On the geometric formulation of Markovian dynamics

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Jesús Clemente-Gallardo
BIFI-Departmento de Física Teórica
Universidad de Zaragoza

Universidad Zaragoza 1542

Email: jcg@unizar.es

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Standing on the shoulders of giants ...

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Dicebat Bernardus Carnotensis nos esse quasi nanos, gigantium humeris insidentes, ut possimus plura eis et remotiora videre, non utique proprii visus acumine, aut eminentia corporis, sed quia in altum subvenimur et extollimur magnitudine gigantea.

Johannes Parvus, Metalogicon, 1159

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Standing on the shoulders of giants ...

Bernard of Chartres used to compare us to dwarfs perched on the shoulders of giants. He pointed out that we see more and farther than our predecessors, not because we have keener vision or greater height, but because we are lifted up and borne aloft on their gigantic stature.

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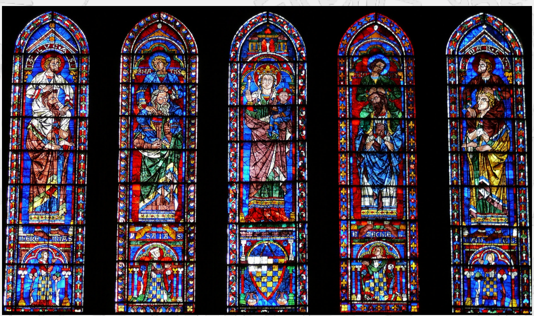

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Content

Joint Work with J. F. Cariñena, J. A. Jover-Galtier, G. Marmo and the students A. Franco and P. Sala

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Geometric formulation of Quantum Mechanics

Let us consider a finite dimensional quantum system. Heinseberg formalism will be defined on a C^*–algebra $\mathcal{A}$, finite dimensional, and therefore isomorphic to $\mathrm{End}(\mathcal{H})$, with the Frobenius norm $|A|^2 = \mathrm{Tr}(A^\dagger A)$ and an involution defined by the hermitian adjoint $A \mapsto A^\dagger$.

The set of hermitian operators is isomorphic to the Lie algebra of the unitary group $\mathfrak{u}(\mathcal{H}) \sim \mathfrak{u}(n)$. As we have a (non-degenerate) scalar product

$$\langle A|B\rangle = \mathrm{Tr}(AB); \qquad \forall A, B \in \mathfrak{u}(n)$$

we can identify $\mathfrak{u}(n)$, $i\mathfrak{u}(n)$ and $\mathfrak{u}^*(n)$

$$i\mathfrak{u}(n) \ni |A\rangle \mapsto \langle A| \in \mathfrak{u}^*(n)$$

Definition of tensor fields

On $\mathfrak{u}^*(n)$ we can consider two tensors

$$R_\rho(df_A, df_B) = \langle \rho, (AB + BA) \rangle$$

and

$$\Lambda_\rho(df_A, df_B) = \langle \rho, [A, B] \rangle$$

where $f_A(\rho) = \rho(A)$, $f_B(\rho) = \rho(B)$

R is a symmetric tensor and Λ is the canonical Lie-Poisson tensor for the unitary algebra. Thus, they allow us to consider the notion of gradient $Y_A = R(df_A, \cdot)$ and Hamiltonian $X_A = \Lambda(df_A, \cdot)$ vector fields
von Neumann equation can also be written as a Hamiltonian vector field on $\mathfrak{u}^*(n)$:

$$\dot{\rho} = -i[H, \rho] \Rightarrow X_H = \Lambda(df_H, \cdot)$$

Property

Hamiltonian vector fields preserve the Poisson and the symetric tensors

$$L_{X_A} R = 0 = L_{X_A} \Lambda; \qquad \forall A \in i\mathfrak{u}(n)$$

Lie-Jordan algebra

A vector space endowed with a Jordan algebra structure $\circ$ and a Lie structure $[\cdot,\cdot]$, such that $\forall a, b, c \in \mathcal{L}$:

- Leibnitz $[a, b \circ c] = [a, b] \circ c + b \circ [a, c]$
- $(a \circ b) \circ c - a \circ (b \circ c) = \hbar^2 [b, [c, a]]$ where $\hbar \in \mathbb{R}$.

Lie-Jordan Banach (LJB) algebras

A Lie-Jordan algebra $\mathcal{L}$ endowed with a norm $\|\cdot\|$ such that $\mathcal{L}$ is complete and satisfies

- $\|a \circ b\| \leq \|a\| \|b\|$
- $\|[a, b]\| \leq |\hbar|^{-1} \|a\| \|b\|$
- $\|a^2\| = \|a\|^2$
- $\|a^2\| \leq \|a^2 + b^2\|$

for any $a, b \in \mathcal{L}$.

Theorem

$(\mathfrak{u}^*(n), R, \Lambda)$ with the Frobenius norm defines a LJB algebra.

The space of physical states

Definition

The **set of density matrices** $\mathcal{D}(\mathcal{H})$ of a system corrresponds to the subset of $\mathfrak{u}^*(\mathcal{H})$ defined by the convex combinations of rank-one projectors. Analogously, $\rho \in \mathfrak{u}^*(\mathcal{H})$ is a density matrix iff

$$\mathrm{Tr}\rho = 1, \qquad \rho \geq 0.$$

The subset of **pure states** corresponds to the submanifold $\mathcal{D}^1(\mathcal{H})$ of rank-one projectors

We adapt the notation from Grabowski, Kus and Marmo and denote by D_Λ and D_R the generalized distributions on $\mathcal{O}^*$ of Hamiltonian and gradient vector fields, respectively.

Proposition GKM

The distribution $D_1 = D_\Lambda + D_R$ on $\mathcal{O}^*$ is involutive and can be integrated to a generalized foliation $\mathcal{F}_1$, whose leaves correspond to the orbits of the action of the general linear group $GL(n, \mathbb{C})$ on $\mathcal{O}^*$, $n = \dim \mathcal{H}$, defined by $(T, \xi) \mapsto T\xi T^*$.

Proposition

Let $\mathcal{P}(\mathcal{A})$ denote the set of real positive linear functionals $\zeta : \mathcal{A} \to \mathbb{C}$, i.e. such that

$$\zeta(a^*) = \overline{\zeta(a)}, \quad \zeta(a^* a) \geq 0, \forall a \in \mathcal{A}. \tag{1}$$

The set $\mathcal{P}(\mathcal{A})$ is a subset of $\mathcal{O}^*$. Furthermore, it is a stratified manifold,

$$\mathcal{P}(\mathcal{A}) = \bigcup_{k=0}^{n} \mathcal{P}^k(\mathcal{A}), \tag{2}$$

where the stratum $\mathcal{P}(\mathcal{A})^k$ is the set of rank k operators in $\mathcal{P}(\mathcal{A})$. Each stratum $\mathcal{P}(\mathcal{A})^k$ is a leaf of the foliation $\mathcal{F}_1$ corresponding to the joint distribution, union of Hamiltonian and gradient vector fields.

Proposition

The set of states $\mathcal{D}(\mathcal{A})$ is a stratified manifold,

$$\mathcal{D}(\mathcal{A}) = \bigcup_{k=1}^{n} \mathcal{D}(\mathcal{A})^k, \quad \text{where} \quad \mathcal{D}(\mathcal{A})^k(\mathcal{A}) = \mathcal{P}(\mathcal{A})^k \bigcap \{\xi \in \mathcal{O}^* | \xi(I) = 1\}. \tag{3}$$

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- Notice that the functional $0 \in \mathcal{P}(\mathcal{A})$ is a fixed point of Y_I. Removing it, we obtain a regular foliation by Y_I of $\mathcal{P}_0(\mathcal{A}) := \mathcal{P}(\mathcal{A}) - \{0\}$.

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- We can thus define the corresponding quotient manifold identifying points in the same leaf; two points ζ, ζ' are equivalent if $\zeta = c\zeta'$, with $c > 0$. The set of states $\mathcal{D}(\mathcal{A})$ is the section of this fibration defined by the elements of trace equal to one.

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We are interested in the characterization of geometrical objects in $\mathcal{D}(\mathcal{A})$ as objects in $\mathcal{P}(\mathcal{A})$ that are projectable with respect to the fibration

$$\pi_{\mathcal{P}}(\zeta) = \frac{1}{f_I(\zeta)}\,\zeta, \quad \zeta \in \mathcal{P}_0(\mathcal{A}),$$

Definition

Let us consider a set of expectation value functions defined, from the linear ones, in the form

$$e_A(\rho) := \pi_{\mathcal{P}}^*(f_a|_{\mathcal{D}(\mathcal{A})})(\zeta) = \frac{f_a(\zeta)}{f_I(\zeta)}, \quad \zeta \in \mathcal{P}_0(\mathcal{A}), \quad a \in \mathcal{O}.$$

Theorem

We obtain thus two tensors on $\mathcal{D}(\mathcal{H})$ as

$$\Lambda_{\mathcal{D}(\mathcal{H})}(de_A, de_B) = e_{[A,B]}$$

$$R_{\mathcal{D}(\mathcal{H})}(de_A, de_B) = e_{[A,B]_+} - e_A e_B = Cov(A, B)$$

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$R_{\mathcal{D}}$ captures the difference between the pointwise and the Jordan product.

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Geometric characterization of the KL equation

GKS and Lindblad determined, in 1976, the form of the infinitesimal generator of a markovian dynamics on the set of states.

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \frac{1}{2}\sum_{j=1}^{n^2}([V_j\rho(t), V_j^\dagger] + [V_j, \rho(t)V_j^\dagger] =$$

$$- i[H, \rho(t)] - \frac{1}{2}\sum_{j=1}^{n^2}([V_j^\dagger V_j, \rho(t)]_+ + \sum_{j=1}^{n^2} V_j\rho(t)V_j^\dagger$$

This equation defines a vector field Z_L on $\mathcal{D}(\mathcal{H})$:

$$\frac{d\rho(t)}{dt} = Z_L(\rho).$$

We can characterize the different terms from a geometrical point of view and write

$$Z_L = X_H + Y_J + K$$

where

- X_H is a Hamiltonian vector field with respect to the Poisson tensor $\Lambda_{\mathcal{D}(\mathcal{H})}$
- Y_J, is the gradient vector field associated with the function $J = \sum_{j=1}^{n^2} V_j^\dagger V_j$ by the symmetric tensor $R_{\mathcal{D}(\mathcal{H})}$.
- K is the vector field associated to the action of the Kraus operators

$$K(\rho) = \sum_{j=1}^{n^2} V_j \rho V_j^\dagger$$

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Dynamics on the space of tensors

We can encode the evolution in a transformation of the algebraic structures of our LJB system. Therefore we shall consider the following equations

$$\frac{d}{dt}\Lambda(t) = L_{Z_L}\Lambda(t); \qquad \Lambda(0) = \Lambda_{\mathcal{D}(\mathcal{H})}$$

$$\frac{d}{dt}R(t) = L_{Z_L}R(t); \qquad R(0) = R_{\mathcal{D}(\mathcal{H})}$$

The system we are interested in is the limit:

$$R_\infty = \lim_{t\to\infty} R(t) = \lim_{t\to\infty} e^{-tL_{Z_L}} R_{\mathcal{D}(\mathcal{H})}; \qquad \Lambda_\infty = \lim_{t\to\infty} \Lambda(t) = \lim_{t\to\infty} e^{-tL_{Z_L}} \Lambda_{\mathcal{D}(\mathcal{H})}$$

Question

Does $(R_\infty, \Lambda_\infty)$ define a LJB algebra?

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Example: 2-level systems

Let us consider the phase damping of a qubit, given by the following Kossakowski-Lindblad operator

$$L\rho = -\gamma(\rho - \sigma_3\rho\sigma_3).$$

The vector field Z_L associated to this operator is:

$$Z_L = -2\gamma \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right).$$

By computing the Lie derivatives with respect to this vector field of $\Lambda_{\mathcal{D}}$ and $R_{\mathcal{D}}$, we obtain the coordinate expressions of the families $\Lambda_{\mathcal{D},t}$ and $R_{\mathcal{D},t}$:

$$\begin{aligned}
\Lambda_{\mathcal{D},t} =& e^{-4\gamma t} x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1}, \\
R_{\mathcal{D},t} =& e^{-4\gamma t} \left(\frac{\partial}{\partial x_1} \otimes \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \otimes \frac{\partial}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \otimes \frac{\partial}{\partial x_3} \\
& - \sum^{3} x_j x_k \partial x_j \otimes \partial x_k.
\end{aligned}$$

In this case, the asymptotic limits $t \to \infty$ of the families do exist.

Proposition

The phase damping evolution of a qubit defines a contraction of the Lie-Jordan algebra of functions on the space of states, determined by the following products:

$$\{x_1, x_3\}_\infty = -x_2, \quad \{x_2, x_3\}_\infty = x_1, \quad \{x_1, x_2\}_\infty = 0,$$
$$(x_1, x_1)_\infty = (x_2, x_2)_\infty = 0, \quad (x_3, x_3)_\infty = 1.$$

The Lie algebra $(\mathrm{span}(x_1, x_2, x_3), \{\cdot, \cdot\}_\infty)$ is isomorphic to the Euclidean Lie algebra. The pair $(\mathrm{span}(x_1, x_2, x_3, 1), (\cdot, \cdot)_\infty)$ is a Jordan algebra. The triple $(\mathrm{span}(x_1, x_2, x_3, 1), (\cdot, \cdot)_\infty, \{\cdot, \cdot\}_\infty)$ is a Lie-Jordan algebra.

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Example: 3-level systems

The model of decoherence for massive particles is given by

$$L(\rho) = -\gamma[X,[X,\rho]],$$

where X is the position operator. This model can be discretized by considering a finite number $d = 3$ of positions $\vec{x}_m$ along a circle. The positions are given by

$$\vec{x}_m = (\cos\phi_m, \sin\phi_m), \quad \phi_m = \frac{2\pi m}{d}, \quad m, = 1, 2, \ldots, d.$$

The operator L in the basis of eigenstates of the position operator takes the form

$$L|m\rangle\langle n| = -\gamma\, |\vec{x}_m - \vec{x}_n|\, |m\rangle\langle n| = -4\gamma \sin^2\left(\frac{\pi(m-n)}{d}\right) |m\rangle\langle n|,$$

for $m, n = 1, 2, \ldots, d$.

On the other hand, the pure decoherence of a d-level system is given by

$$L(\rho) = -\frac{1}{d}\sum_{k=1}^{d-1}\gamma_k(\rho - U_k\rho U_k^*), \quad \gamma_k > 0,\ k = 1, 2, \ldots, d-1,$$

where U_k are the unitary operators given by

$$U_k = \sum_{l=1}^{d-1}\lambda^{-k(l-1)}P_l, \quad \lambda = e^{\frac{2\pi i}{d}},$$

and P_l are the 1-dimensional projectors $|l\rangle\langle l|$.

The evolutions of a 3-level system by either the decoherence model of massive particles or the pure decoherence model define a contraction of the Lie-Jordan algebra of functions. The Poisson and the Jordan brackets of the contracted algebras are

$$\{x_1, x_3\}_\infty = -x_2,\ \{x_2, x_3\}_\infty = x_1,$$
$$\{x_4, x_3\}_\infty = -\frac{1}{2}x_5,\ \{x_5, x_3\}_\infty = \frac{1}{2}x_4,\ \{x_4, x_8\}_\infty = -\frac{\sqrt{3}}{2}x_5,\ \{x_5, x_8\}_\infty = \frac{\sqrt{3}}{2}x_4,$$
$$\{x_6, x_3\}_\infty = \frac{1}{2}x_7,\ \{x_7, x_3\}_\infty = -\frac{1}{2}x_6,\ \{x_6, x_8\}_\infty = -\frac{\sqrt{3}}{2}x_7,\ \{x_7, x_8\}_\infty = \frac{\sqrt{3}}{2}x_6,$$

$$(x_3, x_3)_\infty = \frac{2}{3} + \frac{1}{\sqrt{3}}x_8,\ (x_8, x_8)_\infty = \frac{2}{3} - \frac{1}{\sqrt{3}}x_8,$$
$$(x_1, x_8)_\infty = \frac{1}{\sqrt{3}}x_1,\ (x_2, x_8)_\infty = \frac{1}{\sqrt{3}}x_2,\ (x_3, x_8)_\infty = \frac{1}{\sqrt{3}}x_3,\ (x_4, x_8)_\infty = -\frac{1}{2\sqrt{3}}x_4,$$
$$(x_5, x_8)_\infty = -\frac{1}{2\sqrt{3}}x_5,\ (x_6, x_8)_\infty = -\frac{1}{2\sqrt{3}}x_6,\ (x_7, x_8)_\infty = -\frac{1}{2\sqrt{3}}x_7,$$
$$(x_4, x_3)_\infty = \frac{1}{2}x_4,\ (x_5, x_3)_\infty = \frac{1}{2}x_5,\ (x_6, x_3)_\infty = -\frac{1}{2}x_6,\ (x_7, x_3)_\infty = -\frac{1}{2}x_7.$$

The triple $(\mathrm{span}(x_1, \ldots, x_8, 1), (\cdot, \cdot)_\infty, \{\cdot, \cdot\}_\infty)$ is a Lie-Jordan algebra.

Question: When does the dynamics on the space of tensors converge?

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Proposition (AFR)

Let us consider a 3-level system and assume a markovian dynamics with Kraus operators K_j, $j = 1, 2, \ldots n$ which are diagonalizable and with real spectrum

$$K_j = \begin{pmatrix} a_j & 0 & 0 \\ 0 & b_j & 0 \\ 0 & 0 & c_j \end{pmatrix} \qquad j = 1, 2, \cdots, n.$$

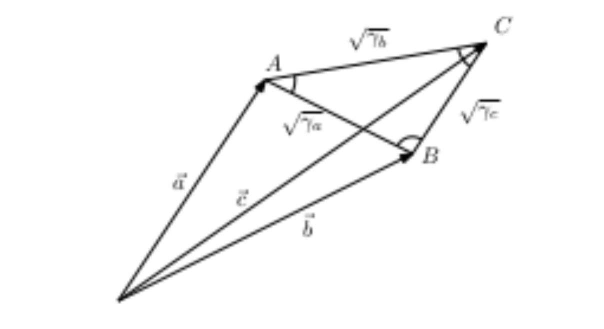
Then, the corresponding Lindblad operator defines a convergent vector field on the space of LJB algebras if and only if $\vec{a} = (a_1, a_2, \cdots, a_n)$, $\vec{b} = (b_1, b_2, \cdots, b_n)$ and $\vec{c} = (c_1, c_2, \cdots, c_n)$ satisfy

$$\|\vec{b} - \vec{a}\|^2 \leq \|\vec{a} - \vec{c}\|^2 + \|\vec{c} - \vec{b}\|^2 \quad \|\vec{a} - \vec{c}\|^2 \leq \|\vec{c} - \vec{b}\|^2 + \|\vec{b} - \vec{a}\|^2$$

$$\|\vec{c} - \vec{b}\|^2 \leq \|\vec{b} - \vec{a}\|^2 \leq +\|\vec{a} - \vec{c}\|^2$$

Equivalently, recalling the cosine theorem

$$\cos \widehat{ACB} \geq 0; \qquad \cos \widehat{CBA} \geq 0 \qquad \cos \widehat{BAC} \geq 0$$



Thanks, Beppe, for too many things to mention all here.