Line bundles over noncommutative spaces

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Abstract:

- A procedure to construct total spaces of principal bundles out of a Fock-space construction

- Pimsner algebras of ‘tautological’ line bundles:

- Gysin-like sequences in KK-theory

- some hint to T-dual noncommutative bundles

- Examples:

Quantum lens spaces as direct sums of line bundles over weighted quantum projective spaces

Irrational Rotation Algebra for quadratic irrationals

A starting motivation: Gysin sequences for U(1)-bundles

among other things it relates:

H-flux (three-forms on the total space E of a bundle)

to

line bundles on the base space M (two-forms classifying line bundles)

also giving an isomorphism between Dixmier-Douady classes on E (gerbes over the base space M) and line bundles on M

The classical Gysin sequence

Long exact sequence in cohomology; for any sphere bundle

In particular, for circle bundles: $\mathrm{U}(1) \to E \xrightarrow{\pi} X$

$$\cdots \longrightarrow H^k(E) \xrightarrow{\pi_*} H^{k-1}(X) \xrightarrow{\cup c_1(E)} H^{k+1}(X) \xrightarrow{\pi^*} H^{k+1}(E) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^3(X) \xrightarrow{\pi^*} H^3(E) \xrightarrow{\pi_*} H^2(X) \xrightarrow{\cup c_1(E)} H^4(X) \longrightarrow \cdots$$

$$H^3(E) \ni H \mapsto \pi_*(H) = F' = c_1(E')$$

$$
\begin{tikzcd}
 & E \times_M E' \arrow[dl] \arrow[dr] & \\
E \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\
 & X &
\end{tikzcd}
$$

$$\cdots \longrightarrow H^3(X) \xrightarrow{\pi^*} H^3(E') \xrightarrow{\pi_*} H^2(X) \xrightarrow{\cup c_1(E')} H^4(X) \longrightarrow \cdots$$

$$F' \cup F = 0 = F \cup F'$$

and since the sequences are exact

$$\Rightarrow \quad \exists \quad H^3(E') \ni H' \mapsto \pi_*(H) = F = c_1(E)$$

T-dual (E,H) and (E',H') Bouwknegt, Evslin, Mathai, 2004

Difficult to generalize to quantum spaces

rather go to K-theory ; a six term exact sequence

Example: projective spaces and lens spaces

A Gysin sequence in topological K-theory:

$$0 \longrightarrow K^1(\mathsf{L}^{(n,r)}) \xrightarrow{\delta} K^0(\mathbb{C}\mathsf{P}^n) \xrightarrow{\alpha} K^0(\mathbb{C}\mathsf{P}^n) \xrightarrow{\pi^*} K^0(\mathsf{L}^{(n,r)}) \longrightarrow 0$$

δ is a 'connecting homomorphism'

α is multiplication by the Euler class $\chi(\mathcal{L}_{-r}) := 1 - [\mathcal{L}_{-r}]$

Projective and lens spaces

$$\mathbb{C}\mathsf{P}^n = \mathsf{S}^{2n+1}/\mathsf{U}(1) \qquad \text{and} \qquad L^{(n,r)} = \mathsf{S}^{2n+1}/\mathbb{Z}_r$$

assemble in principal bundles : $\mathsf{S}^{2n+1} \longrightarrow \mathsf{L}^{(n,r)} \xrightarrow{\pi} \mathbb{C}\mathsf{P}^n$

From the Gysin sequence:

$$K^1(\mathsf{L}^{(n,r)}) \simeq \ker(\alpha) \quad \text{and} \quad K^0(\mathsf{L}^{(n,r)}) \simeq \mathrm{coker}(\alpha)$$

torsion groups

U(1)-principal bundles

The Hopf algebra

$$\mathcal{H} = \mathcal{O}(\mathrm{U}(1)) := \mathbb{C}[z, z^{-1}]/\left\langle 1 - zz^{-1} \right\rangle$$

$$\Delta : z^n \mapsto z^n \otimes z^n \quad ; \qquad S : z^n \mapsto z^{-n} \quad ; \qquad \epsilon : z^n \mapsto 1$$

Let $\mathcal{A}$ be a right comodule algebra over $\mathcal{H}$ with coaction

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$$

$\mathcal{B} := \{x \in \mathcal{A} \mid \Delta_R(x) = x \otimes 1\}$ be the subalgebra of coinvariants

Definition 1. *The datum* $\left(\mathcal{A}, \mathcal{H}, \mathcal{B}\right)$ *is a* quantum principal U(1)-bundle *when the* canonical map *is an isomorphism*

$$\mathrm{can} : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}\,, \quad x \otimes y \mapsto x\, \Delta_R(y)\,.$$

$\mathbb{Z}$-graded algebras

$\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ a $\mathbb{Z}$-graded algebra. A right $\mathcal{H}$-comodule algebra:

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H} \quad x \mapsto x \otimes z^{-n}, \text{ for } x \in \mathcal{A}_n\,,$$

with the subalgebra of coinvariants given by $\mathcal{A}_0$.

Proposition 2. *The triple* $\left(\mathcal{A}, \mathcal{H}, \mathcal{A}_0\right)$ *is a quantum principal* U(1)-*bundle if and only if there exist finite sequences*

$$\{\xi_j\}_{j=1}^{N}\,,\ \{\beta_i\}_{i=1}^{M} \textit{ in } \mathcal{A}_1 \quad \text{and} \quad \{\eta_j\}_{j=1}^{N}\,,\ \{\alpha_i\}_{i=1}^{M} \textit{ in } \mathcal{A}_{-1}$$

such that:

$$\sum\nolimits_{j=1}^{N} \xi_j \eta_j = 1_{\mathcal{A}} = \sum\nolimits_{i=1}^{M} \alpha_i \beta_i\,.$$

Corollary 3. *Same conditions as above. The right-modules* $\mathcal{A}_1$ *and* $\mathcal{A}_{-1}$ *are finitely generated and projective over* $\mathcal{A}_0$*.*

Proof. For $\mathcal{A}_1$: define the module homomorphisms

$$\Phi_1 : \mathcal{A}_1 \to (\mathcal{A}_0)^N, \quad \Phi_1(\zeta) = \begin{pmatrix} \eta_1\,\zeta \\ \eta_2\,\zeta \\ \vdots \\ \eta_N\,\zeta \end{pmatrix} \quad \text{and}$$

$$\Psi_1 : (\mathcal{A}_0)^N \to \mathcal{A}_1, \quad \Psi_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \sum_j \xi_j\, x_j\,.$$

Then $\Psi_1\Phi_1 = \mathrm{Id}_{\mathcal{A}_1}$.

Thus $E_1 := \Phi_1\Psi_1$ is an idempotent in $M_N(\mathcal{A}_0)$. $\square$

The above results show that $\left(\mathcal{A},\mathcal{H},\mathcal{A}_0\right)$ is a quantum principal U(1)-bundle if and only if $\mathcal{A}$ is *strongly* $\mathbb{Z}$-graded, that is

$$\mathcal{A}_n\mathcal{A}_m = \mathcal{A}_{n+m}$$

Equivalently, the right-modules $\mathcal{A}_{(\pm 1)}$ are finitely generated and projective over $\mathcal{A}_0$ if and only if $\mathcal{A}$ is *strongly* $\mathbb{Z}$-graded

C. Nastasescu, F. Van Oystaeyen, *Graded Ring Theory*

K.H. Ulbrich, 1981

More general scheme: Pimsner algebras M.V. Pimsner '97

The right-modules $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ before are ‘line bundles’ over $\mathcal{A}_0$

The slogan:

a line bundle

is a

self-Morita equivalence bimodule

Morita equivalence in one-page

E a (right) Hilbert module over the algebra B

B-valued hermitian structure $\quad \langle\cdot,\cdot\rangle_{\bullet} \quad$ on E

$\mathcal{L}(E)$ adjointable operators; $\mathcal{K}(E) \subseteq \mathcal{L}(E)$ compact operators

With $\xi, \eta \in E$, denote $\theta_{\xi,\eta} \in \mathcal{K}(E)$ defined by $\theta_{\xi,\eta}(\zeta) = \xi \langle\eta,\zeta\rangle_{\bullet}$

$\mathcal{K}(E)$-valued hermitian structure $\quad {}_{\bullet}\langle\cdot,\cdot\rangle \quad$ on E: $\quad {}_{\bullet}\langle\xi,\eta\rangle := \theta_{\xi,\eta}$

The hermitian structures are compatible by construction

$${}_{\bullet}\langle\xi,\eta\rangle\,\zeta = \xi\,\langle\eta,\zeta\rangle_{\bullet}$$

Algebras $\mathcal{K}(E)$ and B are Morita equivalent via the bimodule E.

For a line bundle we are asking that there is an isomorphism $\phi : B \to \mathcal{K}(E)$ and thus E is a B-bimodule

Comparing with before:

$$\mathcal{A}_0 \rightsquigarrow B \qquad \text{and} \qquad \mathcal{A}_{-1} \rightsquigarrow E$$

Look for the analogue of $\mathcal{A} \quad \rightsquigarrow \mathcal{O}_E$ Pimsner algebra

Examples

$B = \mathcal{O}(\mathbb{C}\mathsf{P}^n_q)$ quantum (weighted) projective spaces

$E = \mathcal{L}_{-r} \simeq (\mathcal{L}_{-1})^r$ (powers of) tautological line bundle

$\mathcal{O}_E = \mathcal{O}(\mathsf{L}_q^{(n,r)})$ quantum lens spaces

Define the B-module

$$E_\infty := \bigoplus_{N\in\mathbb{Z}} E^{\otimes_\phi N}, \qquad E^0 = B$$

$E \otimes_\phi E$ the inner tensor product: a B-Hilbert module with B-valued hermitian structure

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_\bullet = \langle \eta_1, \phi(\langle \xi_1, \xi_2 \rangle_\bullet)\eta_2 \rangle_\bullet$$

$E^{-1} = E^*$ the dual module;
its elements are written as λ_ξ for $\xi \in E$: $\lambda_\xi(\eta) = \langle \xi, \eta \rangle_\bullet$

For each $\xi \in E$ a bounded adjointable operator

$$S_\xi : E_\infty \to E_\infty$$

generated by $S_\xi : E^{\widehat{\otimes}_\phi N} \to E^{\widehat{\otimes}_\phi (N+1)}$:

$$\begin{aligned} S_\xi(b) &:= \xi\, b\,, & b \in B\,, \\ S_\xi(\xi_1 \otimes \cdots \otimes \xi_N) &:= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_N\,, & N > 0\,, \\ S_\xi(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-N}}) &:= \lambda_{\xi_2\, \phi^{-1}(\theta_{\xi_1,\xi})} \otimes \lambda_{\xi_3} \otimes \cdots \otimes \lambda_{\xi_{-N}}\,, & N < 0\,. \end{aligned}$$

Definition 4. *The Pimsner algebra $\mathcal{O}_E$ of the pair (ϕ, E) is the smallest subalgebra of $\mathcal{L}(E_\infty)$ which contains the operators $S_\xi : E_\infty \to E_\infty$ for all $\xi \in E$.*

Pimsner: universality of $\mathcal{O}_E$

The Cuntz-Pimsner algebra $\mathcal{O}_E$ is the universal *-algebra generated by B and $\{S_\xi : \xi \in E\}$ with relations

$$S_{\alpha\xi+\beta\eta} = S_{\alpha\xi} + S_{\beta\eta}, \quad aS_\xi b = S_{\phi(a)\xi b}, \quad S_\xi^* S_\eta = \langle \xi, \eta \rangle_\bullet$$

for any $a, b \in B$, $\xi, \eta \in E$, $\alpha, \beta \in \mathbb{C}$

and $\qquad i_{\mathcal{K}}(\phi(a)) = a$

for $a \in B$ with $i_{\mathcal{K}} : \mathcal{K}(E) \to \mathcal{O}_E$ defined by

$$i_{\mathcal{K}}(\theta_{\xi,\eta}) = S_\xi S_\eta^*.$$

There is a natural inclusion

$$B \hookrightarrow \mathcal{O}_E \qquad \text{a generalized principal circle bundle}$$

roughly: as a vector space $\mathcal{O}_E \simeq E_\infty$ and

$$E^{\widehat{\otimes}_\phi N} \ni \eta \mapsto \eta\lambda^{-N}, \qquad \lambda \in \mathsf{U}(1)$$

Conditions for a principal bundle:

being strongly graded for an algebra translates to

a semi-saturated circle action on the algebra

Associated six-terms exact sequences Gysin sequences:
in K-theory:

$$\begin{tikzcd} K_0(B) \arrow[r, "1-[E]"] & K_0(B) \arrow[r, "i_*"] & K_0(\mathcal{O}_E) \arrow[d, "[\partial]"] \\ K_1(\mathcal{O}_E) \arrow[u, "[\partial]"] & K_1(B) \arrow[l, "i_*"'] & K_1(B) \arrow[l, "1-[E]"'] \end{tikzcd} \quad ;$$

the corresponding one in K-homology:

$$\begin{tikzcd} K^0(B) \arrow[d, "[\partial]"] & K^0(B) \arrow[l, "1-[E]"'] & K^0(\mathcal{O}_E) \arrow[l, "i^*"'] \\ K^1(\mathcal{O}_E) \arrow[r, "i^*"] & K^1(B) \arrow[r, "1-[E]"] & K^1(B) \arrow[u, "[\partial]"'] \end{tikzcd} \quad .$$

In fact in KK-theory

Commutative algebras Gabriel, Grensing 2013

Proposition 5. *Let A be a unital, commutative C^*-algebra carrying a circle action. Suppose that the first spectral subspace $E = A_1$ is finitely generated projective over $B = A_0$. Suppose furthermore that E generates A as a C^*-algebra. Then:*

1. *$B = C(X)$ for some compact space X;*

2. *E is the module of sections of some line bundle $L \to X$;*

3. *$A = \mathcal{O}_E = C(P)$, where*

$$\mathsf{U}(1) \to P \to X$$

is the principal $\mathsf{U}(1)$-bundle over X associated with the line bundle L, and the circle action on A comes from the principal $\mathsf{U}(1)$-action on P.

IRA for a quadratic irrational deformation parameter

An ordinary two-torus $\mathbb{T}^2$ with $U_1 = e^{2\pi \mathrm{i} x}$ and $U_2 = e^{2\pi \mathrm{i} y}$

by Fourier expansion the algebra $C^\infty(\mathbb{T}^2)$:

$$a = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n}\, U_1^m U_2^n \ ,$$

with $\{a_{m,n}\} \in S(\mathbb{Z}^2)$ a complex-valued Schwartz function on $\mathbb{Z}^2$

Fix $\theta \in \mathbb{R}$. The algebra $\mathcal{A}_\theta = C^\infty(\mathbb{T}^2_\theta)$ of smooth functions on the noncommutative torus is the associative algebra of all elements of the form above, but now the two generators U_1 and U_2 satisfy

$$U_2 U_1 = e^{2\pi \mathrm{i}\,\theta}\, U_1 U_2$$

Thouless-Kohmoto-Nightingale-Nijs

The Hall conductances for the QHE associated to the energy spectrum of the single particle Hamiltonian operator with periodic potential and magnetic field in the limit of a strong and a weak magnetic field are related by the TKNN-equation

$$N\ t_g + M\ s_g = d_g \qquad\qquad g = 0, \ldots, N_{\mathsf{max}}.$$

The integers t_g and s_g are the Hall conductances associated to the energy spectrum up to the gap g in the limits of a strong or weak magnetic field ($B \gg 1$ or $B \ll 1$) .

If θ a QIN, there is a not trivial fractional transformation

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \qquad \text{such that} \quad \theta_1 := \frac{c + d\theta}{a + b\theta} = \theta.$$

Consider a projection $q \in A_\theta$ such that $\tau(q) = a + b\theta$

Then a Morita equivalence

$$qA_\theta q \longrightarrow qA_\theta \longleftarrow A_\theta$$

As a consequence

$$qA_\theta q \simeq \mathrm{End}_{A_\theta}(qA_\theta) \simeq A_{\theta_1} = A_\theta$$

Denote $\phi : A_\theta \to qA_\theta q$ the above isomorphism and $V_\theta(q) = qA_\theta$

Proposition 6. *The inner tensor product* $V_\theta(q) \otimes_\phi V_\theta(q)$ *is isomorphic to* $V_\theta(q_2)$ *with* q_2 *a projection in* A_θ *s.t.* $\tau(q_2) = (\tau(q))^2$.

Moreover, $V_\theta(q_2)$ *is an equivalence bimodule between* A_θ *and* $q_2 A_\theta q_2 = \mathrm{End}_{A_\theta}(V_\theta(q_2)) \simeq A_{\theta_2} = A_\theta$, *with parameter* θ_2 *given by*

$$\theta_2 = g \cdot \theta_1 = g^2 \cdot \theta.$$

Finally the Pimsner algebra $\mathcal{O}_{(V_\theta(q),\phi)}$ *is generated by* A_θ *and a generator* S_q *with relations, for* $a \in \mathcal{A}_\theta$,

$$aS_q = S_q \phi(a), \qquad S_q^* S_q = q, \qquad S_q S_q^* = \phi^{-1}(q) = 1.$$

Norio Nawata 2012

some q-examples

The quantum spheres

The coordinate algebra $\mathcal{O}(\mathsf{S}_q^{2n+1})$ of quantum sphere S_q^{2n+1}:
$*$-algebra generated by $2n+2$ elements $\{z_i, z_i^*\}_{i=0,\dots,n}$

$+$ relations

$$
\begin{aligned}
z_i z_j &= q^{-1} z_j z_i & \qquad & 0 \le i < j \le n \,, \\
z_i^* z_j &= q z_j z_i^* & & i \neq j \,, \\
[z_n^*, z_n] = 0 \,, \quad [z_i^*, z_i] &= (1-q^2) \sum_{j=i+1}^{n} z_j z_j^* & & i = 0, \dots, n-1 \,,
\end{aligned}
$$

and a sphere relation:

$$1 = z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^* \,.$$

L. Vaksman, Ya. Soibelman, 1991 ; M. Welk, 2000

Quantum quantum weighted projective line and lens spaces:

$B = (W_q(k,l))$ = quantum weighted projective line
the fixed point algebra for a weighted circle action on $\mathcal{O}(S_q^3)$

$$z_0 \mapsto \lambda^k z_0 \,, \quad z_1 \mapsto \lambda^l z_1 \,, \quad \lambda \in \mathrm{U}(1)$$

The corresponding universal enveloping C^*-algebra $C(W_q(k,l))$ does not in fact depend on the label k: isomorphic to the unitalization of l copies of $\mathcal{K}$ = compact operators on $l^2(\mathbb{N}_0)$

$$C(W_q(k,l)) = \widetilde{\oplus_{s=0}^{l}\mathcal{K}}$$

Then: $$K_0(C(W_q(k,l))) = \mathbb{Z}^{l+1} \,, \quad K_1(C(W_q(k,l))) = 0$$

a partial resolution of singularities, since classically

$$K_0(C(W(k,l))) = \mathbb{Z}^2$$

$\mathcal{O}(W_q(k,l))$ is the unital $*$-subalgebra of $\mathcal{O}(S_q^3)$ generated by

$$z_0^l (z_1^*)^k \qquad \text{and} \qquad z_1 z_1^*$$

Alternatively, $\mathcal{O}(W_q(k,l))$ is the universal unital $*$-algebra with generators a, b, subject to the relations

$$b^* = b \ , \quad ba = q^{-2l} \ ab \ ,$$

$$aa^* = q^{2kl} \ b^k \prod_{m=0}^{l-1} (1 - q^{2m} b) \ , \quad a^* a = b^k \prod_{m=1}^{l} (1 - q^{-2m} b) \, .$$

$\mathcal{O}(W_q(1,l))$ was named quantum teardrop by B F

Quantum lens space $= \mathcal{O}_E = \mathcal{O}(L_q(lk;k,l))$

Indeed, a vector space decomposition

$$\mathcal{O}(L_q(lk;k,l)) = \oplus_{N\in\mathbb{Z}}\mathcal{L}_n(k,l)\,,$$

with $E = \mathcal{L}_1(k,l)$ a right finitely projective module

$$\mathcal{L}_1(k,l) := (z_1^*)^k \cdot \mathcal{O}(W_q(k,l)) + (z_0^*)^l \cdot \mathcal{O}(W_q(k,l))$$

Also, $\mathcal{O}(L_q(lk;k,l))$ the fixes point algebra of a cyclic action

$$\mathbb{Z}/(lk)\mathbb{Z} \times S_q^3 \to S_q^3$$

$$z_0 \mapsto \exp(\frac{2\pi \mathrm{i}}{l})\ z_0\,, \quad z_1 \mapsto \exp(\frac{2\pi \mathrm{i}}{k})\ z_1\,.$$

K-theory and K-homology of quantum lens space

Denote the diagonal inclusion by $\iota : \mathbb{Z} \to \mathbb{Z}^l$, $1 \mapsto (1, \ldots, 1)$ with transpose $\iota^t : \mathbb{Z}^l \to \mathbb{Z}$, $\iota^t(m_1, \ldots, m_l) = m_1 + \ldots + m_l$.

Proposition 7. *(Arici, Kaad, G.L. 2015) With $k, l \in \mathbb{N}$ coprime:*

$$K_0\Big(L_q(lk; k, l))\Big) \simeq \mathrm{coker}(1 - E) \simeq \mathbb{Z} \oplus \Big(\mathbb{Z}^l / \mathrm{Im}(\iota)\Big)$$
$$K_1\Big(L_q(lk; k, l))\Big) \simeq \ker(1 - E) \simeq \mathbb{Z}^l$$

as well as

$$K^0\Big(L_q(lk; k, l))\Big) \simeq \ker(1 - E^t) \simeq \mathbb{Z} \oplus \Big(\ker(\iota^t)\Big)$$
$$K^1\Big(L_q(lk; k, l))\Big) \simeq \mathrm{coker}(1 - E^t) \simeq \mathbb{Z}^l \, .$$

Again there is no dependence on the label k.

T-dual Pimsner algebras: a simple example

$$0 \to K_1(\mathsf{L}_q^{(1,r)}) \xrightarrow{\partial} K_0(\mathbb{C}\mathsf{P}_q^1) \xrightarrow{1-[\mathcal{L}_{-r}]} K_0(\mathbb{C}\mathsf{P}_q^1) \longrightarrow K_0(\mathsf{L}_q^{(1,r)}) \longrightarrow 0$$

$$\ker(1-[\mathcal{L}_{-r}]) = < u > = < 1-[\mathcal{L}_{-1}] >$$

$\Rightarrow$

$$K_1(\mathsf{L}_q^{(1,r)}) \ni h \mapsto \partial(h) = h(1-[\mathcal{L}_{-1}]) \simeq 1-[\mathcal{L}_{-h}]$$

and

$$(1-[\mathcal{L}_{-r}])(1-[\mathcal{L}_{-h}]) = 0 = (1-[\mathcal{L}_{-h}])(1-[\mathcal{L}_{-r}])$$

The exactness of the dual sequence for

$$0 \to K_1(\mathsf{L}_q^{(1,h)}) \xrightarrow{\partial} K_0(\mathbb{C}\mathsf{P}_q^1) \xrightarrow{1-[\mathcal{L}_{-h}]} K_0(\mathbb{C}\mathsf{P}_q^1) \longrightarrow K_0(\mathsf{L}_q^{(1,h)}) \longrightarrow 0$$

implies there exists a $r \in K_1(\mathsf{L}_q^{(1,h)})$ such that

$$K_1(\mathsf{L}_q^{(1,h)}) \ni r \mapsto \partial(r) = r(1 - [\mathcal{L}_{-1}]) \simeq 1 - [\mathcal{L}_{-r}]$$

The couples

$\left(\mathsf{L}_q^{(1,r)}, h \in K_1(\mathsf{L}_q^{(1,r)})\right)$ and $\left(\mathsf{L}_q^{(1,h)}, r \in K_1(\mathsf{L}_q^{(1,h)})\right)$

are ‘T-dual’

Summing up:

A general procedure to construct principal circle bundle over a noncommutative space out of a Pimsner algebra connection

A Gysin like sequence relating groups of K-theories.

consequences for

T-duality for noncommutative spaces

and

Chern-Simons theory

Caro Beppe, carissimi auguri !!