\,\mathrm{e}^{2\,u_4 - u_5},\\
u_7' &= (a_7 u_3 + a_8)u_6 \mathrm{e}^{-u_4 + 2\,u_5} + a_7 \mathrm{e}^{u_4 + u_5},\\
u_8' &= (a_8 + a_7 u_3)\mathrm{e}^{-u_4 + 2\,u_5},
\end{aligned}
$$

   which are solved by simple consecutive integrations, once solutions of the Riccati equations are known.

Arbitrary simple Lie algebra $\mathfrak{g}$

- The construction of a ‘canonical’ basis can be done for all simple algebras (classical and exceptional) except the exceptional ones G_2, F_4, and E_8
- The resulting Wei-Norman equations split into several blocks of coupled Riccati and linear equations and ‘trivial’ ones (solvable by consecutive integrations)
- The construction hinges on two things
  - A decomposition of a simple Lie algebra into a sum of commutative algebras (can be done for all simple Lie algebras)
  - A particular property of the adjoint endomorhism (in all simple Lie algebras, except G_2, F_4, and E_8)

Decomposition

- **The Cartan decomposition**

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi_-} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$$

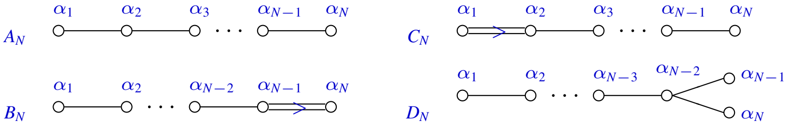
where $\mathfrak{h}$ – the Cartan subalgebra, $\mathfrak{g}$ – the root spaces,

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}.$$

and $\Phi_\pm$ – the set of positive (negative) roots.

- **Decomposition into commutative subalgebras**

For a basis $\{\alpha_1, \ldots, \alpha_N\}$ of the roots system consisting of positive simple roots



define

$$\Phi_k := \left\{ \beta : \beta = \sum_k^N n_i \alpha_i, n_i > 0 \right\}, \quad \mathfrak{a}_k := \operatorname{span}\{X_\beta : \beta \in \Phi_k\}, \quad \widetilde{\mathfrak{a}}_k := \operatorname{span}\{X_{-\beta} : \beta \in \Phi_k\}$$

$\mathfrak{a}_k$ are commutative and

$$\mathfrak{g} = \bigoplus_{k=1}^N \mathfrak{a}_k \oplus \mathfrak{h} \oplus \bigoplus_{j=1}^n \widetilde{\mathfrak{a}}_j$$

Properties of the adjoint endomorphism

- $\mathfrak{g}$ – a simple Lie algebra not equal to G_2, F_4, E_8
- α – a root and $X_\alpha \in \mathfrak{g}$ – the corresponding root vector

  Then

  1. The image of $(\mathrm{ad}_{X_\alpha})^2$ is equal to $\mathfrak{g}_\alpha$
  2. $(\mathrm{ad}_{X_\alpha})^3 = 0$.

- **Remark**: for G_2, F_4, and E_8, there are roots for which $(\mathrm{ad}_{X_\alpha})^3 \neq 0$, but for all roots we have $(\mathrm{ad}_{X_\alpha})^5 = 0$
- **Corollary**: For $X \in \mathfrak{a}_k$ or $X \in \widetilde{\mathfrak{a}}_k$ the matrix of $\exp(\mathrm{ad}_X)$ is a quadratic polynomial in ad_X and it is block diagonal with respect to the decomposition

$$\mathfrak{g} = \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_{k-1} \oplus \underbrace{\left(\mathfrak{b}_k \oplus \mathfrak{h} \oplus \widetilde{\mathfrak{b}}_k\right)}_{\text{one block}} \oplus \widetilde{\mathfrak{a}}_{k-1} \oplus \ldots \oplus \widetilde{\mathfrak{a}}_1 .$$

- The equation $\mathbf{u}' = A^{-1}\mathbf{a}$ separates into blocks:
  - matrix $N \times N$ Riccati equation for $u_1, \ldots, u_N$, corresponding to $\mathfrak{a}_1$,
  - matrix $(N-1) \times (N-1)$ Riccati equation for $u_{N+1}, \ldots, u_{2N-1}$, corresponding to $\mathfrak{a}_2$,

    ⋮

  - scalar Riccati equation for $u_{N(N+1)/2}$, corresponding to $\mathfrak{a}_N$,
  - the remaining equations which, are solved by consecutive integrations (provided the solutions of Riccati equations are known).

Summary

- An algorithm for reducing the highly non linear system of Wei-Norman equations

$$\sum_{k=1}^{n} a_k X_k = \sum_{l=1}^{n} u_l' \prod_{k<l} \exp(u_k \mathrm{ad}_{X_k}) \cdot X_l,$$

for the parameters u_k to a hierarchy of Riccati matrix equations and integrals.

- It is known that every system of Riccati equations is related to a Lie group action and solution of this system is equivalent to solution of the system of the form

$$\frac{d}{dt} K(t) = M(t)K(t), \quad K(0) = I,$$

but not every system of this form is equivalent to Riccati equation system. Here we provide an explicit construction of a hierarchy of Riccati matrix equations equivalent to the equation on a Lie group for all classical groups (and the exceptional ones E_6 and E_7)

- For the exceptional algebras G_2, F_4,a nd E_8 the nonlinearities are of 4-th order (c.f. the property of the adjoint endomorphism)

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