# Numerical studies of the Bethe–Salpeter equation in Minkowski space

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# Outline



In collaboration with


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PRD **85**, 036009 (2012) – PRD **89**, 016010 (2014) – EPJC **75**, 398 (2015)

# Motivations

- [Salpeter & Bethe, PR **84**, 1232 (1951)]
- To achieve a fully covariant description for a few-body system, in Minkowski space, within a field-theoretical framework
- Applications for bound states of $\bar{q}q$ and $qqq$ systems $\rightarrow$ spectra & PDF
- Applications for nuclear systems $\rightarrow$ solution for continuum states

# Motivations

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### Solution in Euclidean space using the Wick rotation

- correct binding energy
- difficult to obtain other quantities, like EM form factors, etc
- Meson and baryon masses from QCD [Roberts *et al.*, EPJ Special Topics **140**, 53 (2007)]
- Solution in Minkowsky space difficult due to the singularities of the kernel

### Solution in Minkowski space

- [Kusaka, Simpson, & Williams, PRD **56**, 5071 (1997)]
- [Sauli & Adam, PRD **67**, 085007 (2003)]
- [Maris & Tandy, NPB **161**, 136 (2006)]
- [Karmanov & Carbonell, EPJA **27**, 1 (2006), PRD **90**, 056002 (2014)]
- [Sauli, `hep-th:1505.03778`]
- We follow the prescription of Karmanov & Carbonell using a light-front projection

# Notation

This contribution: direct solution of the BSE in Minkowski space for two fermion bound states

Preliminary results – ladder approximation – no self-energy in the propagators

Aim: extend our numerical treatment applied so far only for two boson bound and scattering states

## Notation

- Two spin 1/2 particles interacting via
  1. Scalar particle $\mathcal{L} = g\overline{\psi}\psi\phi$
  2. Pseudoscalar particle $\mathcal{L} = ig\overline{\psi}\gamma^5\psi\phi$
  3. Vector particle $\mathcal{L} = g\overline{\psi}\gamma_\mu\psi V^\mu$
- $m =$ mass of the fermion
- $\mu =$ mass of the exchanged particle

- $\kappa^2 = m^2 - M^2/4$, alert$M = 2m - B$ "energy" of the system
- $B=$ binding energy
- Bound states $\kappa^2 > 0$
- Zero energy scattering $\kappa^2 = 0$
- Positive energy scattering $\kappa^2 < 0$

# The BS amplitude

Example: $q\overline{q}$ bound system in a $J^{\pi C} = 0^{-+}$ state of total momentum $p$

$$\Phi_{\alpha,\beta}(k,p) = \int d^4x_1\, d^4x_2 e^{i(\frac{p}{2}+k)x_1 + i(\frac{p}{2}-k)x_2} \langle 0| T\{\psi^H_\alpha(x_1)\overline{\psi}^H_\beta(x_2)\}|B,p,0^{-+}\rangle$$

- $\psi^H(x)$ field of the fermion
- $|B,p,0^{-+}\rangle$ bound state of total momentum $p$
- $\alpha$, $\beta$ 4-spinor indeces
- The amplitude $\Phi$ is a $4\times 4$ matrix

- Under Lorentz $\Phi(p,k) = S(\Lambda)\Phi(\Lambda p, \Lambda k)S(\Lambda)^{-1}$
- Under parity $\Phi(p,k) = -\gamma_0\Phi(\tilde{p},\tilde{k})\gamma^0$ $(\tilde{p} \equiv \{p^0, -\vec{p}\})$
- Under charge conjugation $\Phi(p,k) = \mathcal{C}\Phi(p,-k)^t\mathcal{C}^t$ $(\mathcal{C} = \gamma^0\gamma^2)$

$$\Phi(k,p) = A + B\gamma^5 + C\gamma^\mu p_\mu + D\gamma^\mu k_\mu + E\gamma^\mu\gamma^5 p_\mu + F\gamma^\mu\gamma^5 k_\mu + G\sigma^{\mu\nu}p_\mu k_\nu + H\gamma^5\sigma^{\mu\nu}p_\mu k_\nu$$

$$A \equiv A(k^2, k\cdot p), \ldots$$

$A = C = D = G = 0$ $\quad$ $B, E, H$ $(F)$ even (odd) under $k \to -k$

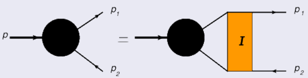
[Llewellyn Smith, AP **53**, 521 (1969)]

# The BS amplitude (2)

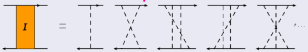
The BS Amplitude $\Phi(k,p)$ fulfills an homogeneous integral equation

$$\Phi(k,p) = S(k+\frac{p}{2}) \int \frac{d^4k'}{(2\pi)^4} \Phi(k',p)\mathcal{I}(k,k',p)S(k-\frac{p}{2})$$

Graphical solution




I $\equiv$ kernel given by the infinite sum of irreducible Feynmann graphs – Iterations produce all the expected contributions




Free propagator of a fermion

$$S(k) = i\,\frac{\not k + m}{k^2 - m^2 + i\epsilon}$$

Scalar kernel in ladder approximation

$$i\mathcal{I}_S^{(Ld)}(k,k') = \frac{i(-ig)^2}{(k-k')^2 - \mu^2 + i\epsilon}$$

Regularization of the vertices

$$g \to gF(k-k') \qquad F(q) = \frac{\mu^2 - \Lambda^2}{q^2 - \Lambda^2 + i\epsilon}$$

# The equation for the BS amplitude

Let us rewrite

$$\Phi(k,p) = S_1\phi_1(k,p) + S_2\phi_2(k,p) + S_3\phi_3(k,p) + S_4\phi_4(k,p)$$

$$S_1 = \gamma^5 \qquad S_2 = \frac{1}{M}\not{p}\gamma^5 \qquad S_3 = \frac{k\cdot p}{M^3}\not{p}\gamma^5 \qquad S_4 = \frac{i}{M^2}\sigma_{\mu\nu}p_\mu k_\nu\gamma^5$$

Convenient decomposition
$Tr(S_i S_j) = N_i\delta_{i,j}$
$M$ "mass" of the bound state $= 2m - B$

$$\begin{aligned}\phi_i(k,p) &\equiv \phi_i(k^2, p\cdot k)\\ \phi_{1,2,4}(k,p) &= \phi_{1,2,4}(-k,p)\\ \phi_3(k,p) &= -\phi_3(-k,p)\end{aligned}$$

Substituting this expression in the BSE in ladder approximation

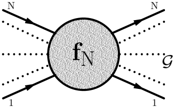
$$\phi_i(k,p) = \sum_j \int \frac{d^4k'}{(2\pi)^4}\frac{c_{ij}(k,k',p)}{\left[(\frac{p}{2}+k)^2 - m^2 + i\epsilon\right]\left[(\frac{p}{2}-k)^2 - m^2 + i\epsilon\right]}\frac{g^2F(k-k')^2}{(k-k')^2-\mu^2+i\epsilon}\phi_j(k',p)$$

$$c_{ij}(k,k',p) = \frac{1}{N_i}Tr\left[S_i(\frac{\not{p}}{2}+\not{k}+m)\Gamma S_j'\Gamma(\frac{\not{p}}{2}-\not{k}-m)\right]$$

$\Gamma = ig, -g\gamma^5, ig\gamma^\mu$ vertices for the cases S, PS, and V
$S_j'$ constructed with $k'$

# Nakanishi perturbation-theory integral representation (PTIR)

In the sixties, Nakanishi (PR **130**, 1230 (1963)) proposed an integral representation for $N$-leg transition amplitudes, based on the parametric formula for the Feynman diagrams.




Generic contribution to the transition amplitude is given by

$$\mathcal{G}(p_1,\ldots,p_N,p'_1,\ldots,p'_N) \propto \prod_{r=1}^{k}\int d^4q_r\, \frac{1}{(\ell_1^2-m_1^2)(\ell_2^2-m_2^2)\ \ldots\ (\ell_n^2-m_n^2)}$$

where one has $n$ propagators and $k$ loops

The sum over all Feynman diagram $\mathcal{G}$ for a full $N$-leg transition amplitude can be formally written as

$$f_N(s) = \sum_{\mathcal{G}} f_{\mathcal{G}}(s) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1-\sum_h z_h)\ \phi_N(z_1,z_2,\ldots,\gamma)}{\gamma - \sum_h z_h s_h}$$

the dependence upon the external momenta, $p_1, p_2 \ldots p_N$, $p'_1, p'_2 \ldots p'_N$ traded off in favour of all the independent scalar products $s \equiv \{s_1, s_2, \ldots s_h, \ldots\}$ one can construct.

# Nakanishi PTIR - II

1. In case of our "three-leg" functions

$$\phi_i(k,p) = \int_0^1 dz \int_0^\infty d\gamma \frac{g_i(z,\gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk\cdot p - i\epsilon}$$

2. Projection of the BSE onto the null plane, i.e. integration of both sides over $k^- = k^0 + k_z$ [Karmanov & Carbonell, EPJA **27**, 1 (2006)]

$$\int_0^\infty d\gamma' \frac{g_i(\gamma',z)}{(\gamma+\gamma'+m^2z^2+(1-z^2)\kappa^2)^2} = g^2 \int_0^\infty d\gamma' \int_{-1}^1 dz' \sum_j V_{ij}(\gamma,z,\gamma',z') g_j(\gamma',z')$$

3. Some of the $c_{ij} \sim (k^-)^4$, special care has to be taken in these cases

$$V_{ij}(\gamma,z,\gamma',z') \sim \delta(z-z')\mathcal{V}_{ij}(\gamma,z,\gamma') + \cdots$$

Carbonell & Karmanov included a regularizing factor to eliminate the contribution of the $\delta$

# Numerical results for the bound state in ladder approx.

We have carried out a comprehensive investigation, in ladder approximation of the three interaction models

- In ladder approximation $V_{ij}$ proportional to $g^2$
- Standard procedure: we fix $M = 2m - B$, $B =$ binding energy, and find the smallest eigenvalue $(g^2)$
- The eigenvector gives the Nakanishi functions $g_i$, from which we can compute $\phi_i$
- Study for different binding energies $0 < B/m \leq 2$ and mass of the exchanged particle, $\mu/m$

## Method of solution

Expansion on a basis for the $z$ and $\gamma$ variables

$$g_i(\gamma, z) = \sum_{\ell=1}^{L_i} \sum_{m=1}^{M_i} A_{\ell m} \, F_\ell(z) \, G_m(\gamma)$$

# Convergence

$$g_i(\gamma, z) = \sum_{\ell=1}^{L_i} \sum_{m=1}^{M_i} A_{\ell m} \; F_\ell(z) \; G_m(\gamma)$$

$\mu/m = 0.15, \; B/m = 0.01$

| $L_1, M_1$ | $L_2, M_2, L_3, M_3, L_4, M_4$ | $g^2$ |
|---|---|---|
| 4 | | 14.92 |
| 8 | | 15.00 |
| 12 | | 15.00 |
| 12 | 4 | 7.831 |
| 12 | 8 | 7.842 |
| 12 | 12 | 7.844 |
| 16 | 16 | 7.844 |

$\mu/m = 0.15, \; B/m = 0.50$

| $L_1, M_1$ | $L_2, M_2, L_3, M_3, L_4, M_4$ | $g^2$ |
|---|---|---|
| 4 | | 125.5 |
| 8 | | 125.3 |
| 12 | | 125.3 |
| 12 | 4 | 95.68 |
| 12 | 8 | 88.82 |
| 12 | 12 | 88.94 |
| 16 | 16 | 88.96 |

# Comparison with other calculations

$\mu/m = 0.15$

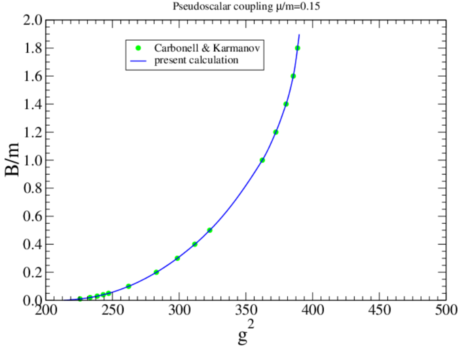
| B/m | $g^2$ (CK) | $g^2$ |
|---|---|---|
| 0.01 | 7.813 | 7.844 |
| 0.05 | 15.35 | 15.34 |
| 0.10 | 23.12 | 23.12 |
| 0.20 | 38.32 | 38.32 |
| 0.50 | 86.95 | 88.96 |

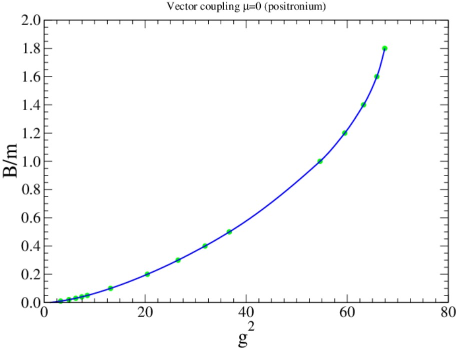
$\mu/m = 0.50$

| B/m | $g^2$ (CK) | $g^2$ |
|---|---|---|
| 0.01 | 25.23 | 25.32 |
| 0.05 | 39.19 | 39.17 |
| 0.10 | 52.82 | 52.81 |
| 0.20 | 78.25 | 78.26 |
| 0.50 | 157.4 | 157.4 |

Values of $g^2$ obtained by solving the eigenequation
Gegenbauer $\times$ Laguerre expansion of the Nakanishi wf
CK: from Carbonell & Karmanov, EPJA **46**, 387 (2010) (spline expansion of the Nakanishi wf).

# Results for the PS and V cases

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# Valence Probabilities and LF Distributions

Once the Nakanishi weight functions is evaluated, one can straightforwardly obtain the BS amplitude and normalize it.

$$\int \frac{d^4k}{(2\pi)^4}\, \bar{\Phi}^{(Ld)}(k,p)\left[M^2(\kappa^2 - k^2) + 2(k\cdot p)^2\right]\Phi^{(Ld)}(k,p) = i2M^2$$

$\rightarrow$ valence wave function & LF distributions

Results for two scalar particles exchanging another massive scalar particles [PRD **89**, 016010 (2014)]

Valence wave function

$\xi$ fraction of longitudinal momentum of particle 1

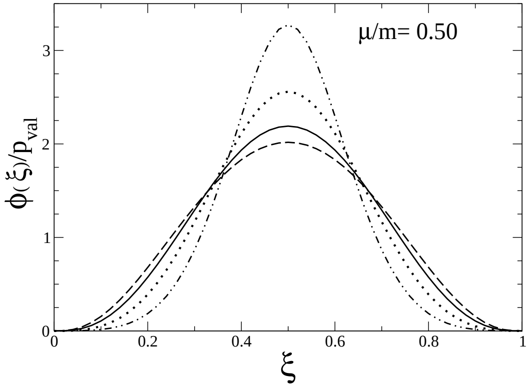
$$\xi = \frac{1}{p^+}\left(\frac{1}{2}p^+ + k^+\right)$$

$$\psi_{n=2}(\xi, k_\perp) = \frac{p^+}{\sqrt{2}}\,\xi\,(1-\xi)\int \frac{dk^-}{2\pi}\Phi(k,p)$$

$\mu/m = 0.50$

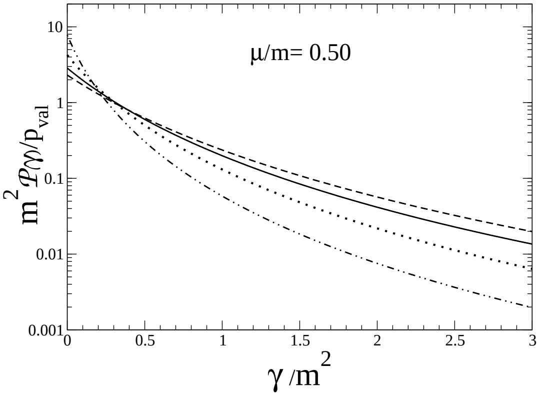
| B/m | $P_{val}$ |
|---|---|
| 0.001 | 0.98 |
| 0.01 | 0.96 |
| 0.10 | 0.87 |
| 0.20 | 0.83 |
| 0.50 | 0.77 |
| 1.00 | 0.74 |
| 2.00 | 0.72 |

## Results for two scalar particles exchanging another massive scalar particles [PRD **89**, 016010 (2014)]




The longitudinal LF-distribution, $\phi(\xi) = \int dk_\perp^2 \; |\psi_{n=2}(\xi, k_\perp)|^2$, vs the longitudinal-momentum fraction $\xi = k^+/M$. Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. N.B. $\int_0^1 d\xi \; \phi(\xi) = P_{val}$

## Results for two scalar particles exchanging another massive scalar particles [PRD **89**, 016010 (2014)]




The transverse LF-distribution $\mathcal{P}(\gamma) = \int d\xi \; |\psi_{n=2}(\xi, k_\perp)|^2$ vs the adimensional variable $\gamma/m^2$ ($\gamma = k_\perp^2$). Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. N.B. $\int_0^\infty d\gamma \; \mathcal{P}(\gamma) = P_{val}$.

# Conclusions & Perspectives

- The cross-fertilization between the Light-Front framework and the Nakanishi PTIR paves the path toward a new class of non perturbative calculations, within a rigorous field-theoretical framework (the Bethe-Salpeter Equation in Minkowski space)
- The LF framework has well-known advantages in performing analytical integrations, that within the canonical approach appear highly non trivial.
- Good preliminary results for a two-fermion systems interacting via S, PS, V couplings
- Calculations in progress for
    - energy $> 0$ (inhomogeneous BSE)
    - study of excited states (preprint 2016)
    - the crossed-box contribution (A. Iannone, Master thesis)
    - Fermion+boson bound state (D. Colasante)
    - LF distribution for the two-fermion system