

# Ab initio calculations for non-strange and strange few-baryon systems

- Applied ab initio methods
- Non-strange sector: continuum observables
- Strange sector: hypernuclear bound states

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- Applied ab initio methods
- Non-strange sector: continuum observables
  - Resonances
  - S-Factor in presence of Coulomb potential
- Strange sector: hypernuclear bound states
  - Benchmark calculation with other ab initio methods

# Applied ab initio methods

- A-body system: **A position vectors  $\mathbf{r}_i$** , removal of center of mass coordinate leads to **(A-1) Jacobi vectors  $\boldsymbol{\eta}_i$  ( $\eta_i, \theta_i, \phi_i$ )**
- Expansion of ground-state wave function or LIT state on a complete set:  
**Hyperspherical Harmonics (HH)**
- 3(A-1) coordinates of HH basis: hyperradius  $\rho$ , 2(A-1) angular coordinates  $\Theta_i$  and  $\phi_i$ , (A-2) hyperspherical angles: **1 hyperradius + (3A - 4) angles**
- HH basis states: eigen states of grand-angular momentum operator depending on the **(3A - 4) angles** times a hyperradial basis state
- Different HH versions: normally symmetrized basis states, but also a nonsymmetrized HH (**NSHH**) basis is possible
- Acceleration of convergence: effective interaction (**EIHH**)  
**Short-range two-body correlations (CHH)**

# Applied ab initio methods

- Solve Schrödinger or LIT equation with **N basis states** and **increase N** up to the point that a sufficient **convergence** is obtained

# LIT method

The LIT of a function  $R(E)$  is defined as follows

$$\Rightarrow L(\sigma) = \int dE \mathcal{L}(E, \sigma) R(E),$$

where the kernel  $\mathcal{L}$  is a Lorentzian,

$$\Rightarrow \mathcal{L}(E, \sigma) = \frac{1}{(E - \sigma_R)^2 + \sigma_I^2}$$

For inclusive reactions the LIT  $L(\sigma)$  is calculated by solving an equation of the form

$$(H - \sigma) \tilde{\Psi} = S,$$

where  $H$  is the Hamiltonian of the system under consideration and  $S$  is an asymptotically vanishing source term related to the operator inducing the specific reaction.

The solution  $\tilde{\Psi}$  is localized and the LIT is given by

$$L(\sigma) = \langle \tilde{\Psi} | \tilde{\Psi} \rangle.$$

Alternative way:

$$L(\sigma) = -\frac{1}{\sigma_I} \text{Im} \left( \langle S | \frac{1}{\sigma_R + i\sigma_I - H} | S \rangle \right).$$

The source term  $S$  for inclusive reactions has the form

$$\Rightarrow |S\rangle = \theta|0\rangle,$$

where the operator  $\theta$  induces a specific electroweak reaction.

The corresponding response function is given by

$$\Rightarrow R(E_f) = \int dE_f |\langle f|\theta|0\rangle|^2 \delta(E_f - E_0 - \omega)$$

Ingredients of the solution of the LIT equation via an expansion on a basis of dimension  $N$ :

$N$  eigenstates with eigenenergies

$$\phi_n$$

$$E_n$$

and strength

$$S_n = |\langle \phi_n | \theta | 0 \rangle|^2$$

leading to the following LIT

$$L(\sigma) = \sum_{i=1}^N \frac{S_n}{(\sigma_R - E_n)^2 + \sigma_I^2}$$

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$$\Rightarrow L(\sigma) = \sum_{i=1}^N \frac{S_n}{(\sigma_R - E_n)^2 + \sigma_I^2}$$

# Inversion of the LIT

- LIT is calculated for a fixed  $\sigma_l$  in many  $\sigma_r$  points
- Express the searched response function formally on a basis set with N basis functions  $f_n(E)$  and open coefficients  $c_n$  with correct threshold behaviour for the  $f_n(E)$  (e.g.,  $f_n = f_{thr}(E) \exp(-\alpha E/n)$ )
- Make a LIT transform of the basis functions and determine coefficients  $c_n$  by a fit to the calculated LIT
- Increase N up to the point that a sufficient convergence is obtained (uncontrolled oscillations should not be present)

# Resonances

## $^4\text{He}$ isoscalar monopole resonance

Isoscalar monopole response function  $M(q, E_f = E_0 + \omega)$

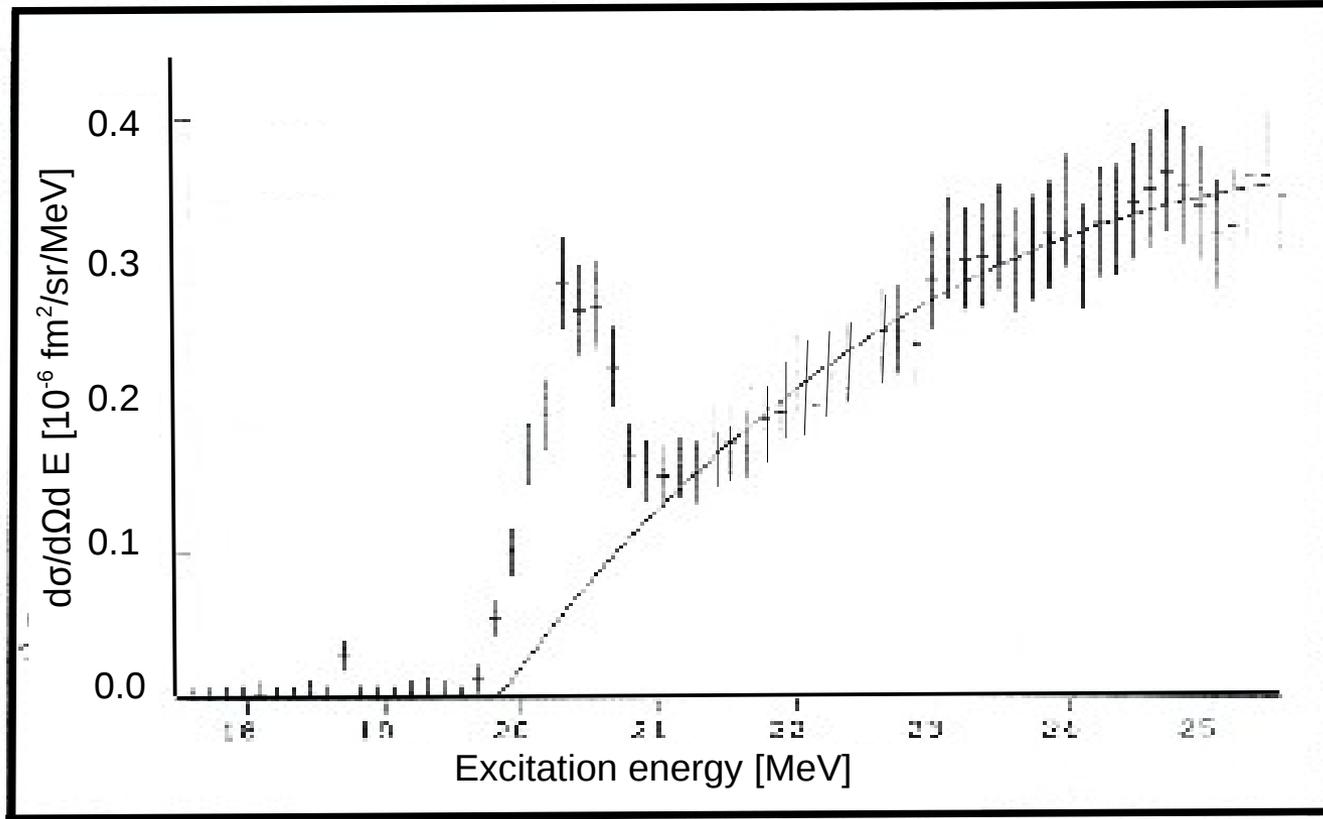
with transition operator  $\theta(q) = \frac{G_E^s(q^2)}{2} \sum_{i=1}^A j_0(qr_i)$

$G_E^s(q^2)$ : nucleon isoscalar electric form factor

$j_0$ : spherical Bessel function of  $0^{\text{th}}$  order

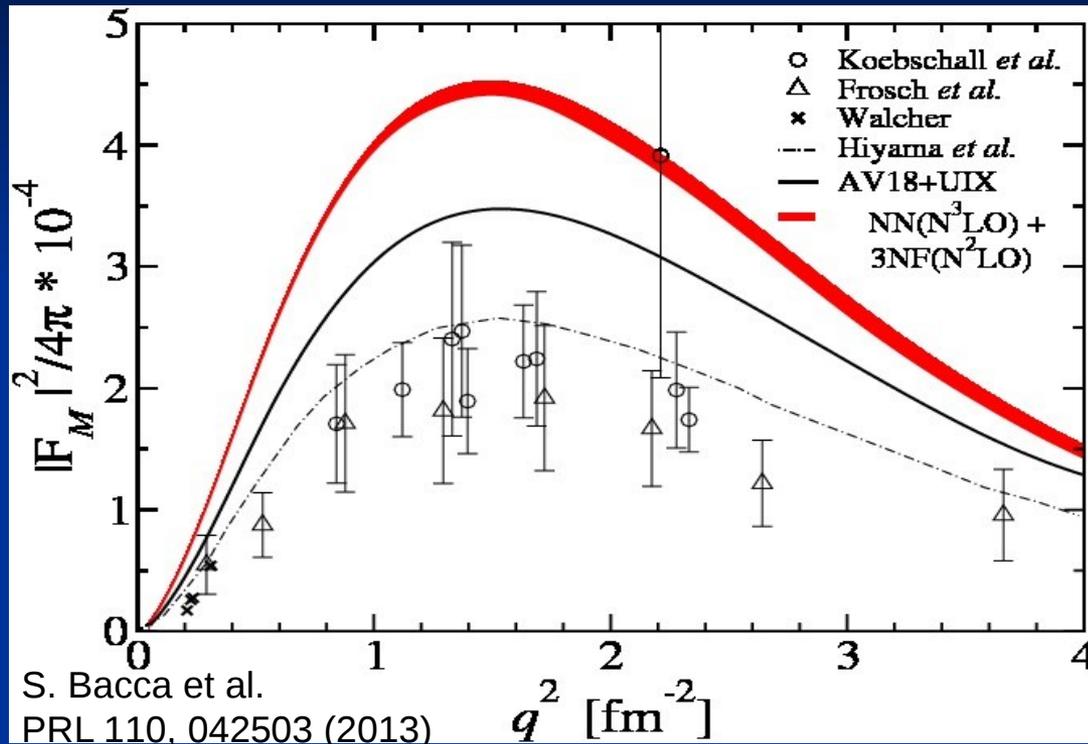
# $0^+$ Resonance in the $^4\text{He}$ compound system

G. Köbschall et al./ Quasi bound state in  $^4\text{He}$  - Nucl. Phys. A405, 648 (1983)



Resonance at  $E_R = -8.2$  MeV, i.e. above the  $^3\text{H}$ -p threshold. **Strong evidence** in electron scattering off  $^4\text{He}$ ,  $\Gamma = 270 \pm 50$  keV

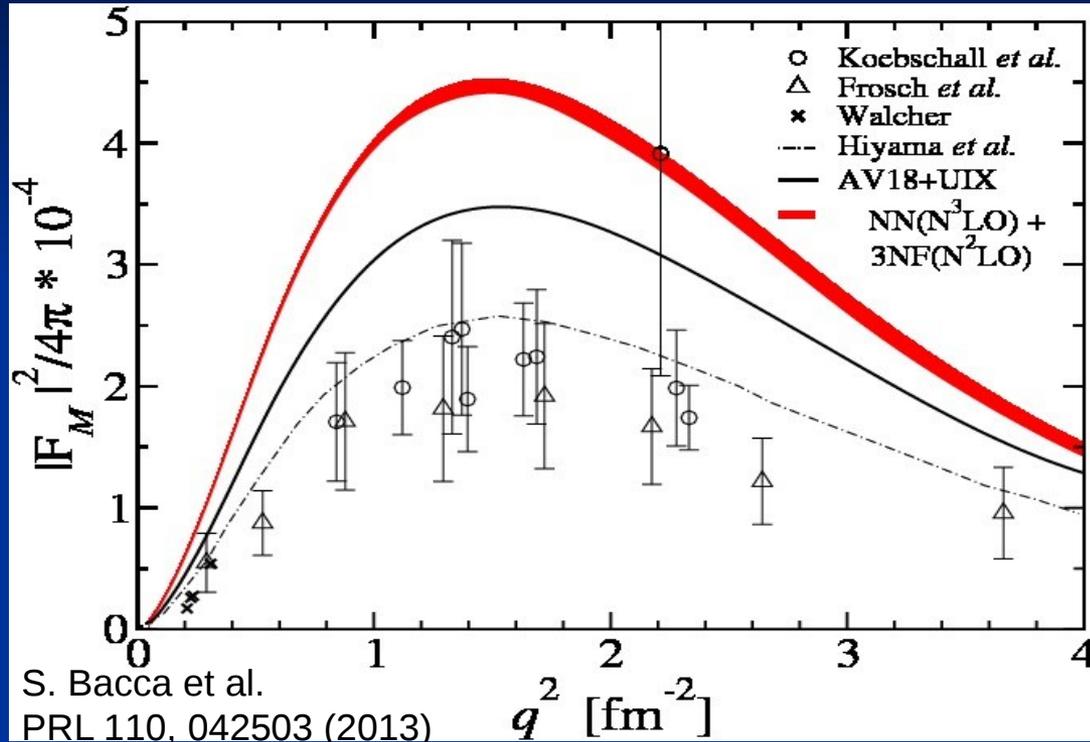
# Comparison to experimental results



LIT/EIHH Calculation for AV18+UIX and Idaho-N3LO+N2LO

dash-dotted: AV8' + central 3NF (Hiyama et al.)

# Comparison to experimental results



Observable is strongly dependent on potential model

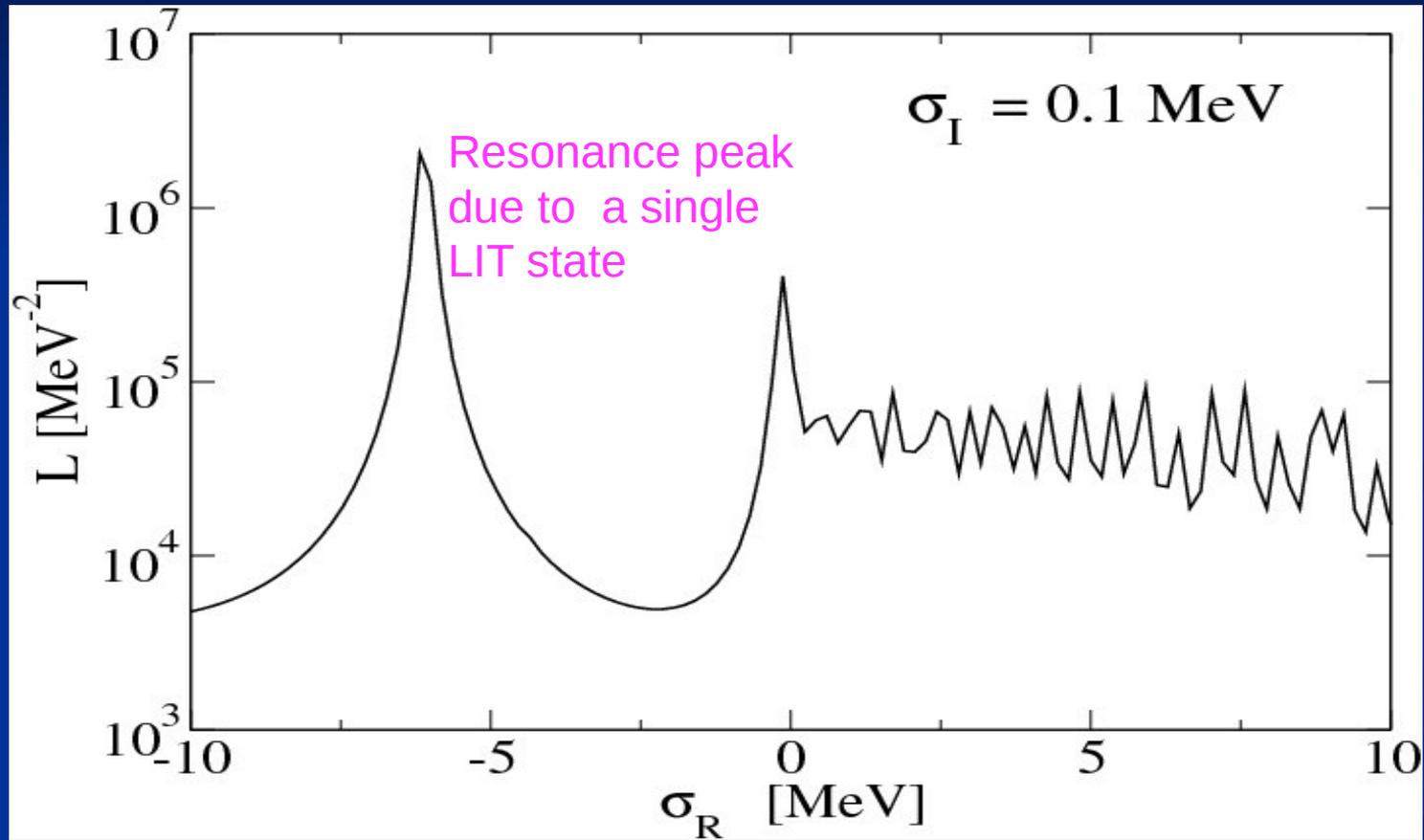
Why were we unable to determine the width of the  $^4\text{He}$   
isoscalar monopole resonance?

To answer this let us check our very first LIT calculation from 1997:

$^4\text{He}(e,e')$  inelastic longitudinal response function

with a central NN potential

Unpublished result from a CHH calculation with the TN potential (V. Efros, WL, G. Orlandini, PRL 78,432 (1997))



To study the problem better let us consider first instead of a four-body reaction a simpler three-body reaction:



at low energies

LIT calculation with central MTI/III NN potential in unretarded dipole approximation

**Aim:** Increase low-energy density of LIT states

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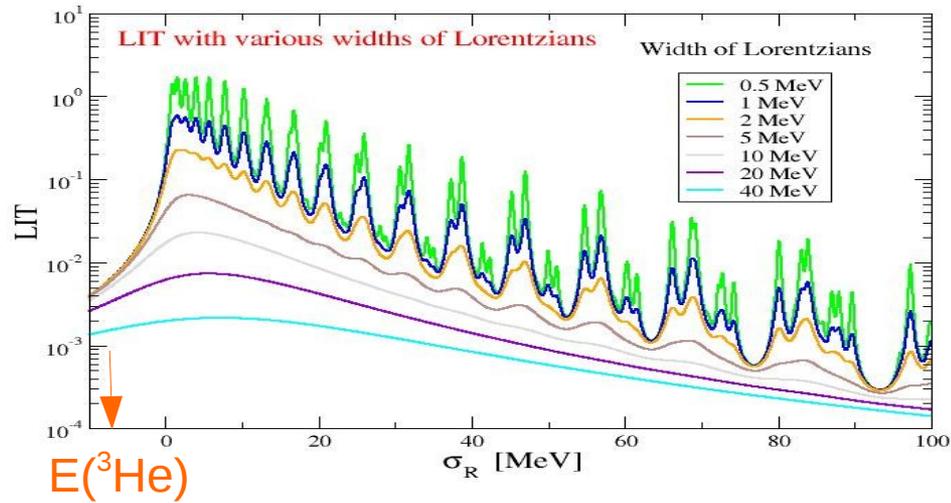
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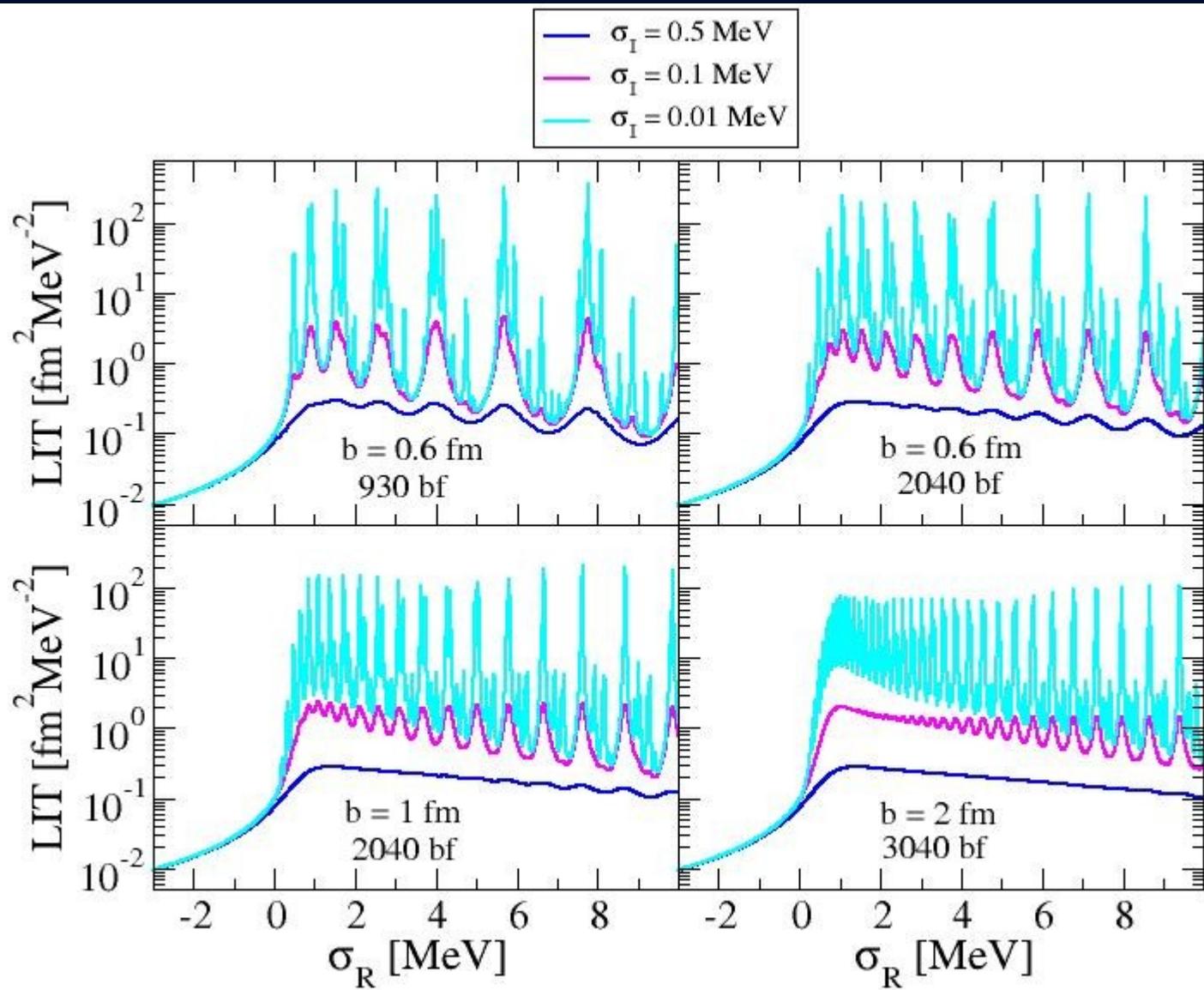
**Aim:** Increase low-energy density of LIT states

Answer seems to be simple increase number of basis states both for hyperspherical and hyperradial states

**Also note:** hyperradial basis states consist in an expansion on Laguerre polynomials times a spatial cutoff  $\exp(-\rho/b)$   
Increase of **b** shifts spectrum to lower energies



30 hyperspherical  
 31 hyperradial  
 $\Rightarrow$  930 basis states  
 $b = 0.6$  fm



# Observation

The LIT is a method with a controlled resolution

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But in present LIT calculation below three-body breakup threshold not a single LIT state! Similar problem as in the previous four-body case

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Solution: use instead of the HH basis a somewhat modified basis

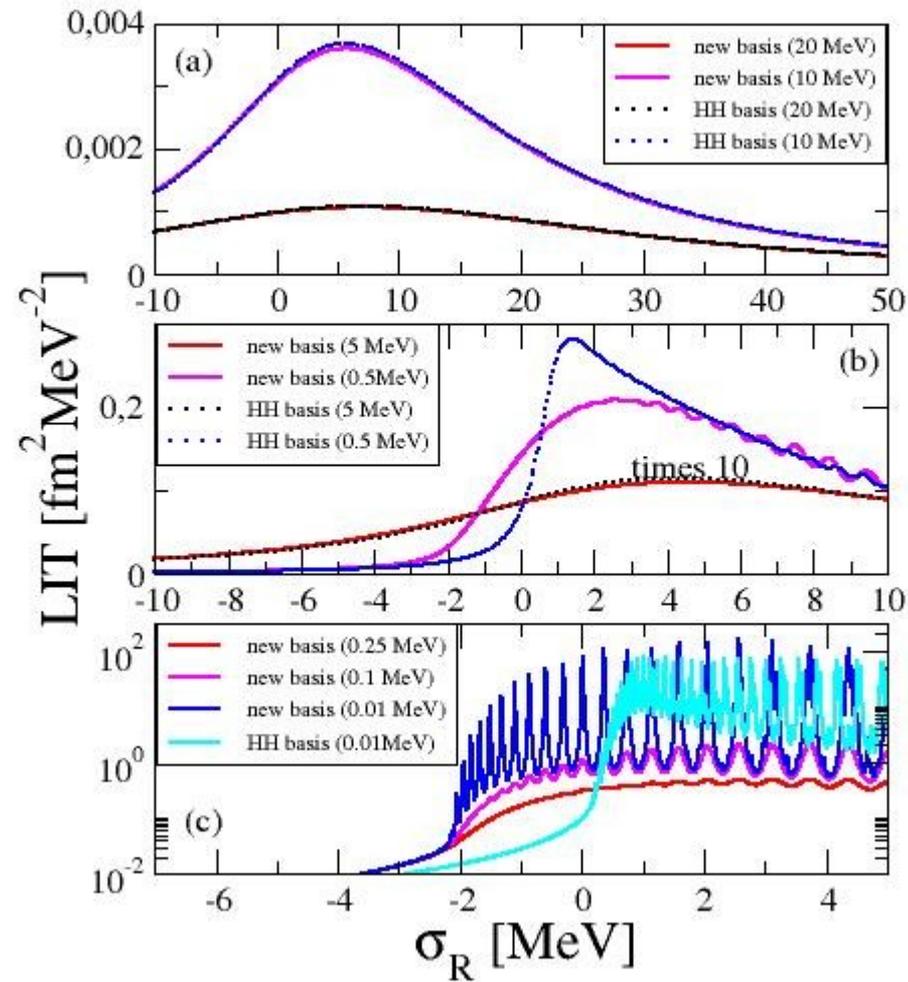
# New A-body basis

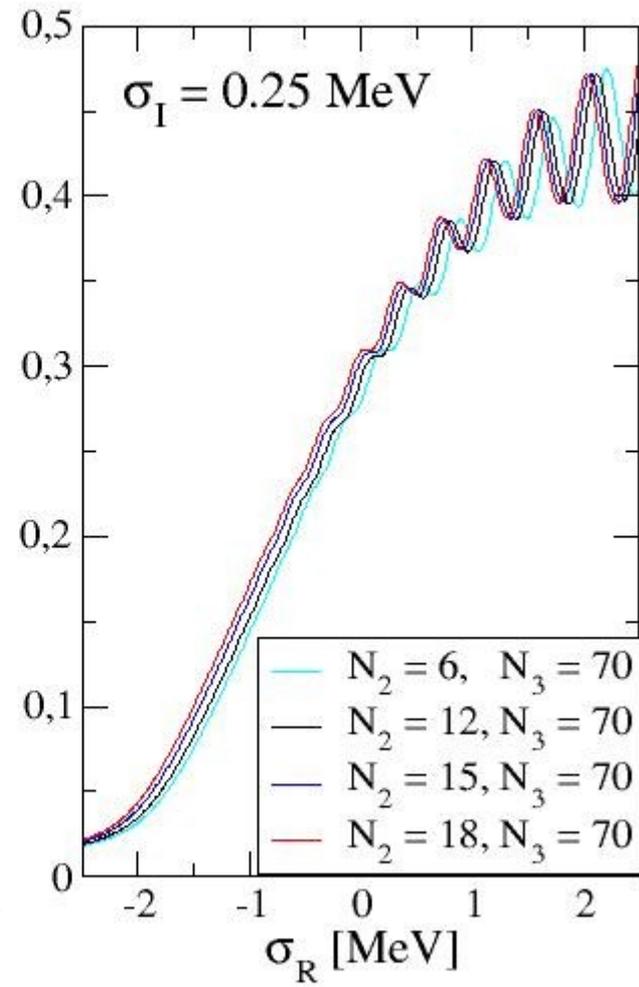
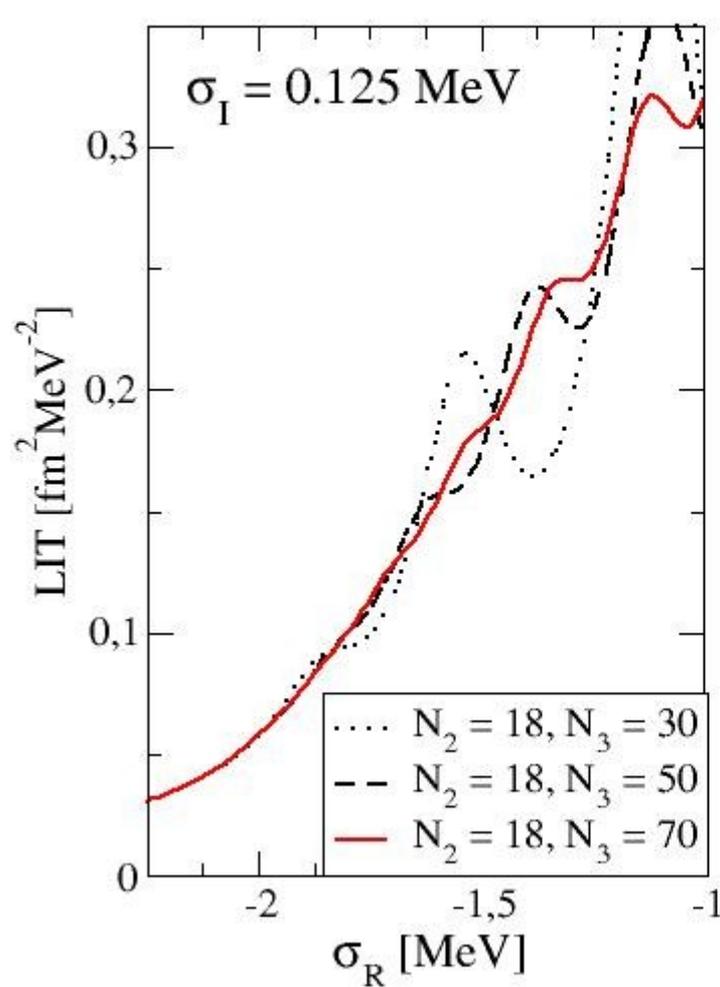
Note one of the (A-1) Jacobi vectors can be written in the following form:

$$\eta = \mathbf{r}_A - \mathbf{R}_{\text{cm}}(1,2,\dots,A-1)$$

This is the coordinate one would use for the scattering of a nucleon with a (A-1)-nucleon system. In other words the relevant coordinate for a two-body breakup. Therefore

A-body HH basis  $\longrightarrow$  (A-1)-body HH basis times expansion on  $\eta$   
radial part: Laguerre polynomials  
angular part:  $Y_{LM}(\theta_\eta, \phi_\eta)$

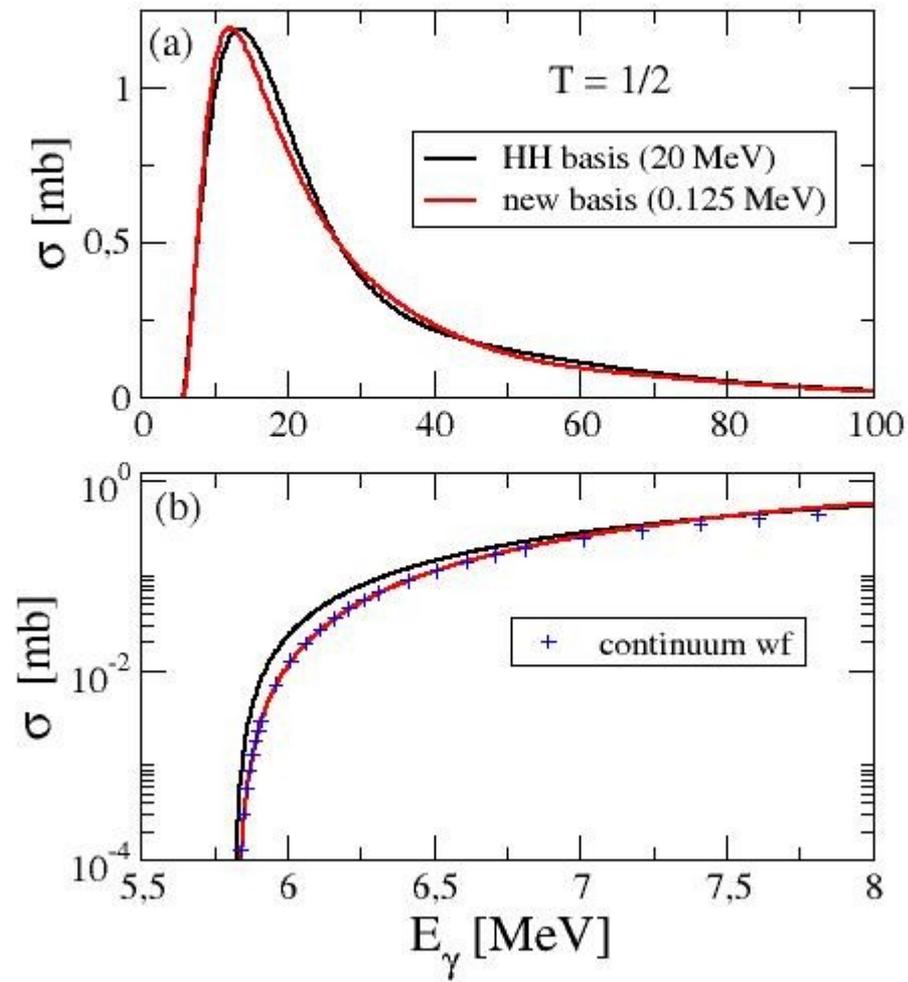


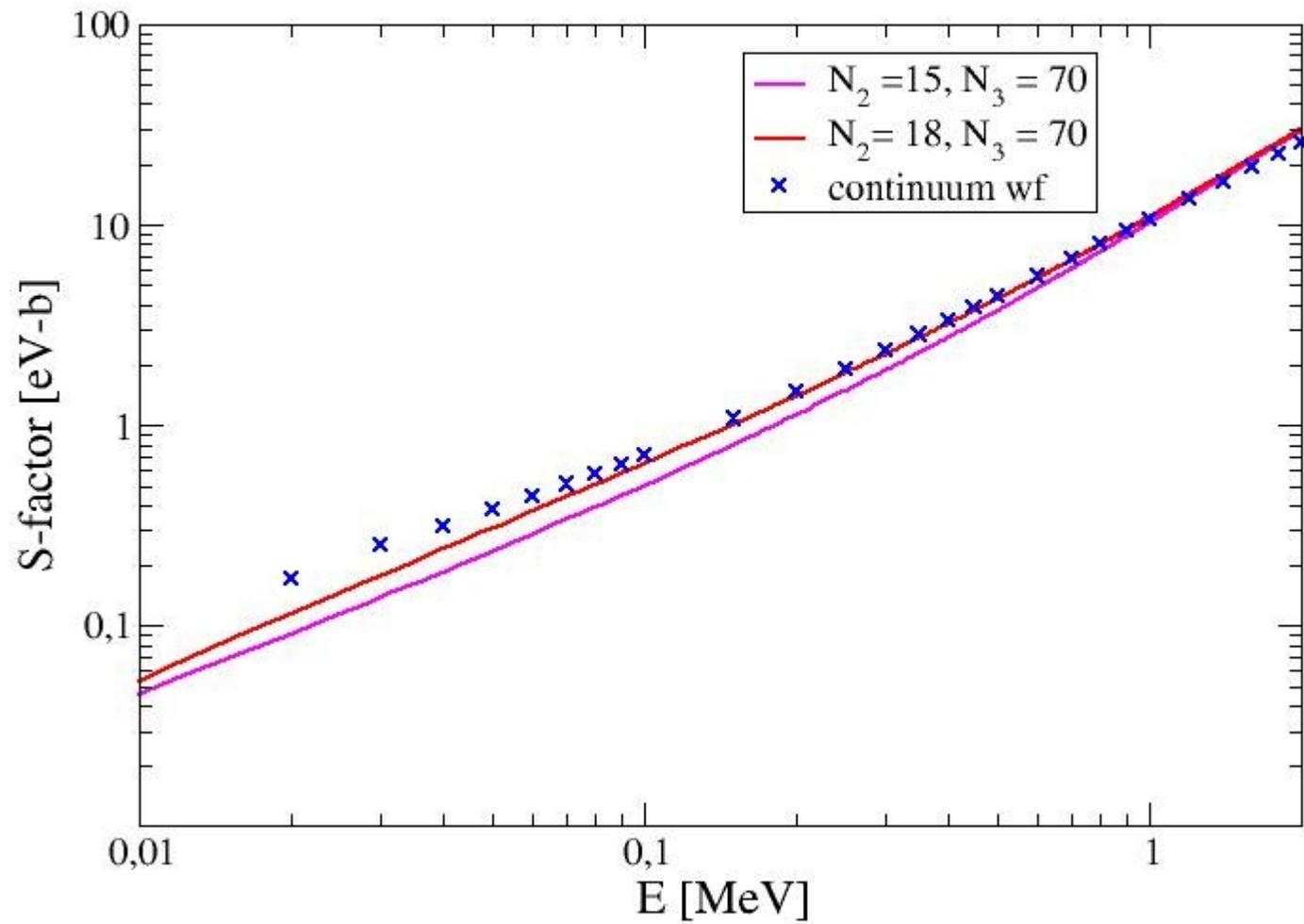


# Inversions

Implement correct threshold behaviour for  ${}^3\text{He} + \gamma \rightarrow \text{d} + \text{p}$

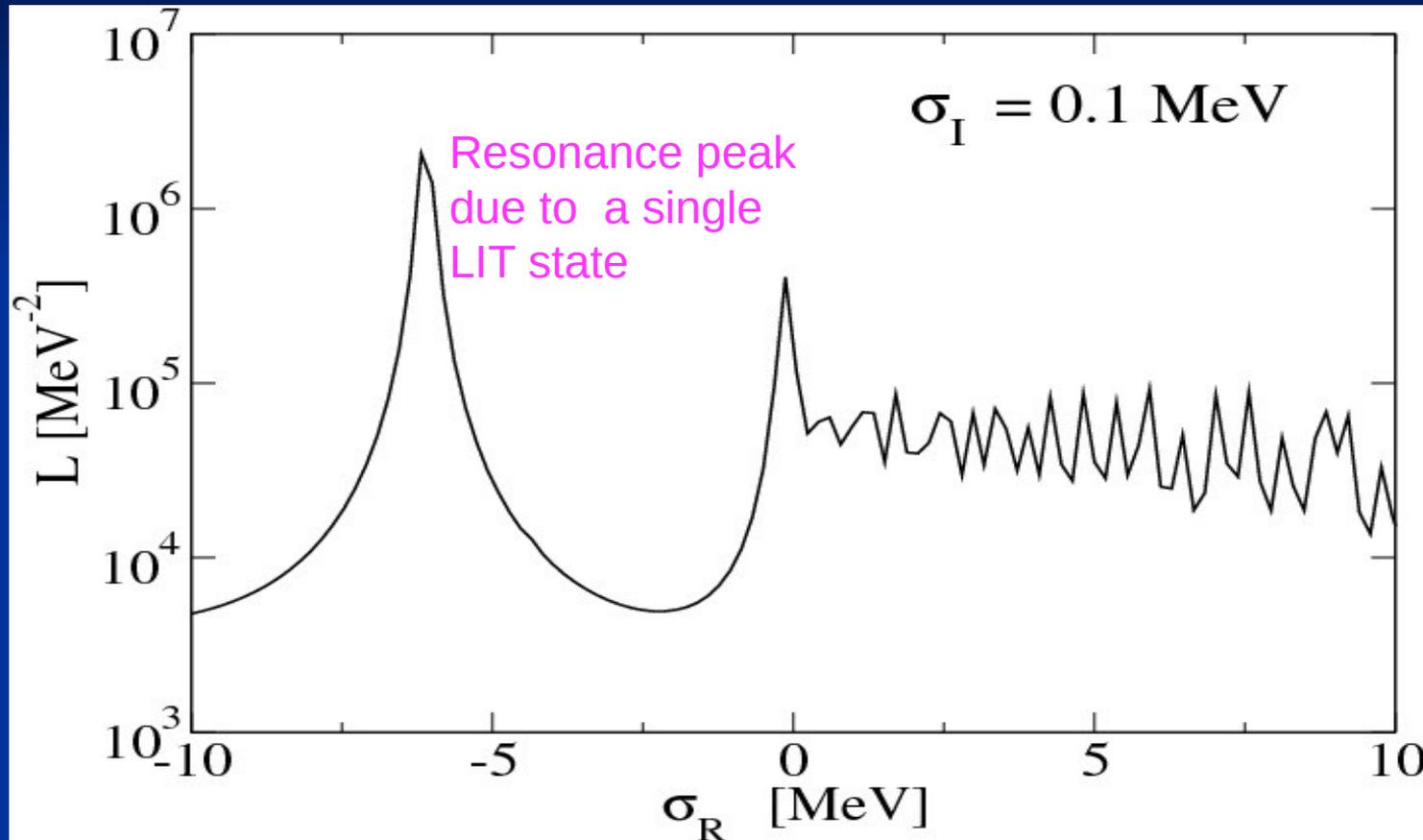
Due to Coulomb potential: usual Gamow factor



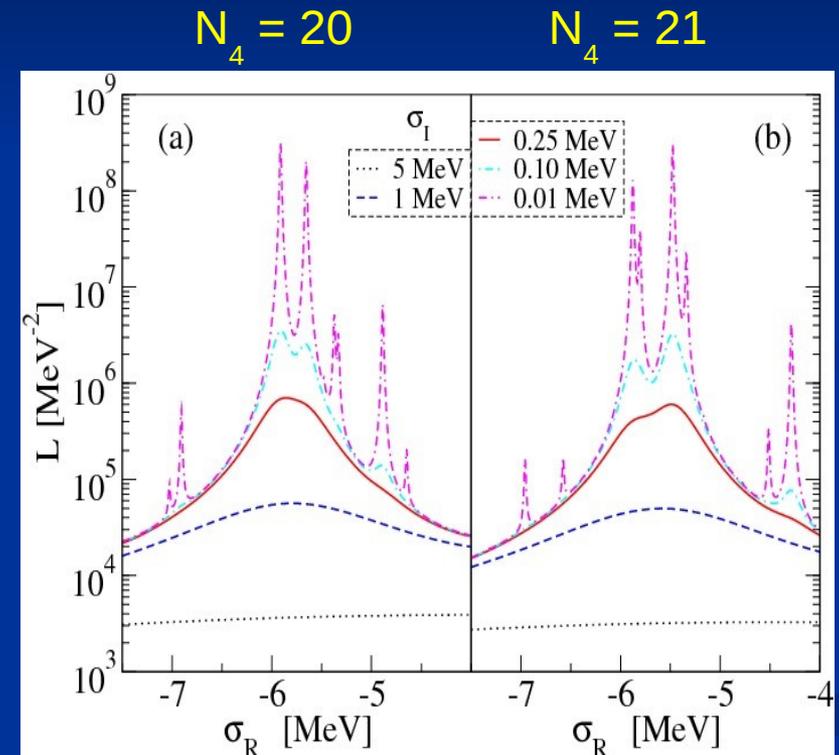
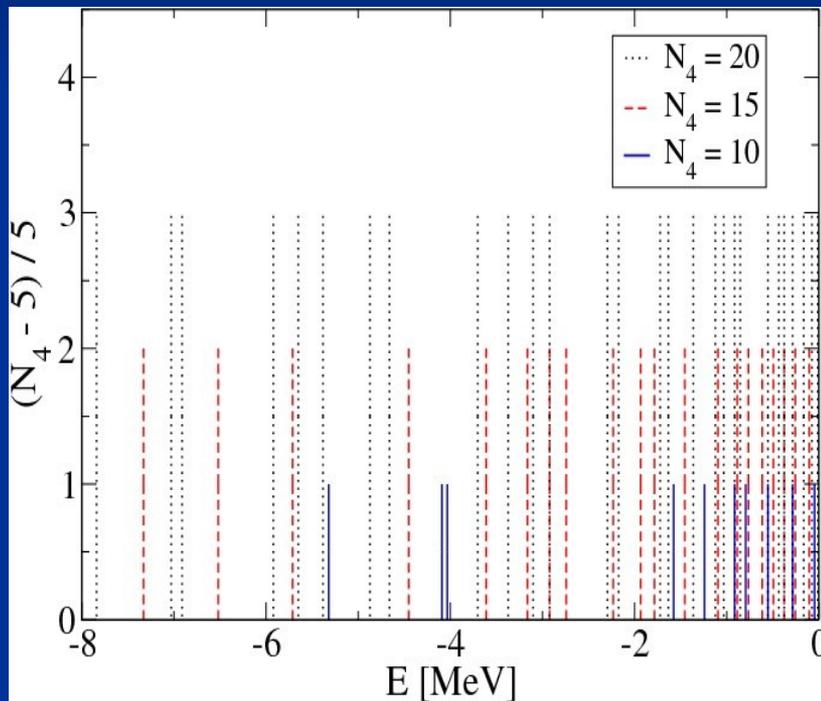


# Back to the $^4\text{He}$ resonance

Unpublished result from a CHH calculation with the TN potential (V. Efros, WL, G. Orlandini, PRL 78,432 (1997))

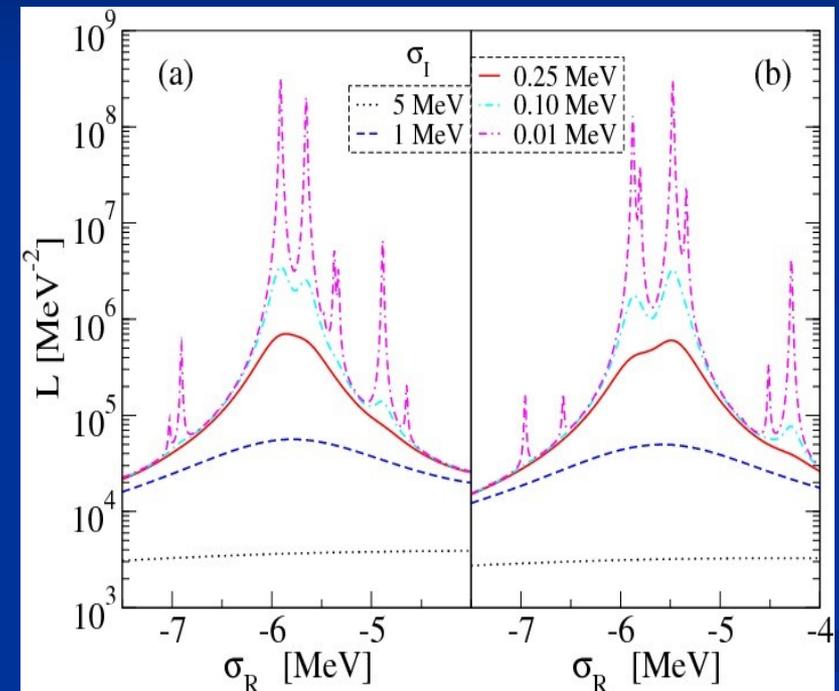
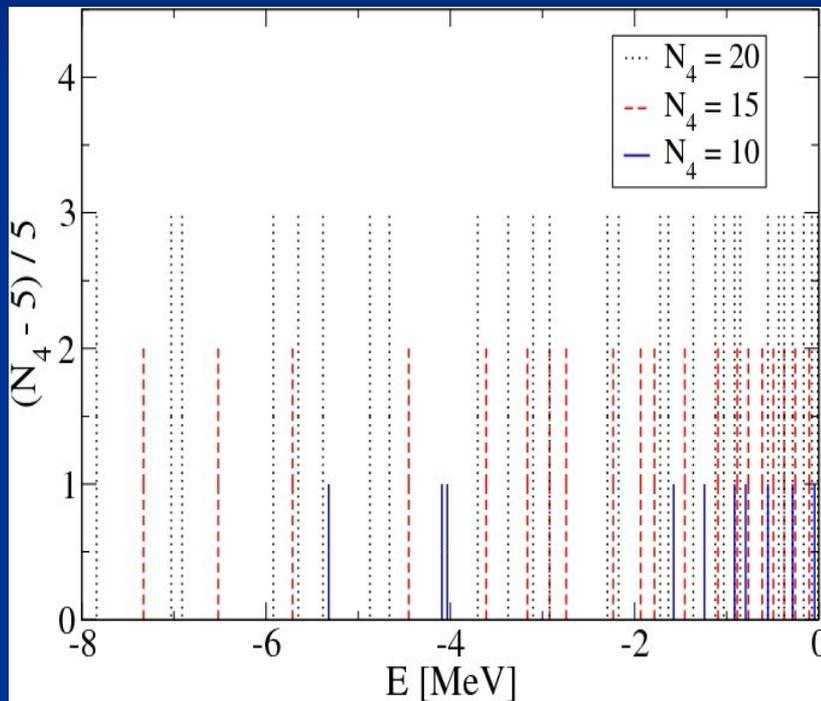


# Results with new basis



LIT

# Results with new basis



**Inversion:  $\Gamma = 180(70)$  keV**

WL, PRC 91, 054001 (2015)

# Benchmark calculations for hypernuclei

- Quick introduction to hypernuclei
- Short outline of our NSHH method
- Preliminary benchmark results: comparison with  
AFDMC (D. Lonardonì, F. Pederiva)  
Faddeev (A. Nogga)  
(GEM: E. Hiyama)

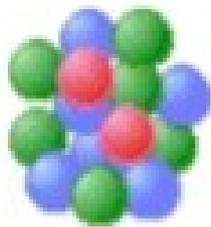
# Nuclei with Strangeness

$$m_{\Lambda} = 1116 \text{ MeV}$$

$$m_{\Sigma^+} = 1189$$

$$m_{\Sigma^0} = 1193$$

$$m_{\Omega^-} = 1673$$

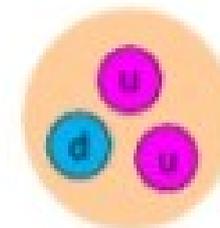


neutron



No charge

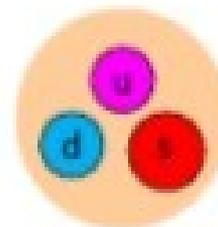
proton: 3 quarks



Mass: 938 MeV

+charge

hyperon: including strangeness quark



$\Lambda, \Xi$



$\Xi$



$\Omega$

$$\tau_{\Lambda} = 263 \text{ ps}$$

$$\tau_{\Sigma^+} = 80 \text{ ps}$$

$$\tau_{\Sigma^0} = 7.4 \cdot 10^{-20} \text{ s}$$

$$\tau_{\Omega^-} = 82 \text{ ps}$$

# Hypernuclear Chart

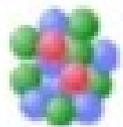
$N_u - N_d - N_s$



Stable strangeness in neutron stars ( $\rho > 3 - 4 \rho_0$ )  
 Strange hadronic matter ( $A \rightarrow \infty$ )

$p, n, \Lambda, \Xi, \Sigma$

Higher density



$\Lambda$



$p$   $n$

Strangeness

$\Lambda\Lambda, \Xi$  Hypernuclei

$\Lambda, \Sigma$  Hypernuclei

$Z$

-2

-1

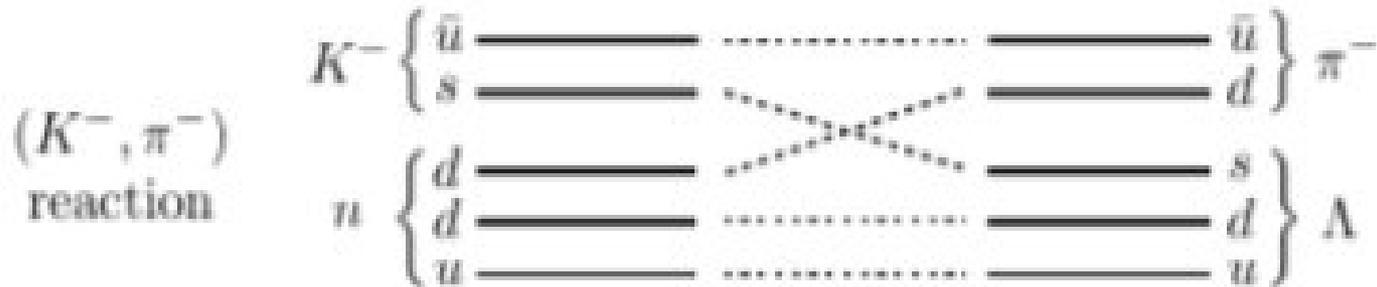
0

$N$

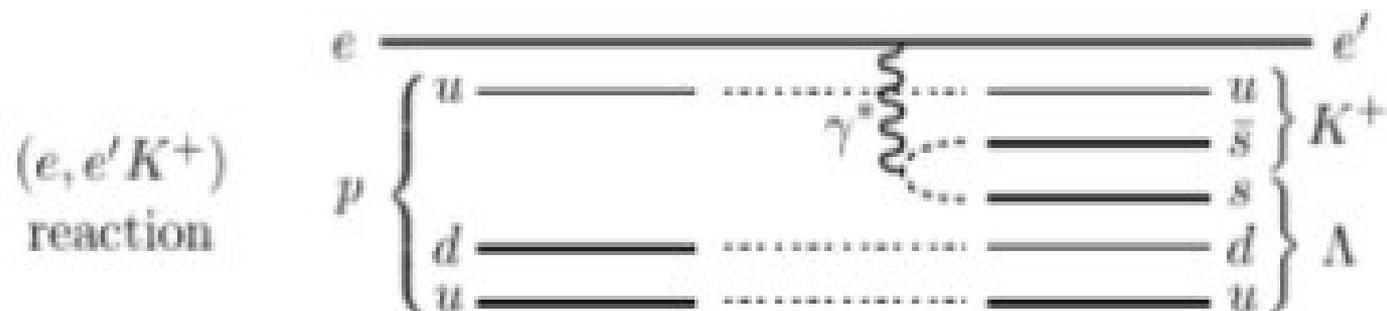
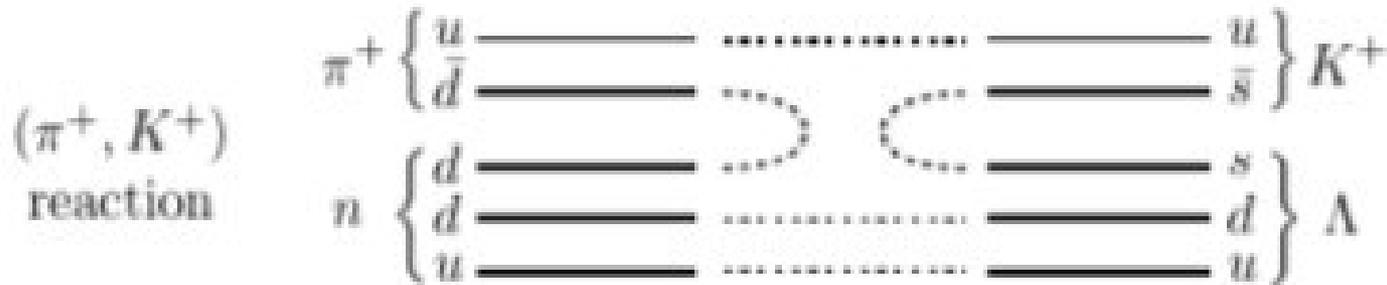
3-dimensional nuclear chart

# Production of Hypernuclei

## Strangeness exchange reaction



## Associated production reaction



# Experimental Present and Future Perspectives

- Despite extensive investigations, single  $\Lambda$  hypernuclei knowledge is **far** from that of ordinary nuclei;
- Only **one** bound  $\Sigma$ -hypernucleus detected!
- **No**  $\Xi$  hypernuclei detected (some indications of weak attraction);
- **No** experimental information about  $\Omega$  hypernuclei;
- **Four**  $\Lambda\Lambda$ -hypernuclei energies measured ( ${}_{\Lambda\Lambda}^6\text{He}$ ,  ${}_{\Lambda\Lambda}^{10}\text{Be}$ ,  ${}_{\Lambda\Lambda}^{12}\text{Be}$ ,  ${}_{\Lambda\Lambda}^{13}\text{B}$ );

# Non-Symmetrized HH method

**Problem:** selection of **antisymmetric** states (we deal with fermions):

⇒ We add to  $\hat{H}$  the **Casimir operator** of the **permutation group  $S_N$** , which selects "by himself" the interesting states:

$$\hat{H}' = \hat{H} + \gamma \hat{C}(A) \quad ; \quad \hat{C}(A) = \sum_{i>j} \hat{P}_{ij}$$

Its action on the vectors:

$$\hat{C}(A)\psi_s = \frac{A(A-1)}{2}\psi_s = \lambda_s\psi_s ;$$

$$\hat{C}(A)\psi_m = \lambda_m\psi_m ;$$

$$\hat{C}(A)\psi_a = -\frac{A(A-1)}{2}\psi_a = \lambda_a\psi_a ,$$

⇒ with a proper choice of  $\gamma$  the g.s. energy  $E_A^0$  becomes the **lowest eigenvalue of  $H'$**  (similar procedure for excited states).

Hypernuclei are systems made of **two different species** of particles.

- Different **masses**:

- mass weighted coordinates  $\Rightarrow$  dependence inside **transpositions**:

$$B_{ij} \rightarrow B_{ij}(m_1, m_i, m_j) ;$$

- relative coordinate **rescaling** - mass dependence inside potential.

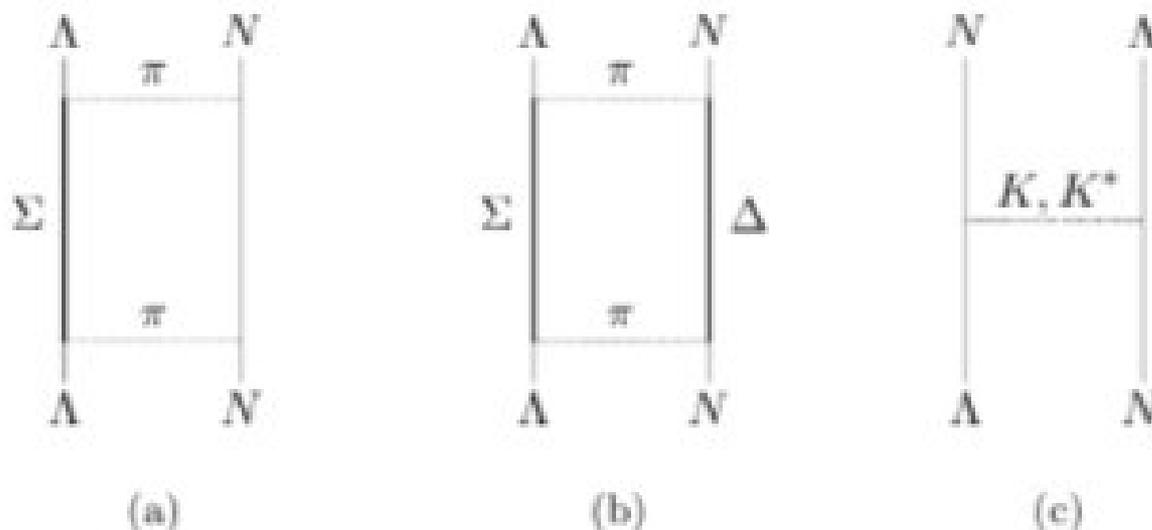
- Assuming 2-body potential: 3 types of different **interactions**:

- **NN** - nuclear core;
- **YN** - nucleon-hyperon couples;
- **YY** - hyperonic part.

## 2-body Bodmer Usmani interaction

The  $\Lambda$  particle has  $T = 0$  so there is no OPE term:

$$v_{\Lambda N}(r) = v_0(r) + \frac{V_\sigma}{4} T_\pi^2(r) \sigma_\Lambda \cdot \sigma_N,$$



$$v_0(r) = \frac{W_c}{1 + e^{\frac{r-\bar{r}}{a}}} - \bar{v} T_\pi^2(r)$$

$$T_\pi(r) = \left[ 1 + \frac{3}{\mu_\pi r} + \frac{3}{(\mu_\pi r)^2} \right] \frac{e^{-\mu_\pi r}}{\mu_\pi r} (1 - e^{-cr^2})^2$$

# Bodmer Usmani benchmark results

$V_{NN} + V_{YN}$	System	AFDMC	NSHH	FY
AV4'	${}^2\text{H}$	-2.245(15)	[-2.245(1)]	-2.245(1)
AV4'+U	${}^3_{\Lambda}\text{H}$	-2.45(5)	-2.530(3)	-2.537(1)
	$B_{\Lambda}$	0.21(5)	0.285(3)	0.292(1)
AV4'	${}^3\text{H}$	-8.92(5)	-8.983(7)	
AV4'+U	${}^4_{\Lambda}\text{H}$	-11.95(5)	-12.023(3)	
	$B_{\Lambda}$	3.03(7)	3.039(8)	
AV4'	${}^4\text{H}$	-32.85(5)	-32.695(6)	
AV4'+U	${}^5_{\Lambda}\text{H}$	-39.50(5)	-39.543(10)	
	$B_{\Lambda}$	6.65(7)	6.848(12)	

# NSC97f realistic interaction

We employed the **NSC97f realistic potential**<sup>3</sup> which simulates the Nijmegen scattering phase shifts:

$$\begin{aligned} {}^S V_{NY-NY'}(r) = & \sum_i \left( {}^S V_{NY-NY'}^C e^{-(r/\beta_i)^2} \right. \\ & + {}^S V_{NY-NY'}^T \mathbf{S}_{12} e^{-(r/\beta_i)^2} \\ & \left. + {}^S V_{NY-NY'}^{LS} \mathbf{LS} e^{-(r/\beta_i)^2} \right) \end{aligned}$$

- $Y \rightarrow \Lambda, \Sigma$ ;
- C  $\rightarrow$  central, T  $\rightarrow$  tensor, LS  $\rightarrow$  spin-orbit;
- gaussian radial functions with fitted parameters.

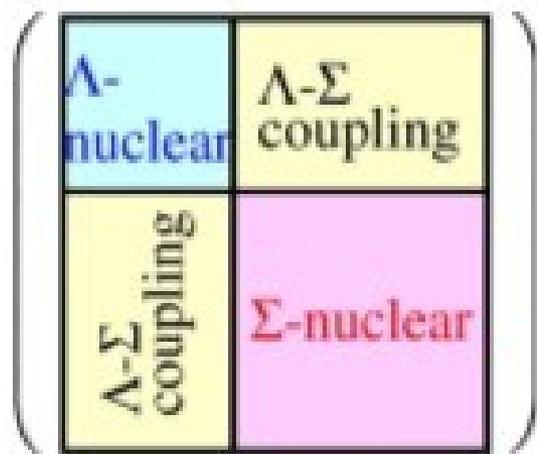
Explicit use of  **$\Sigma$  degree of freedom**  $\Rightarrow$  need for **extension** of the HYP-NSHH method.

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<sup>3</sup>E.Hiyama et al., Th.A.Rijken, Phys. Rev. C89, 061302 (2014).

# Lambda-Sigma mixing

⇒ extension of the basis including  $\Lambda/\Sigma$  degree of freedom:



- definition of transformation between two Jacobi sets **differing by one mass**;
- **extension of Lee-Suzuki** procedure including  $\Lambda/\Sigma$  degree of freedom.

# NSC97f interaction benchmark

$V_{NN} + V_{YN}$	System	NSHH	FY	GEM
AV8'	${}^2\text{H}$	[-2.226(1)]	-2.226(1)	
AV8'+NSC97f	${}^3_{\Lambda}\text{H}$	-2.41(2)	-2.415(1)	
	$B_{\Lambda}$	0.19(2)	0.189(1)	0.19(1)
AV8'	${}^3\text{H}$	-7.76(0)		
AV8'+NSC97f	${}^4_{\Lambda}\text{H}$	-10.05(7)		
	$B_{\Lambda}$	2.29(7)		2.33

Results were obtained in collaboration with

S. Deflorian and F. Ferrari-Rufino (Trento PhD students)

N. Barnea (Jerusalem), V. Efros (Moscow), G. Orlandini (Trento)