Non-dissipative corrections to energy-momentum tensor for a relativistic fluid

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Energy momentum tensor at equilibrium

The equation of relativistic hydrodynamics:

$$\partial_\mu T^{\mu\nu} = 0 \quad \partial_\mu j^\mu = 0$$

At equilibrium the energy momentum tensor and charge current are:

$$T^{\mu\nu}(x) = (\rho + p)u^\mu u^\nu - g^{\mu\nu} p$$

$$j^\mu = n u^\mu$$

The basic assumption of an hydrodynamical evolution is the local equilibrium condition

$$\rho(x) = \rho_{eq}(T(x), \mu(x)) \quad p(x) = p_{eq}(T(x), \mu(x))$$
$$n(x) = n_{eq}(T(x), \mu(x))$$

Maximum Entropy Principle

Looking for the maximum of von Neumann entropy

$$S = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$$

with fix energy density and momentum density for a given space-like hyper surface

$$-\mathrm{tr}(\widehat{\rho} \log \widehat{\rho}) + \int_{\Sigma(\tau)} \mathrm{d}\Sigma \, n_\mu \left[\left(\langle \widehat{T}^{\mu\nu}(x) \rangle - T^{\mu\nu}(x) \right) \beta_\nu - \left(\langle \widehat{j}^\mu(x) \rangle - j^\mu(x) \right) \xi \right]$$

$$n_\mu \mathrm{tr}(\hat{T}^{\mu\nu} \hat{\rho}_{LE}) \equiv n_\mu T^{\mu\nu}_{LE} = n_\mu T^{\mu\nu} \qquad n_\mu \mathrm{tr}(\hat{j}^\mu \hat{\rho}_{LE}) \equiv n_\mu j^\mu_{LE} = n_\mu j^\mu$$

Generalized equilibrium density matrix

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma} \mathrm{d}\Sigma_{\mu} \left(\widehat{T}^{\mu\nu} \beta_{\nu} - \zeta \widehat{j}^{\mu} \right) \right]$$

- if the vector field β_{μ} is a Killing vector field
- if ζ is a constant

$$\nabla_{\mu}\beta_{\nu} + \nabla_{\nu}\beta_{\mu} = 0 \qquad \nabla_{\nu}\zeta = \nabla_{\nu}\left(\frac{\mu}{\sqrt{\beta^2}}\right) = 0$$

The density matrix is stationary, i.e. independent from the choice of the hyper-surface

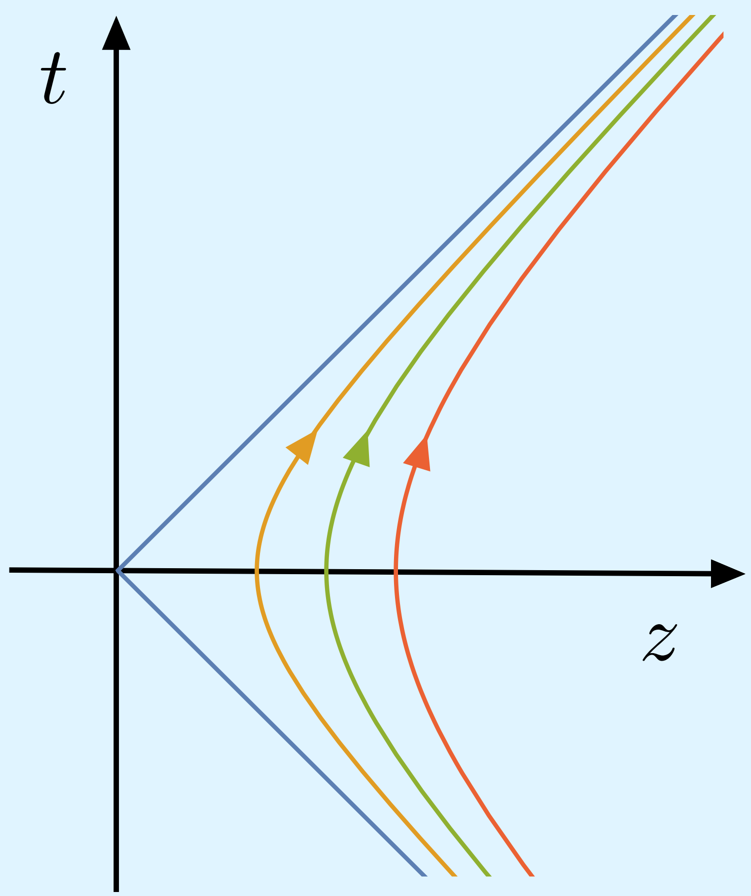
Equilibrium with rotation and acceleration

The general solution in Minkovsky space-time depends on 10 constant parameters:

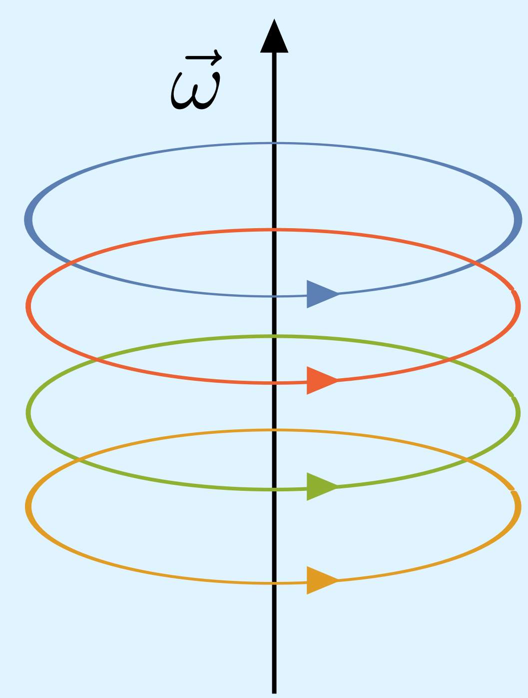
$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu} x_\nu \qquad \varpi^{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at) \qquad \beta^\mu = \frac{1}{T_0}(1, \boldsymbol{\omega} \times \mathbf{x})$$

acceleration



angular velocity



Basis vectors

The vorticity tensor can be decompose using the four velocity

$$\varpi^{\mu\nu} = \alpha^{\mu} u^{\nu} - \alpha^{\nu} u^{\mu} + \epsilon^{\mu\nu\rho\sigma} w_{\rho} u_{\sigma}$$

Four velocity

$$u_{\mu} = \beta_{\mu} / \sqrt{\beta^2}$$

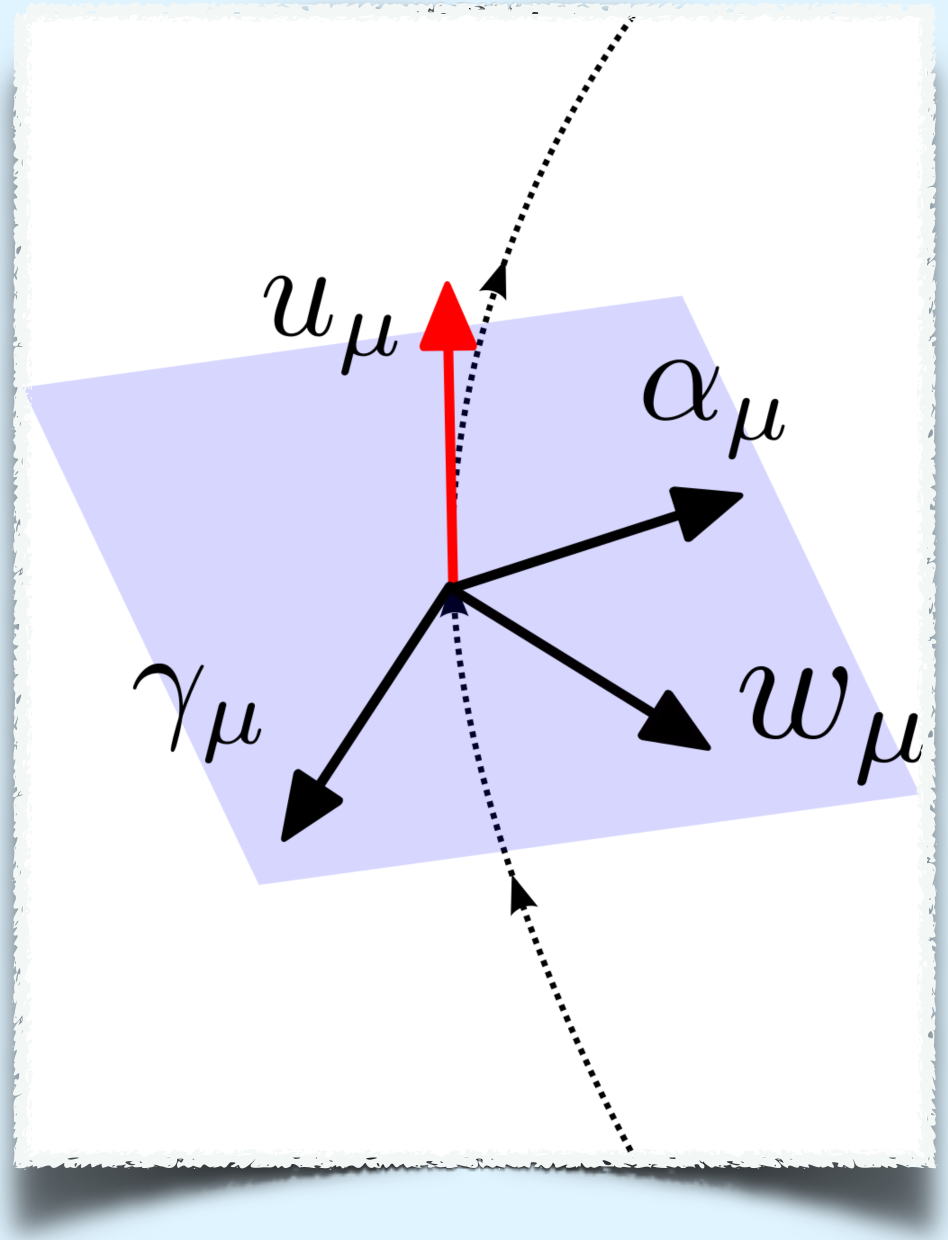
Acceleration over temperature

$$\alpha_{\mu} = \beta^{\nu} \varpi_{\mu\nu}$$

Angular velocity over temperature

$$w_{\mu} = \frac{1}{2} u^{\rho} \epsilon_{\rho\sigma\nu\mu} \varpi^{\sigma\nu}$$

$$\gamma_{\mu} = w^{\nu} \alpha^{\rho} u^{\sigma} \epsilon_{\mu\nu\rho\sigma}$$



Mean value of the energy-momentum tensor

The density operator can be written in terms of the generator of the Poincaré Group

$$\rho = \frac{1}{Z} \exp\left[-b_\mu P^\mu + \frac{1}{2}\varpi_{\mu\nu}J^{\mu\nu} + \zeta Q\right] = \frac{1}{Z(\beta(x))} \exp\left[-\beta_\mu(x) P^\mu + \zeta Q\right] + o(\varpi)$$

It is possible to find an expansion in terms of correlation function of the angular momentum and boost operators

$$\langle O(x)\rangle = \langle O(x)\rangle_{\beta(x)} + \alpha^\rho \langle K_\rho O(x)\rangle_{\beta(x)} + w^\rho \langle J_\rho O(x)\rangle_{\beta(x)}$$
$$+ \frac{1}{2}\alpha^\rho\alpha^\sigma \langle K_\rho K_\sigma O(x)\rangle_{\beta(x)} + \frac{1}{2} w^\rho w^\sigma \langle J_\rho J_\sigma O(x) + \frac{1}{2}\alpha^\rho w^\sigma \langle K_\rho J_\sigma O(x)\rangle_{\beta(x)}$$

F. Becattini, L. Bucciantini , E. G., L. Tinti Eur.Phys.J. C75 (2015)
F. Becattini, E. G.Phys.Rev. D92 (2015) 045037

Mean value of the energy-momentum tensor

Because of the rotation invariance only 7 coefficient are different from zero

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu)$$

$$\partial_\mu T^{\mu\nu} = 0$$

The conservation of the energy momentum leads:

$$U_\alpha = -\beta \frac{\partial}{\partial\beta}\left(D_\alpha + A\right) - \left(D_\alpha + A\right)$$

$$U_w = -\beta \frac{\partial}{\partial\beta}\left(D_w + W\right) - D_w + 2A - 3W$$

$$2G = 2\left(D_\alpha + D_w\right) + A + \beta \frac{\partial}{\partial\beta} W + 3W$$

Free complex scalar field

We consider a free complex scalar field at finite temperature and chemical potential:

- The improved energy-momentum:

$$T_{\alpha\beta} = (1-2\xi)\left(\partial_\alpha\phi^\dagger\partial_\beta\phi + \partial_\beta\phi^\dagger\partial_\alpha\phi\right) - (1-4\xi)g_{\alpha\beta}\partial\phi^\dagger\cdot\partial\phi + m^2 g_{\alpha\beta}\phi^\dagger\phi$$
$$+ 2\xi\left(g_{\alpha\beta}\phi^\dagger\Box\phi + \Box\phi^\dagger\phi - \phi^\dagger\partial_\alpha\partial_\beta\phi - \partial_\alpha\partial_\beta\phi^\dagger\phi\right)$$

- conserved current:

$$j_\alpha = \mathrm{i}(\phi^\dagger\partial_\alpha\phi - \partial_\alpha\phi^\dagger\phi)$$

Acceleration and rotation

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$$U_w = \frac{(1-4\xi)}{12\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} p^4 \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu) \right)$$

$$U_\alpha = \frac{1}{48\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} (p^2+m^2)(m^2+4p^2(1-6\xi)) \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu) \right)$$

$$W = \frac{(2\xi-1)}{24\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} p^4 \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu) \right)$$

$$A = \frac{1}{48\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} \left(2p^4(1-6\xi) + p^2m^2(3-12\xi)\right) \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu) \right)$$

$$G = \frac{1}{92\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} \left(p^4(1+6\xi) + 3p^2m^2\right) \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu) \right)$$

$$D_\alpha = \frac{1}{144\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} \left(8p^4(6\xi-1) + 3m^2(24\xi-5)\right) \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu) \right)$$

$$D_w = \frac{\xi}{6\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} p^4 \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu) \right)$$

Acceleration and rotation (massless case)

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$\xi = \text{generic}$

$$U_w = \frac{(1-4\xi)}{12\beta^4}$$
$$U_\alpha = \frac{(1-6\xi)}{12\beta^4}$$
$$W = \frac{(2\xi-1)}{12\beta^4}$$
$$A = \frac{(1-6\xi)}{12\beta^4}$$
$$G = \frac{(1+6\xi)}{36\beta^4}$$
$$D_\alpha = \frac{(6\xi-1)}{18\beta^4}$$
$$D_w = \frac{\xi}{6\beta^4}$$

$\xi = 1/6$

$$U_w = \frac{1}{36\beta^4}$$
$$U_\alpha = 0$$
$$W = -\frac{1}{18\beta^4}$$
$$A = 0$$
$$G = \frac{1}{18\beta^4}$$
$$D_\alpha = 0$$
$$D_w = \frac{1}{36\beta^4}$$

$\xi = 0$

$$U_w = \frac{1}{12\beta^4}$$
$$U_\alpha = \frac{1}{12\beta^4}$$
$$W = -\frac{1}{12\beta^4}$$
$$A = \frac{1}{12\beta^4}$$
$$G = \frac{1}{36\beta^4}$$
$$D_\alpha = -\frac{1}{18\beta^4}$$
$$D_w = 0$$

Acceleration and rotation (Bessel Series)

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$$U_w = \frac{m^4}{2\pi^2}\sum_{n=1}^{+\infty}\frac{2\cosh(yn)}{2x^2}(1-4\xi)K_2(xn)$$

$$U_\alpha = \frac{m^4}{2\pi^2}\sum_{n=1}^{+\infty}\frac{2\cosh(yn)}{24x^2}\left[(n^2x^2+24\xi)K_2(nx)+3(1-8\xi)nxK_3(nx)\right]$$

$$W = \frac{m^4}{2\pi^2}\sum_{n=1}^{+\infty}\frac{2\cosh(yn)}{2x^2}(2\xi-1)K_2(nx)$$

$$A = \frac{m^4}{2\pi^2}\sum_{n=1}^{+\infty}\frac{2\cosh(yn)}{4x^2}\left[(4\xi-2)K_2(nx)+(1-\xi)nxK_3(nx)\right]$$

$$G = \frac{m^4}{2\pi^2}\sum_{n=1}^{+\infty}\frac{2\cosh(yn)}{6x^2}\left[(6\xi-3)K_2(nx)+nxK_3(nx)\right]$$

$$D_\alpha = \frac{m^4}{2\pi^2}\sum_{n=1}^{+\infty}\frac{2\cosh(yn)}{24x^2}\left[(12-48\xi)K_2(nx)+(24\xi-5)nxK_3(nx)\right]$$

$$D_w = \frac{m^4}{2\pi^2}\sum_{n=1}^{+\infty}\frac{2\cosh(yn)}{x^2}\xi K_2(nx)$$

$x = \beta m$ and $y = \beta\mu$

Dirac Field

We consider a Dirac field at finite temperature and chemical potential and we compute the coefficients using two different energy-momentum tensor

- The symmetric

$$T_{\alpha\beta} = \frac{\mathrm{i}}{4}\left[\bar{\psi}\gamma_\alpha\partial_\beta\psi - \partial_\beta\bar{\psi}\gamma_\alpha\psi\bar{\psi}\gamma_\beta\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\beta\psi\right]$$

- The canonical

$$T_{\alpha\beta} = \frac{\mathrm{i}}{2}\left[\bar{\psi}\gamma_\alpha\partial_\beta\psi - \partial_\beta\bar{\psi}\gamma_\alpha\psi\right]$$

- The currents

$$j_\alpha = \bar{\psi}\gamma_\alpha\psi \qquad j^5_\alpha = \bar{\psi}\gamma^5\gamma_\alpha\psi$$

Coefficients for the symmetric tensor

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$$U_w = -\frac{1}{8\pi^2\beta^2}\int_0^\infty \mathrm{d}p\,(3p^2 + m^2)\left(n_F^{(1)}(E_p - \mu) + n_F^{(1)}(E_p + \mu)\right)$$

$$U_\alpha = \frac{1}{24\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2 + m^2}}(p^2 + m^2)^2\left(n_F^{(2)}(E_p - \mu) + n_F^{(2)}(E_p + \mu)\right)$$

$$W = 0$$

$$A = 0$$

$$G = -\frac{1}{24\pi^2\beta^2}\int_0^\infty \mathrm{d}p\,(4p^2 + m^2)\left(n_F^{(1)}(E_p - \mu) + n_F^{(1)}(E_p + \mu)\right)$$

$$D_\alpha = -\frac{1}{24\pi^2\beta^2}\int_0^\infty \mathrm{d}p\,(p^2 + m^2)\left(n_F^{(1)}(E_p - \mu) + n_F^{(1)}(E_p + \mu)\right)$$

$$D_w = -\frac{1}{8\pi^2\beta^2}\int_0^\infty \mathrm{d}p\,p^2\left(n_F^{(1)}(E_p - \mu) + n_F^{(1)}(E_p + \mu)\right)$$

Coefficients massless case

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu \alpha^\nu + W w^\mu w^\nu + G(u^\mu \gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

Symmetric

$$U_w = \frac{1}{8\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$U_\alpha = \frac{1}{24\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$W = 0$$
$$A = 0$$
$$G = \frac{1}{18\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$D_\alpha = \frac{1}{72\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$D_w = \frac{1}{24\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$

Canonical

$$U_w = \frac{1}{8\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$U_\alpha = \frac{1}{24\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$W = 0$$
$$A = 0$$
$$G_1 = \frac{2}{9\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2}) = -2G_2$$
$$D_\alpha = \frac{1}{72\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$D_w = \frac{1}{24\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$

Coefficients symmetric (Bessel series)

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$$U_w = -\frac{m^2}{16\pi^2\beta^2}\sum_{n=1}^{\infty} xn(-1)^n\Big(3K_3(xn) + K_1(xn)\Big)\cosh(yn)$$

$$U_\alpha = -\frac{m^4}{6\pi^2}\sum_{n=1}^{\infty} n^2(-1)^n\left(K_0(xn) + 2\frac{K_1(xn)}{xn} + 3\frac{K_2(xn)}{x^2n^2}\right)\cosh(yn)$$

$$W = 0$$

$$A = 0$$

$x = \beta m$ and $y = \beta\mu$

$$G = -\frac{m^2}{12\pi^2\beta^2}\sum_{n=1}^{\infty} xn(-1)^n K_3(xn)\cosh(yn)$$

$$D_\alpha = -\frac{m^2}{48\pi^2\beta^2}\sum_{n=1}^{\infty} xn(-1)^n\Big(K_3(xn) + 3K_1(xn)\Big)\cosh(yn)$$

$$D_w = -\frac{m^2}{16\pi^2\beta^2}\sum_{n=1}^{\infty} xn(-1)^n\Big(K_3(xn) - K_1(xn)\Big)\cosh(yn)$$

Coefficients canonical (T=0)

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

For T=0 the distribution function becomes a step function

$$U_\alpha = \frac{3E_{\rm F}^5 - 4m^2 E_{\rm F}^3}{12\pi^2\beta^2 p_{\rm F}^3},\ D_\alpha = -\frac{E_{\rm F}^2}{12\pi^2\beta^2},\ A = 0,$$
$$U_w = -\frac{E_{\rm F}^2 + 2p_{\rm F}^2}{4\pi^2\beta^2},\ D_w = -\frac{p_{\rm F}^2}{4\pi^2\beta^2},\ W = 0,\ G = -\frac{E_{\rm F}^2 + 3p_{\rm F}^2}{12\pi^2\beta^2}$$

Axial current

In fermion case we have also the axial current that is conserved in massless case.

$$j^5_\alpha = \bar{\psi}\gamma^5\gamma_\alpha\psi \qquad \partial^\alpha j^5_\alpha = 2m\mathrm{i}\bar{\psi}\gamma^5\psi$$

The only contribution is due to the vorticity (have the same parity and time reversal)

$$\langle j^5_\alpha \rangle = \langle j^5_\alpha \rangle_\beta + w^\rho \Delta_{\rho\alpha} j^5$$

$$j^5 = \frac{1}{2\pi^2\beta}\int_0^\infty \mathrm{d}p \frac{2p^2+m^2}{\sqrt{p^2+m^2}}\left(n_F(E_p-\mu)+n_F(E_p+\mu)\right)$$

The (non-)conservation of axial current

The divergence of the axial current leads the following relation

$$\partial^\alpha \langle j^5_\alpha \rangle = -\alpha \cdot w \left(\frac{3}{\beta} j^5 + \frac{\partial}{\partial \beta} j^5 \right)$$

The left right hand side must be equal to

$$\langle \partial^\alpha j^5_\alpha \rangle = -2m\mathrm{i}\alpha \cdot w \langle K_3 J_3 \bar{\psi} \gamma_5 \psi \rangle = \alpha \cdot w l^5$$

Vanish in massless case, in order to have a conserved current.

$$l^5 = \frac{1}{2\pi^2 \beta} \int_0^{+\infty} \mathrm{d}p \frac{m^2}{\sqrt{p^2 + m^2}} \left(n_F^{(1)}(E_p - \mu) + n_F^{(1)}(E_p + \mu) \right) = -\left(\frac{3}{\beta} j^5 + \frac{\partial}{\partial \beta} j^5 \right)$$

Conclusions and Outlook

- The ideal form of the energy momentum tensor get extra correction due to vorticity and acceleration.
- The second order coefficients involving vorticity and acceleration are generally different from zero for a free case despite of the other transport coefficient.
- They depend on the particular theory and also on the choice of the energy momentum tensor operator.
- They are generally negligible, but can be relevant in some extreme situation.