Nonlinear equations from linear ones

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Geometria è Fisica, a Geometrical Vision of Physics, July 11, 2016
Matrix equations

- $G$ – a $n$-dimensional Lie group,

- $g$ – its Lie algebra.

- Equation

$$\frac{d}{dt} K(t) = M(t)K(t), \quad K(0) = I,$$

where

- $\mathbb{R} \ni t \mapsto M(t) \in g$ – a given curve in $g$,

- $K(t)$ – a curve in $G$, which is a solution to the equation.
Motivation

In quantum mechanical applications the Schrödinger equation for an n-level system governed by a time-dependent Hamiltonian reads:

\[ i \frac{d\psi}{dt} = H(t)\psi. \]  

(1)

Writing the solution of with an initial condition \( \psi(0) \) as

\[ \psi(t) = U(t)\psi(0) \]

and substituting \( H(t) = iM(t) \), we obtain:

\[ \frac{d}{dt}U(t) = M(t)U(t), \quad U(0) = I, \]

where \( U(t) \in G = U(n) \).

This equations can be also treated as a classical control system on the Lie group \( G \).
Wei-Norman method

- $G - n$-dimensional Lie group
- $g$ - its Lie algebra (simple, complex)
- $\mathbb{R} \ni t \mapsto M(t) \in g$ - a curve in $g$.
- $K(t)$ - a curve in $G$ given by the differential equation:

$$\frac{d}{dt} K(t) = M(t)K(t), \quad K(0) = I.$$ 

- $X_k, k = 1, \ldots, n$ is some basis in $g$, then:

$$M(t) = \sum_{k=1}^{n} a_k(t)X_k.$$ 

- We look for the solution $K(t)$ in the form

$$K(t) = \prod_{k=1}^{n} \exp (u_k(t)X_k).$$
Wei-Norman method

- Differentiating \((' = d/dt)\) and commuting...

\[
K' = \sum_{l=1}^{n} u'_l \prod_{k<l} \text{Ad}_{\exp(u_k x_k)} X_l K,
\]

where \(\text{Ad}\) is the adjoint action of \(G\) on \(g\),
\[
\text{Ad}_g X := gXg^{-1}, \quad g \in G, \quad X \in g.
\]

- Using

\[
\text{Ad}_{\exp(f \cdot X)} = \exp(f \cdot \text{ad}_X),
\]

where \(\text{ad}_X = [X, \cdot ]\) is the adjoint action of \(g\) on itself, we obtain:

- ... we get

\[
K' = \sum_{l=1}^{n} u'_l \prod_{k<l} \exp(u_k \text{ad}_{x_k}) X_l K.
\]
Wei-Norman method

Comparing

\[ \frac{d}{dt} K(t) = \sum_{l=1}^{n} u'_l \prod_{k \prec l} \exp(u_k \text{ad}_{X_k}) \cdot X_l \cdot K(t) \]

\[ M(t) = \sum_{k=1}^{n} a_k(t)X_k \]

... we obtain equations for the (unknown) coefficients \( u_j \)

\[ \mathbf{a} = A \mathbf{u}', \quad \mathbf{u}' = A^{-1} \mathbf{a}. \]

where

\[ A_{jl} = A_{jl}^{(l)}, \quad A^{(l)} = \prod_{k \prec l} \exp(u_k \text{ad}_{X_k}) \]

It can be shown, that \( A \) is invertible, at least locally
Choice of basis

- $A$ depends on the choice of (an ordered) basis in $\mathfrak{g}$

- Example: $\mathfrak{sl}(2, \mathbb{C})$ (J. Cariñena, J. Grabowski, G. Marmo, Lie-Scheffers Systems: A Geometric Approach) for

$$X_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

we get

$$u'_1 = a_1 e^{-u_2} - a_3 u_1^2 e^{u_2}, \quad u'_2 = a_2 + 2a_3 u_1 e^{u_2}, \quad u'_3 = a_3 e^{u_2}$$

whereas, for

$$X_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

a much 'nicer' Riccati system

$$u'_1 = a_1 + a_2 u_1 + a_3 u_1^2, \quad u'_2 = a_2 + 2a_3 u_1, \quad u'_3 = a_3 e^{u_2}$$

- How to find such a ‘canonical’ ordered basis for an arbitrary Lie group?
Example: spin in rotating magnetic field

- Time dependent Hamiltonian:

\[
H(t) = -\frac{B \cos(\omega t)}{2} \sigma_x - \frac{B \sin(\omega t)}{2} \sigma_y = -\frac{B}{2} \begin{bmatrix} 0 & \exp(-i\omega t) \\ \exp(i\omega t) & 0 \end{bmatrix}, \quad U' = -iHU
\]

- In the above ‘nice’ basis the equations read:

\[
u'_1(t) = \frac{iB}{2} \left( e^{-i\omega t} + e^{i\omega t} (u_1(t))^2 \right), \quad u'_2(t) = iBe^{i\omega t} u_1(t), \quad u'_3(t) = \frac{iB}{2} e^{i\omega t} e^{u_2(t)}
\]

- ... and are easily solved

\[
U = \begin{bmatrix}
\left( \cos \left( \frac{\Omega t}{2} \right) + \frac{i \sin \left( \frac{\Omega t}{2} \right) \omega}{\Omega} \right) e^{-i \frac{\Omega t}{2}} & -\frac{B}{2\Omega} \left( e^{-it\Omega} - 1 \right) e^{i \frac{\Omega - \omega}{2} t} \\
\frac{B}{2\Omega} \left( e^{it\Omega} - 1 \right) e^{-i \frac{\Omega - \omega}{2} t} & \left( \cos \left( \frac{\Omega t}{2} \right) - \frac{i \sin \left( \frac{\Omega t}{2} \right) \omega}{\Omega} \right) e^{i \frac{\omega t}{2}}
\end{bmatrix}, \quad \Omega = \sqrt{B^2 + \omega^2}
\]
Example: \( \text{SL}(3, \mathbb{C}) \)

1. A system of two coupled Riccati equations:

\[
\begin{align*}
    u_1' &= a_1 + (2a_5 - a_4)u_1 + a_6u_2 - a_8u_1^2 - a_7u_1u_2, \\
    u_2' &= a_2 + a_3u_1 + (a_4 + a_5)u_2 - a_8u_1u_2 - a_7u_2^2,
\end{align*}
\]

2. A scalar Riccati equation for \( u_3 \):

\[
    u_3' = (a_3 - a_8u_2) + (2a_4 - a_5 + a_8u_1 - a_7u_2)u_3 + (a_7u_1 - a_6)u_3^2,
\]

3. The rest

\[
\begin{align*}
    u_4' &= a_4 - a_6u_3 + a_7(u_1u_3 - u_2) \\
    u_5' &= a_5 - a_8u_1 - a_7u_2 \\
    u_6' &= (a_6 - a_7u_1)e^{2u_4 - u_5} \\
    u_7' &= (a_7u_3 + a_8)u_6e^{-u_4 + 2u_5} + a_7e^{u_4 + u_5} \\
    u_8' &= (a_8 + a_7u_3)e^{-u_4 + 2u_5},
\end{align*}
\]

which are solved by simple consecutive integrations, once solutions of the Riccati equations are known.
Arbitrary simple Lie algebra $\mathfrak{g}$

- The construction of a ‘canonical’ basis can be done for all simple algebras (classical and exceptional) except the exceptional ones $G_2$, $F_4$, and $E_8$.

- The resulting Wei-Norman equations split into several blocks of coupled Riccati and linear equations and ‘trivial’ ones (solvable by consecutive integrations).

- The construction hinges on two things:
  - A decomposition of a simple Lie algebra into a sum of commutative algebras (can be done for all simple Lie algebras).
  - A particular property of the adjoint endomorhism (in all simple Lie algebras, except $G_2$, $F_4$, and $E_8$).
Decomposition

- **The Cartan decomposition**

\[
g = \bigoplus_{\alpha \in \Phi_-} g_\alpha \oplus h \oplus \bigoplus_{\alpha \in \Phi_+} g_\alpha
\]

where \( h \) – the Cartan subalgebra, \( g \) – the root spaces,

\[
g_\alpha := \{ X \in g : [H, X] = \alpha(H)X \ \forall H \in h \}.
\]

and \( \Phi_{\pm} \) – the set of positive (negative) roots.

- **Decomposition into commutative subalgebras**

For a basis \( \{\alpha_1, \ldots, \alpha_N\} \) of the roots system consisting of positive simple roots

\[
\begin{align*}
A_N & : \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \cdots \quad \alpha_{N-1} \quad \alpha_N \\
B_N & : \quad \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{N-2} \quad \alpha_{N-1} \quad \alpha_N
\end{align*}
\]

\[
\begin{align*}
C_N & : \quad \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{N-1} \quad \alpha_N \quad \alpha_{N-1} \\
D_N & : \quad \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{N-2} \quad \alpha_{N-1} \quad \alpha_N
\end{align*}
\]

define

\[
\Phi_k := \left\{ \beta : \beta = \sum_{i=1}^{N} n_i \alpha_i, n_i > 0 \right\}, \quad \alpha_k := \text{span} \{ X_\beta : \beta \in \Phi_k \}, \quad \tilde{\alpha}_k := \text{span} \{ X_{-\beta} : \beta \in \Phi_k \}
\]

\( \alpha_k \) are commutative and

\[
g = \bigoplus_{k=1}^{N} \alpha_k \oplus h \oplus \bigoplus_{j=1}^{n} \tilde{\alpha}_j
\]
Properties of the adjoint endomorphism

- $g$ – a simple Lie algebra not equal to $G_2$, $F_4$, $E_8$
- $\alpha$ – a root and $X_\alpha \in g$ – the corresponding root vector

Then

1. The image of $(\text{ad}_{X_\alpha})^2$ is equal to $g_\alpha$
2. $(\text{ad}_{X_\alpha})^3 = 0$.

- **Remark:** for $G_2$, $F_4$, and $E_8$, there are roots for which $(\text{ad}_{X_\alpha})^3 \neq 0$, but for all roots we have $(\text{ad}_{X_\alpha})^5 = 0$

- **Corollary:** For $X \in a_k$ or $X \in \tilde{a}_k$ the matrix of $\exp(\text{ad}_X)$ is a quadratic polynomial in $\text{ad}_X$ and it is block diagonal with respect to the decomposition

$$g = a_1 \oplus \ldots \oplus a_{k-1} \oplus \left( b_k \oplus h \oplus \tilde{b}_k \right) \oplus \tilde{a}_{k-1} \oplus \ldots \oplus \tilde{a}_1.$$

- The equation $u' = A^{-1}a$ separates into blocks:
  - matrix $N \times N$ Riccati equation for $u_1, \ldots, u_N$, corresponding to $a_1$,
  - matrix $(N-1) \times (N-1)$ Riccati equation for $u_{N+1}, \ldots, u_{2N-1}$, corresponding to $a_2$,
  - \ldots
  - scalar Riccati equation for $u_{N(N+1)/2}$, corresponding to $a_N$,
  - the remaining equations which, are solved by consecutive integrations (provided the solutions of Riccati equations are known).
Summary

- An algorithm for reducing the highly non linear system of Wei-Norman equations

\[ \sum_{k=1}^{n} a_k X_k = \sum_{l=1}^{n} u'_l \prod_{k<l} \exp(u_k \text{ad}X_k) \cdot X_l, \]

for the parameters \( u_k \) to a hierarchy of Riccati matrix equations and integrals.

- It is known that every system of Riccati equations is related to a Lie group action and solution of this system is equivalent to solution of the system of the form

\[ \frac{d}{dt} K(t) = M(t)K(t), \quad K(0) = I, \]

but not every system of this form is equivalent to Riccati equation system. Here we provide an explicit construction of a hierarchy of Riccati matrix equations equivalent to the equation on a Lie group for all classical groups (and the exceptional ones \( E_6 \) and \( E_7 \))

- For the exceptional algebras \( G_2, F_4, \) and \( E_8 \) the nonlinearities are of 4-th order (c.f. the property of the adjoint endomorphism)
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