Geometria è Fisica, a geometrical vision of Physics

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Geometry and Experience

Geometry owes its existence to the need which was felt of learning something about the behaviour of real objects. The very word geometry , which, of course, means Earth measuring, proves this. For Earth measuring has to do with the possibilities of the dispositions of certin natural objects with respect to one another, namely with parts of the Earth, measuring lines, measuring-wands, etc.

Thus, geometry is evidently a natural science; we may in fact regard it as the most ancient branch of Physics. Its affirmations rest essentially on induction from experience, but not on logical inferences only.
We will call this geometry "practical geometry", and should distinguish it from "purely axiomatic geometry".

Geometry and Experience

The question wether the practical geometry of the universe is Euclidean or not has a clear meaning, and its answer can only be furnished by experience.

The question wether the space-time continuum has a Euclidean, Riemannian, or any other structure is a question of proper physics which must be answered by experiemce.

Without the above interpretation the decisive step in the transition to generally covariant equations would not have been taken.

Geometrical formulation

A theory is geometrically formulated if it allows for nonlinear transformations of the carrier space.

For example, consider the Schrodinger equation:

$$-\hbar \frac{\partial}{\partial t}\psi \;=\; -\frac{\hbar^2}{2m}\,\Delta\psi + V\,\psi$$

With a nonlinear transformation of the wave function:

$$\psi(x\,;t) \;=\; \sqrt{\psi^*(x\,;t)\,\psi(x\,;t)}\,\mathrm{e}^{\imath s(x\,;t)}$$

the complex linear Schrodinger equation splits into two nonlinear real differential equations.

Geometrical formulation

From this point of view, linearity becomes synonymous of simplicity (hystorically determined notion).

This transition allows for the following questions:

Once a nonlinear transformation has been performed, how to recover the linear structure?

Is there a tensorial description of linearity?

Is the linear structure unique?

Geometrical formulation

Linearity is connected with superposition rules.
How to define superposition in nonlinear context?

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k$$

$$x = \frac{x_1(x_3 - x_2) + kx_2(x_1 - x_3)}{x_3 - x_2 + k(x_1 - x_3)}$$

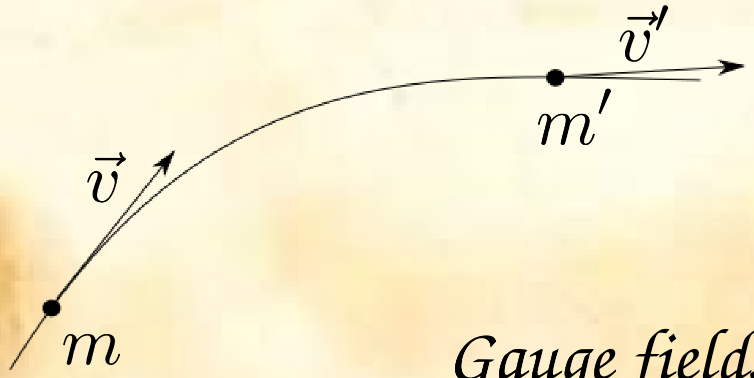
Linearity is connected with convexity:

- *Probabilities, probability amplitudes;*
- *Information geometry;*
- *Tensors from potential functions;*
- *Lagrangians: affine potential for equations of motion.*

Geometrical formulation

Field theories (Gauge theories)

"Transport rules" to define differences of "vectors" with different "foot-points"



Gauge fields equations = equations for the "transport rule"

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Geometry from Dynamics, Classical and Quantum

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Geometry is a mathematical language close to our intuition built on our experience of the world. As such, it induces a feeling of familiarity, at least for many mathematically motivated people. And though this familiarity might be a source of misleading ideas, it makes various technical tools more understandable and practical than abstract mathematical constructions. But also, as a language, geometry is often able to express mathematical ideas at various levels of rigor, depth and concreteness. A book written in this language can be a combination of forms. It can be a rigorous essay, a historical account, a story and even a poem. The present book is all of these. The authors write with the same enthusiasm and mastery that an experienced novelist could use to express his life experiences, with passion, memories and novelty. The material in the book is closely related to the extensive research of the authors, if not directly emanating from this. The subjects covered have a common theme—the geometric ideas, concepts and tools that are deeply related to and strongly provide the framework of classical dynamics and quantum mechanics. For the whole presentation, reference points are linearity, symmetry, invariant structures and integrability.

The first chapter presents an elementary framework and examples of dynamical systems, both linear and nonlinear, which may be analyzed by elementary means, but their study hints at the methods developed in the following parts. This especially concerns the motivation of how the geometric language emerges in the analysis of dynamics.

In the next two chapters, the backbone of the geometric concepts is introduced. In chapter 2 the game is played by the linear systems. It is shown how, from the elementary algebraic description of linear systems, a picture springs forth full of all those seeds needed to grow a geometric description: vector fields, forms and tensors. At this point the so-called easy geometrization principle is stated as a handy way to bridge the conceptual gap from algebra to geometry. Naturally here the first statements are given concerning the integrability of dynamical systems. The analysis leads to the topic of integration of Lie algebras and its relevance for the general theory.

The next chapter is the starting point for the geometrization of dynamical systems. The basic subject is differentiable manifolds but the need to deal with nonlinearity necessitates a study of the more general framework of differentiable spaces. These are deeply related to the dual approach of algebras of functions and their connection to the observables of the dynamical systems. One of the technicalities is related to the connection of algebraic tensors to the corresponding geometrical ones. This issue is related to the so-called holonomic tensorization principle. Symmetries and invariant structures are first studied here in a deeper manner.

Chapter 4 is the starting point for the formulation of the thesis that geometry arises from dynamics. Thus at this stage the formulation of the problem is connected with specific geometric concepts and invariant structures. These are the Poisson structures. Hamiltonian systems, separability, symmetries and constants of motion naturally emerge as key concepts along with the statement of the Feynman Inverse Problem. Some concrete examples enlighten the exposition.

The next chapter is a bridge to the classical tools of Hamiltonian and Lagrangian dynamics. First with the vehicle of linear Hamiltonian systems, the basic concepts and tools of symplectic linear spaces are presented in a fairly complete way, along with a clarification of the meaning of invariant symplectic structures. Then as a natural step, the concept of symplectic manifolds is introduced and its relation with Hamiltonian systems is presented. Symmetries and constants of motion for Hamiltonian systems complete the study in the framework of the cotangent bundle of the configuration space. The example of a harmonic oscillator gives an opportunity for the reader to establish a better understanding of the general structure through the familiarity of a concrete case. The switch to the tangent bundle framework and the associated Lagrangian formulation of dynamics offer a fruitful background for the presentation of the geometric tools that are fully analyzed and developed in the following chapters. Again symmetries, constants of motion and an extensive discussion of inverse problems complete this long chapter on classical formulation.

The connection with quantum systems starts in the next chapter. In fact, as the authors point out, here the mathematical framework consists of Hilbert spaces and Hermitian structures as they are used in quantum mechanics, but the nature of the systems is not physically quantum. This is good because all those subtleties of foundation and interpretation are avoided so that they do not complicate the pure mathematical properties of various structures. The main subjects here are invariant Hermitian structures and metrics as symmetric objects. And, just as it was shown long ago that complex numbers are an existential need for quantum mechanics, complex structures and complex exterior calculus naturally enter in this topic. Thus Kähler manifolds become indispensable for the study of the space of states of a quantum system. The quantum dynamical evolution is formulated in this geometric setting and the GNS construction gives the most general approach for the construction of the state space and analysis of symmetry in the algebraic formulation of quantum mechanics. And, as in the rest of this book, various alternative formulations are presented.

Chapter 7 deals with reduction of dynamics. This produces equations with a reduced number of degrees of freedom but with a more complicated form. Thus in general a linear system is reduced to a nonlinear one and the integrability of the linear setting induces integrability properties at the nonlinear level. This is formulated in a geometric manifold but in terms of invariant submanifolds or in terms of invariant subalgebras of functions. These concepts are clarified with many examples with concrete constructions. The chapter is completed with a short exposition of reduction in quantum mechanics. The treatment of well-known examples completes the picture of this approach to systems.

Next, in chapter 8, comes the topic of integrable systems. Stated in general terms, the practical question of the solvability of the equations of a dynamical system may be stated at various levels of concreteness and rigor. In the realm of quantum mechanics this would mean separability of the Schrödinger equation, while in the area of nonlinear PDEs it would be associated with the existence of Lax pairs, solitonic solutions and the like. In the middle, in classical mechanics, the Liouville-Arnold integrability definition and the KAM type of arguments are the main players. In this chapter there is a unifying thread of formulating all of these in geometric language. The exposition is very informative, especially with the offering of concrete examples like Toda and Calogero models.

The last chapter deals with a generalization of the idea of the previous one in the form of extending the possibility of nonlinear superposition rules. The key tools come from the Lie-Scheffer Theorem, which establishes conditions on the structure of the vector fields associated to the nonlinear equations. If their commutation relations close, thus giving a Lie algebra, then nonlinear superpositions are possible. An extensive motivating introduction is given using linear systems. The chapter uses various useful examples like Ricatti equations to clarify the technical points and the practical tools.

The book has an extensive appendix covering many concepts needed in the text.

The general impression given is that, as an old mathematician once said, the book contains everything and a little more. It is offered for a comprehensive exposition of the use of geometrical tools in the study of both classical and quantum systems. It would be very useful to a motivated student or a researcher wishing to adopt the geometrical framework in his/her work. Each chapter contains an extensive bibliography, old and current, doing justice to the various possible directions of study.

Unfortunately there are some weak points. First there are numerous misprints in the text and in the formulas. Regarding the latter, the alert student would not have serious problems. Also there are some incorrect cross-references of paragraphs and formulas. A more serious problem is the poor index. In a first reading, an uninitiated student would have difficulty locating the definition of a concept, a property or a theorem. Apart from enriching the index, making the defined concepts boldface and providing detailed contents of the appendix would have helped a lot. Finally, a matter of style: most of the comments help to clarify things, but some comments seem a bit overly enthusiastic in reference to the paragraphs that follow, and this might perplex some readers. The book would be very useful for an audience of mathematical physicists and applied mathematicians and, if improved in a future edition, we expect it to become a reference text. *Demetris P. K. Ghikas*