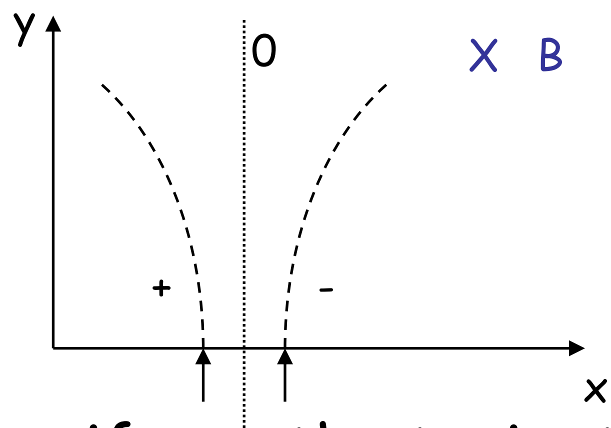
A fast introduction to the tracking and to the Kalman filter

Alghero 2009 Alberto Rotondi Pavia

The tracking

To reconstruct the particle path to find the origin (vertex) and the momentum

The trajectory is usually curved by the Lorentz force



$$\underline{F} = \frac{q}{c}\left(\underline{v} \times \underline{B}\right)$$

Even when B is uniform, the trajectory is NOT an helix, due to

- energy loss
- multiple scattering

The track is defined as a set of points usually on detector planes (real and/or virtual)

The track

the five track coordinates: 1/p, v',w',v,w

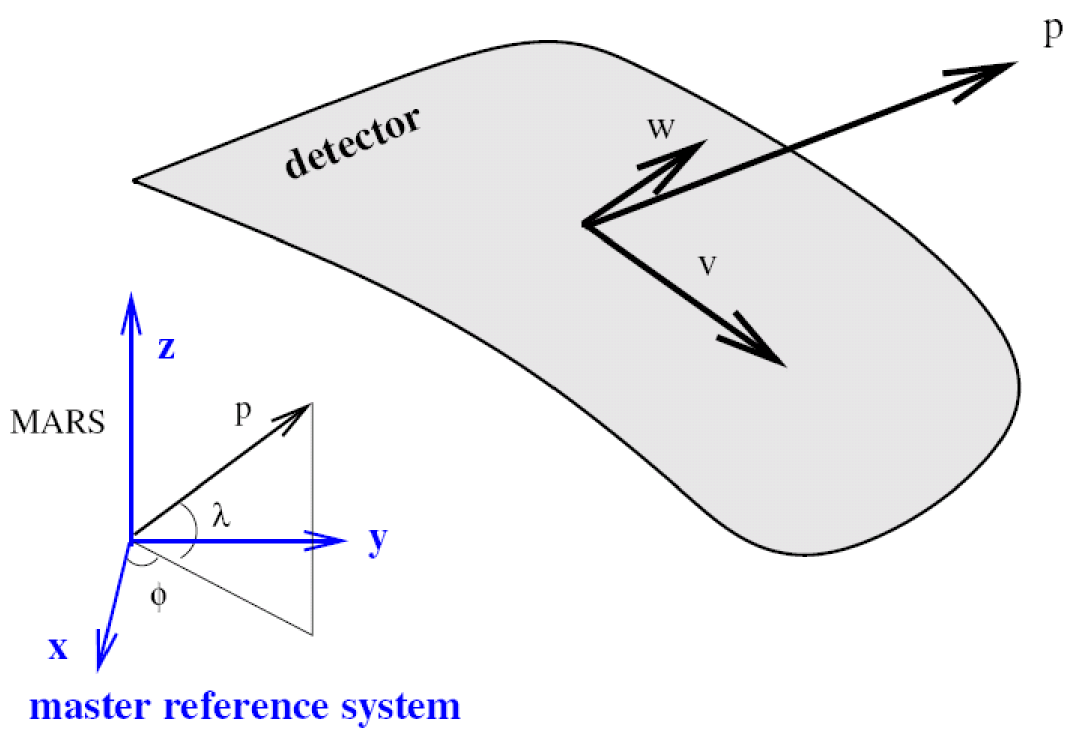


Figure 1: The five track parameters

Since on a detector plane we have two coordinates (v,w) and three momentum components (p_u, p_v, p_w) the track

is a 5-dimensional mathematical entity

Fitting method	Helix	Spline	Kalman
Magnetic field dishomogeneity Material effect	NO NO	YES NO	YES YES

Tracking neglecting inhomogeneous magnetic field and the medium effects

Global fit HELIX

Tracking in inhomogeneous magnetic field neglecting the medium effects

Global fit SPLINES

H. Wind, NIM 115(1974)431

Tracking in inhomogeneous magnetic field with energy loss and multiple scattering

Progressive fit KALMAN ...

R. Frühwirth, NIM A262(1987)444

The Helix (no matter, uniform mag. Field)

- Motion of charged particle determined by Lorentz force 

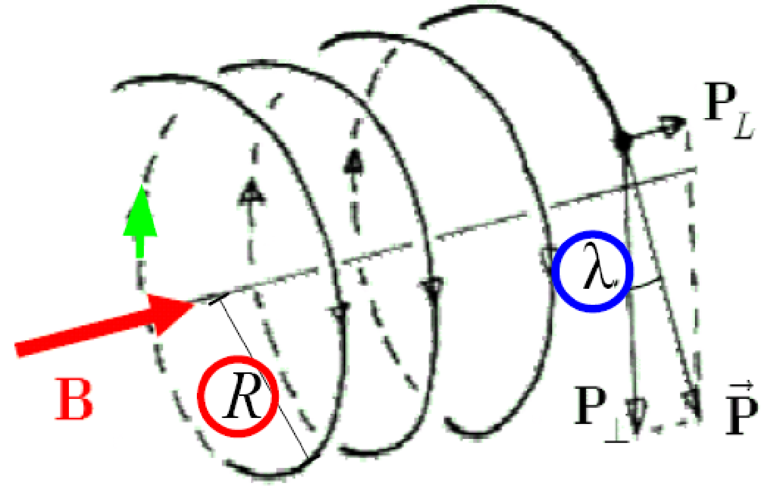
$$m_0\gamma\frac{d^2\vec{r}}{dt^2}=e\frac{d\vec{r}}{dt}\times\vec{B} \qquad \frac{d^2\vec{r}}{ds^2}=\frac{e}{p}\frac{d\vec{r}}{ds}\times\vec{B} \qquad (ds=v\,dt)$$

- Solution in homogeneous field: Helix

$$x(s)=x_0+R\left[\cos\left(\Phi_0+\frac{h\cos\lambda}{R}s\right)-\cos\Phi_0\right]$$

$$y(s)=y_0+R\left[\sin\left(\Phi_0+\frac{h\cos\lambda}{R}s\right)-\sin\Phi_0\right]$$

$$z(s)=z_0+s\sin\lambda$$



$$P_T=P\cos\lambda=0.3\,B\,R \quad ([P]=\mathrm{GeV},[B]=\mathrm{T},[R]=\mathrm{m})$$ (transverse momentum)

$h=\pm1$ sense of rotation of helix ⇒ sign of charge

B // z → two planes:
xy : circle
z - s : straight line

s = track length
$P(x_0, y_0, z_0)$ = starting point
λ = dip angle
Φ_0 = azimuthal angle
R_H = radius
h = -*sign*(qB_z)

Spline fit

H. Wind, NIM 115 (1974), 431

No medium effects, dishomogeneous magnetic field is taken intoaccount

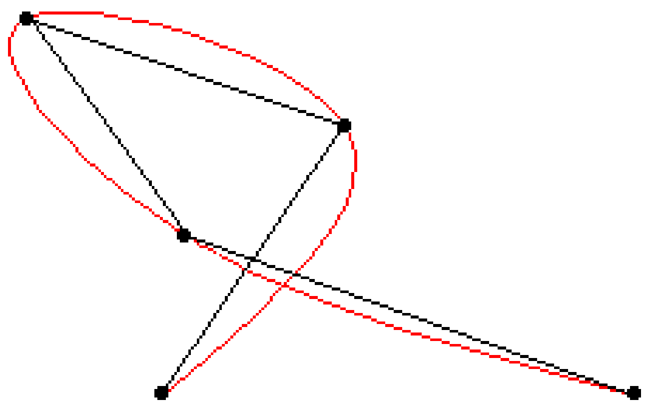
- The spline is a smooth segmented polynomial

- Cubic spline through *n+1* points $y_0, \dots, y_n$:

$$Y_i(t) = a_i + b_i t + c_i t^2 + d_i t^3$$

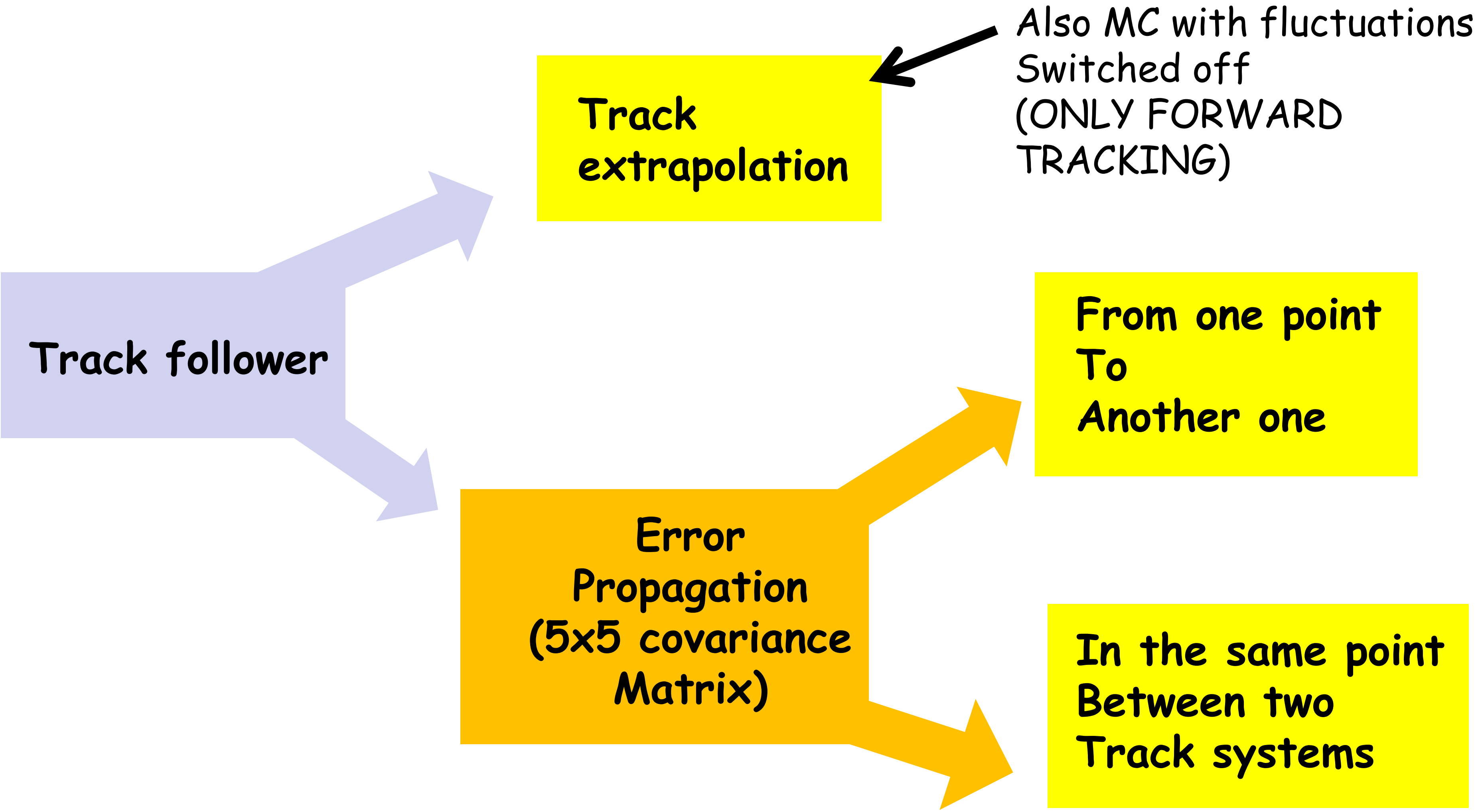
parameters

$i = 0,\dots,n$
$t \in [0,1]$



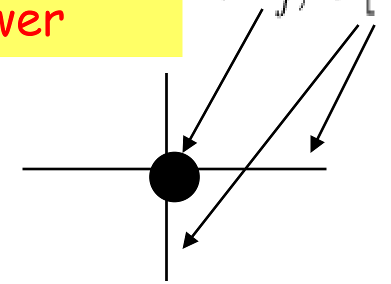
- The parameters are found by constraining the pieces of splines to be connected in the measured points assuring the continuity up to the 2 derivative

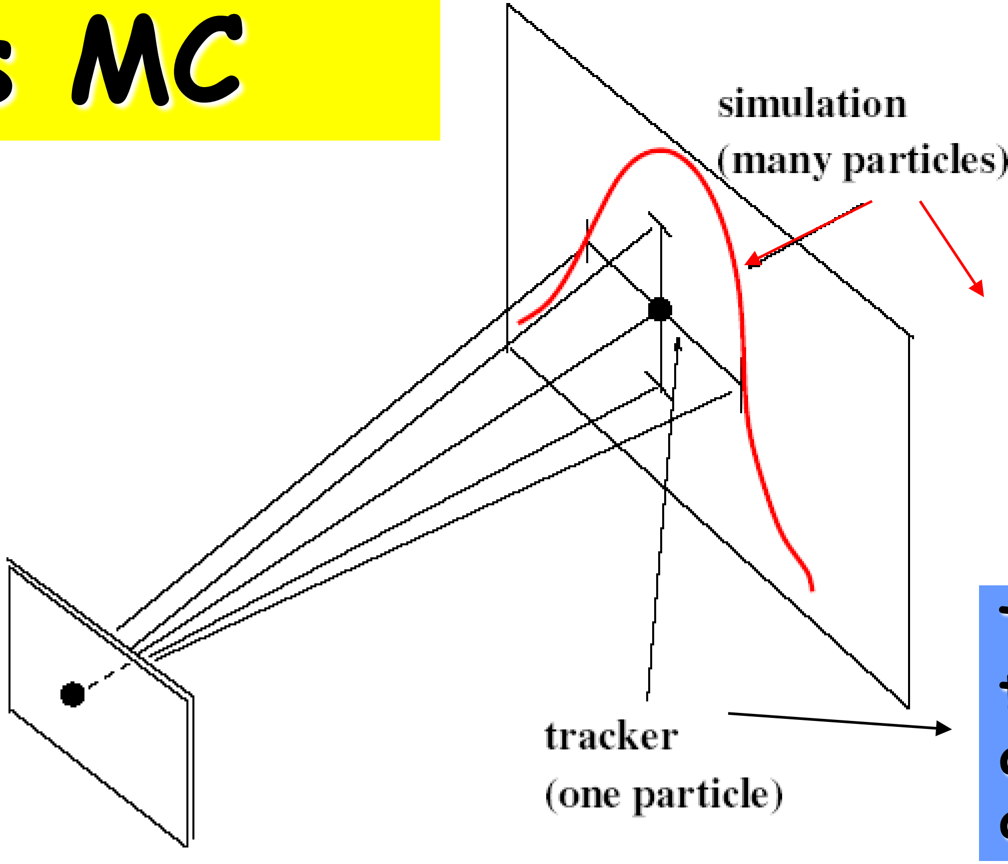
What is the track follower?



Tracking vs MC

$e_i, \boldsymbol{\sigma}[e_i]\,\Delta x, \text{medium} \rightarrow$ **track follower** $\rightarrow e_j, \boldsymbol{\sigma}[e_j]$





MC= at each step the trajectory is sampled as a random value

Tracking= at each step the trajectory is calculated as a mean value with an associate error

Energy loss affects both tracking (averages) and error propagation (covariance matrix), multiple scattering affects the error propagation only.

GEANE

V. Innocente et al. *Average Tracking and Error Propagation Package*, CERN Program Library W5013-E (1991).

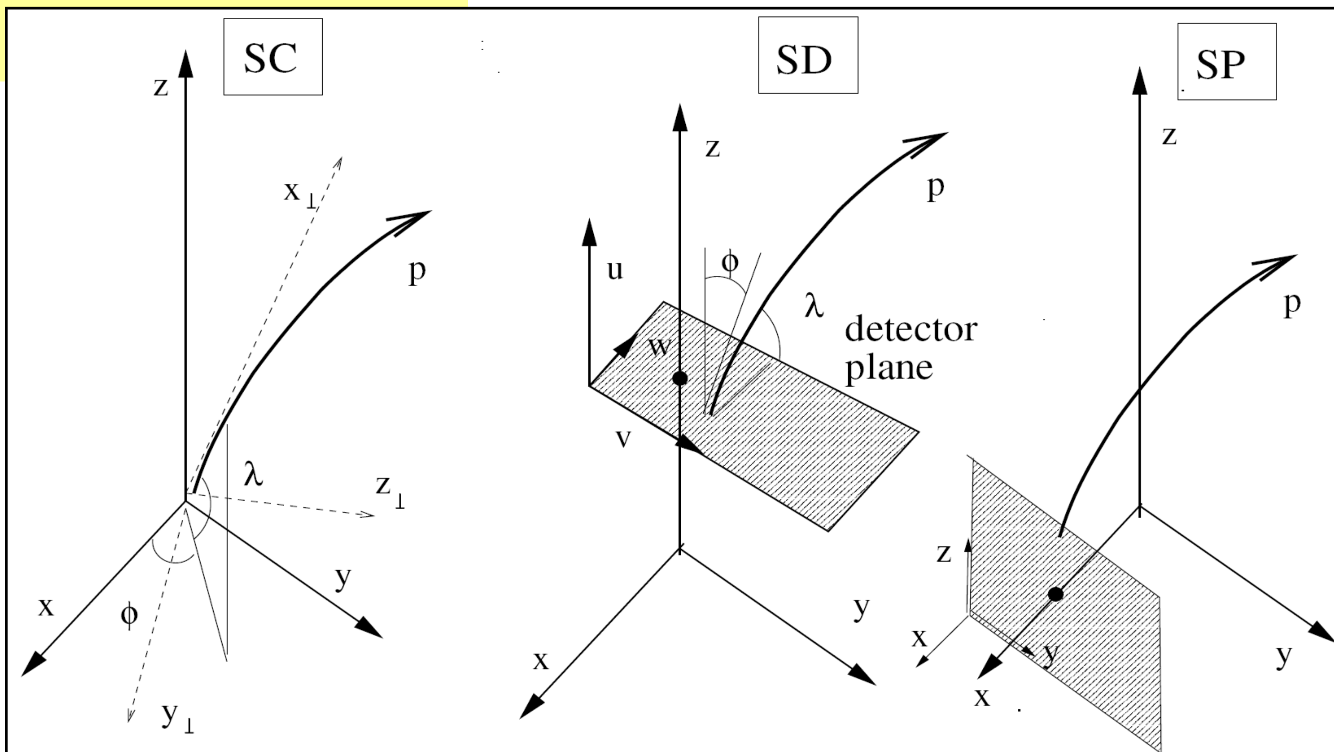
Two main tasks:

- **Track propagation**: the same MC geometry banks are used.
- **Error propagation**:
- from one point to another one
- In the same point between different systems

MARS → SC

$$\begin{pmatrix} x_\perp \\ y_\perp \\ z_\perp \end{pmatrix} = \begin{pmatrix} \cos\lambda\cos\varphi & \cos\lambda\sin\varphi & \sin\lambda \\ -\sin\varphi & \cos\varphi & 0 \\ -\sin\lambda\cos\varphi & -\sin\lambda\sin\varphi & \cos\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

GEANE = tracking with the geometry of GEANT3 + a lot of mathematics for the transport matrix calculation



Track propagation

Tracking:

$$e_j[k_i] = \boldsymbol{G}[k_i] \ ,$$

$\boldsymbol{G}$ is the software part that calculates the trajectory taking into account magnetic field and energy loss.

Error propagation:

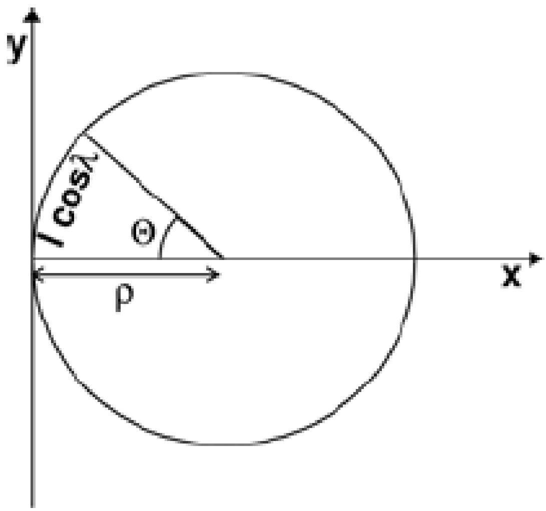
If $\boldsymbol{\sigma}[k_i]$ is the covariance matrix on the prediction k_i, the error on the extrapolated point e_i is given by the standard error propagation:

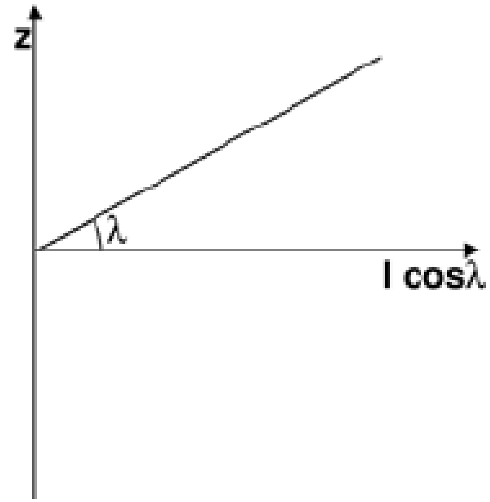
$$\boldsymbol{\sigma}[e_j] = \boldsymbol{T}_{ij}\boldsymbol{\sigma}[k_i]\boldsymbol{T}_{ij}^T + \boldsymbol{W}_{ij}^{-1} \qquad T_{ij}(l_2, l_1) = \frac{\partial e^i(l_2)}{\partial e^j(l_1)} \ ,$$

$\boldsymbol{T}_{ij}$ is the transport (derivative or gradient) matrix

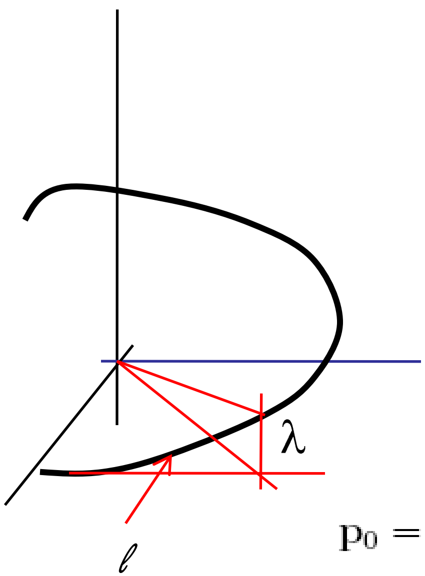
$\boldsymbol{W}_{ij}$ contains the errors (fluctuations) due to multiple scattering and energy loss. The calculation of this matrix is terribly complicated, so that usually people search for already existing and reliable products.

Track propagation II





a piece of helix
field along z-axis
M(s) is the position vector



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix} - \begin{pmatrix} \rho\cos\Theta \\ \rho\sin\Theta \end{pmatrix} \text{ and } \begin{cases} z &= l\cdot\cos\lambda\cdot\tan\lambda \\ \Theta\cdot\rho &= l\cdot\cos\lambda \end{cases}$$

it results:

$$\Theta\cdot\rho = l\cdot\cos\lambda$$

$$\mathrm{M}(\Theta) = \rho \begin{pmatrix} 1-\cos\Theta \\ \sin\Theta \\ \Theta\tan\lambda \end{pmatrix}$$

$$\mathrm{p}_0 = \mathrm{p}\mathrm{T}_0 = \mathrm{p}\begin{pmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{pmatrix}$$

$$\frac{\partial \mathrm{M}(l)}{\partial l} = \mathrm{T}(l) = \cos\lambda \begin{pmatrix} \sin\Theta \\ \cos\Theta \\ \tan\lambda \end{pmatrix}$$

Track propagation III

Now the tracking can be performed, with the unique assumption for the field to be constant within one step, so that for an arbitrary magnetic field the track can be written as a series of helix pieces (one for each step). To perform the tracking let' s define an orthogonal right-handed triplet of axes (n_i, b_i, h_i):

$$\mathbf{h}_i = \frac{\mathbf{H}_i}{|\mathbf{H}_i|}$$

$$\mathbf{n}_i = \frac{\mathbf{T}_i \times \mathbf{h}_i}{|\mathbf{T}_i \times \mathbf{h}_i|}$$

$$\mathbf{b}_i = \mathbf{h}_i \times \mathbf{n}_i$$

$$\mathbf{M}_{i+1} = \mathbf{M}_i + \rho[(1-\cos\Theta_i)\cdot\mathbf{n}_i + \sin\Theta_i\cdot\mathbf{b}_i + \Theta_i\tan\lambda_i\cdot\mathbf{h}_i]$$

$$\mathbf{T}_{i+1} = \cos\lambda_i[\sin\Theta_i\cdot\mathbf{n}_i + \cos\Theta_i\cdot\mathbf{b}_i + \tan\lambda_i\cdot\mathbf{h}_i]$$

$$\mathbf{T} = \frac{\mathbf{p}}{p}$$

$$\mathbf{N} = \frac{\mathbf{H}\times\mathbf{T}}{|\mathbf{H}\times\mathbf{T}|}$$

$$\mathbf{R} = \mathbf{T}\times\mathbf{N}$$

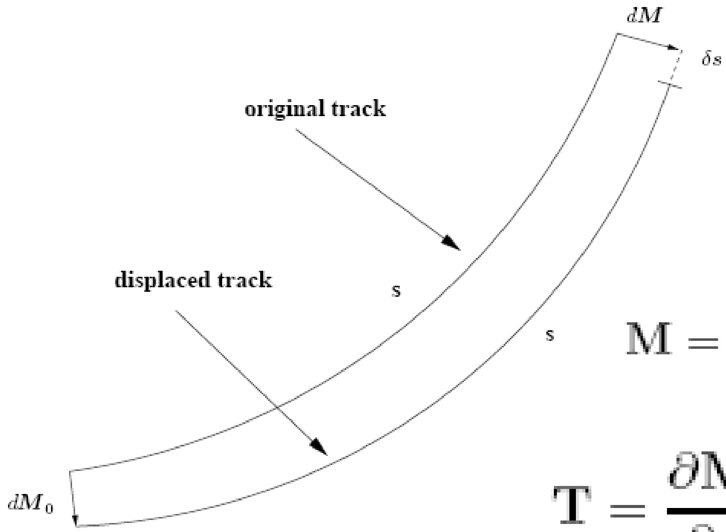
The matrix to change from (n_i, b_i, h_i) to (N_i, R_i, T_i) is

$$\begin{pmatrix} N \\ R \\ T \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sin\lambda & \cos\lambda \\ 0 & \cos\lambda & \sin\lambda \end{pmatrix} \begin{pmatrix} n \\ b \\ h \end{pmatrix}$$

The helix can then be parametrized as follows ([7] and [9]):

$$\mathbf{M} = \mathbf{M}_0 + \frac{\gamma}{Q}(\theta - \sin\theta)\cdot\mathbf{H} + \frac{\sin\theta}{Q}\cdot\mathbf{T}_0 + \frac{\alpha}{Q}(1-\cos\theta)\cdot\mathbf{N}_0$$

Track propagation IV



A. Stradlie and W. Wittek, CMS note 2006/001 and Nucl. Instr.and Meth A566(2006)687

$$\mathbf{M} = \mathbf{M}_0 + \frac{\gamma}{Q}(\theta - \sin\theta)\cdot\mathbf{H} + \frac{\sin\theta}{Q}\cdot\mathbf{T}_0 + \frac{\alpha}{Q}(1-\cos\theta)\cdot\mathbf{N}_0,$$

$$\mathbf{T} = \frac{\partial\mathbf{M}}{\partial s} = \gamma(1-\cos\theta)\cdot\mathbf{H} + \cos\theta\cdot\mathbf{T}_0 + \alpha\sin\theta\cdot\mathbf{N}_0.$$

with M being the position vector of the point on the helix at path length s from the reference point M_0 (at $s=0$), $\mathrm{H}=\mathrm{B}/|\mathrm{B}|$ being a normalized magnetic field vector, $\mathrm{T}=\mathrm{p}/|\mathrm{p}|$ being a normalized tangent vector to the track, $\mathrm{N}=(\mathrm{H}\times\mathrm{T})/\alpha$ with $\alpha=|\mathrm{H}\times\mathrm{T}|$, $\gamma=\mathrm{H}\cdot\mathrm{T}$, $Q=-|\mathrm{B}|\,q/p$ with $p=|\mathrm{p}|$ being the absolute value of the 3-momentum vector, $q=\pm1$ denoting the charge of the particle, and $\theta=Q\cdot s$. The numerical value of $|\mathrm{B}|$ is

$$\begin{aligned} d\mathbf{M} &= \frac{\partial\mathbf{M}}{\partial\mathbf{M}_0}\cdot d\mathbf{M}_0 + \frac{\partial\mathbf{M}}{\partial\mathbf{T}_0}\cdot d\mathbf{T}_0 + \frac{\partial\mathbf{M}}{\partial(q/p_0)}\cdot\delta(q/p_0) + \frac{\partial\mathbf{M}}{\partial s}\cdot\delta s, \\ d\mathbf{T} &= \frac{\partial\mathbf{T}}{\partial\mathbf{T}_0}\cdot d\mathbf{T}_0 + \frac{\partial\mathbf{T}}{\partial(q/p_0)}\cdot\delta(q/p_0) + \frac{\partial\mathbf{T}}{\partial s}\cdot\delta s, \end{aligned}$$

Track propagation V

The Jacobian of the transformation from a curvilinear frame $(q/p, \lambda, \phi, x_\perp, y_\perp)$ at $s_0 = 0$ to the same set of parameters at path length s is then derived by forming the differentials $d\mathrm{M}$ and $d\mathbf{T}$, introducing the specific constraints given by the curvilinear frames,

$$d\mathrm{M}_0 = \mathbf{U}_0 \cdot \delta x_{\perp 0} + \mathbf{V}_0 \cdot \delta y_{\perp 0}, \tag{17}$$

$$d\mathbf{T}_0 = \frac{\partial \mathbf{T}_0}{\partial \lambda_0} \cdot \delta\lambda_0 + \frac{\partial \mathbf{T}_0}{\partial \phi_0} \cdot \delta\phi_0 = \mathbf{V}_0 \cdot \delta\lambda_0 + \cos\lambda_0 \cdot \mathbf{U}_0 \cdot \delta\phi_0, \tag{18}$$

$$d\mathrm{M} = \mathbf{U} \cdot \delta x_\perp + \mathbf{V} \cdot \delta y_\perp, \tag{19}$$

$$d\mathbf{T} = \mathbf{V} \cdot \delta\lambda + \cos\lambda \cdot \mathbf{U} \cdot \delta\phi. \tag{20}$$

Also, since $d\mathrm{M}$ now is defined to be a variation in a plane perpendicular to the track, the functional dependence of δs on the variations of position, direction and momentum at the starting point can be evaluated by multiplying Eq. (3) with $\mathbf{T}$ and using the constraint $d\mathrm{M} \cdot \mathbf{T} = 0$. One obtains

$$\delta s = -\mathbf{T} \cdot d\mathrm{M}_0 - \mathbf{T} \cdot \left(\frac{\partial \mathrm{M}}{\partial \mathbf{T}_0} \cdot d\mathbf{T}_0 \right) - \left(\mathbf{T} \cdot \frac{\partial \mathrm{M}}{\partial (q/p_0)} \right) \cdot \delta(q/p_0). \tag{21}$$

The first task: track propagation

$$\frac{\partial(q/p)}{\partial(q/p_0)} = 1,$$

$$\frac{\partial\lambda}{\partial(q/p_0)} = -\alpha Q\cdot\left(\frac{q}{p}\right)^{-1}\cdot(\mathbf{N}\cdot\mathbf{V})\cdot[\mathbf{T}\cdot(\mathbf{M}_0-\mathbf{M})],$$

$$\frac{\partial\lambda}{\partial\lambda_0} = \cos\theta\cdot(\mathbf{V}_0\cdot\mathbf{V})+\sin\theta\cdot((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{V}) + (1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{V}) + \alpha(\mathbf{N}\cdot\mathbf{V})[-\sin\theta(\mathbf{V}_0\cdot\mathbf{T})+\alpha(1-\cos\theta)(\mathbf{V}_0\cdot\mathbf{N}) - (\theta-\sin\theta)(\mathbf{H}\cdot\mathbf{T})(\mathbf{H}\cdot\mathbf{V}_0)],$$

$$\frac{\partial\lambda}{\partial\phi_0} = \cos\lambda_0\{\cos\theta\cdot(\mathbf{U}_0\cdot\mathbf{V})+\sin\theta\cdot((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{V}) + (1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{V}) + \alpha(\mathbf{N}\cdot\mathbf{V})[-\sin\theta(\mathbf{U}_0\cdot\mathbf{T})+\alpha(1-\cos\theta)(\mathbf{U}_0\cdot\mathbf{N}) - (\theta-\sin\theta)(\mathbf{H}\cdot\mathbf{T})(\mathbf{H}\cdot\mathbf{U}_0)]\},$$

$$\frac{\partial\lambda}{\partial x_{\perp 0}} = -\alpha Q(\mathbf{N}\cdot\mathbf{V})(\mathbf{U}_0\cdot\mathbf{T}),$$

$$\frac{\partial\lambda}{\partial y_{\perp 0}} = -\alpha Q(\mathbf{N}\cdot\mathbf{V})(\mathbf{V}_0\cdot\mathbf{T}),$$

$$\frac{\partial\phi}{\partial(q/p_0)} = -\frac{\alpha Q}{\cos\lambda}\cdot\left(\frac{q}{p}\right)^{-1}\cdot(\mathbf{N}\cdot\mathbf{U})\cdot[\mathbf{T}\cdot(\mathbf{M}_0-\mathbf{M})],$$

$$\frac{\partial\phi}{\partial\lambda_0} = \frac{1}{\cos\lambda}\{\cos\theta\cdot(\mathbf{V}_0\cdot\mathbf{U})+\sin\theta\cdot((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{U}) + (1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{U}) + \alpha(\mathbf{N}\cdot\mathbf{U})[-\sin\theta(\mathbf{V}_0\cdot\mathbf{T})+\alpha(1-\cos\theta)(\mathbf{V}_0\cdot\mathbf{N}) - (\theta-\sin\theta)(\mathbf{H}\cdot\mathbf{T})(\mathbf{H}\cdot\mathbf{V}_0)]\},$$

$$\frac{\partial\phi}{\partial\phi_0} = \frac{\cos\lambda_0}{\cos\lambda}\{\cos\theta\cdot(\mathbf{U}_0\cdot\mathbf{U})+\sin\theta\cdot((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{U}) + (1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{U}) + \alpha(\mathbf{N}\cdot\mathbf{U})[-\sin\theta(\mathbf{U}_0\cdot\mathbf{T})+\alpha(1-\cos\theta)(\mathbf{U}_0\cdot\mathbf{N}) - (\theta-\sin\theta)(\mathbf{H}\cdot\mathbf{T})(\mathbf{H}\cdot\mathbf{U}_0)]\},$$

$$\frac{\partial\phi}{\partial x_{\perp 0}} = -\frac{\alpha Q}{\cos\lambda}(\mathbf{N}\cdot\mathbf{U})(\mathbf{U}_0\cdot\mathbf{T}),$$

$$\frac{\partial\phi}{\partial y_{\perp 0}} = -\frac{\alpha Q}{\cos\lambda}(\mathbf{N}\cdot\mathbf{U})(\mathbf{V}_0\cdot\mathbf{T}),$$

$$\frac{\partial x_\perp}{\partial(q/p_0)} = \left(\frac{q}{p}\right)^{-1}[\mathbf{U}\cdot(\mathbf{M}_0-\mathbf{M})],$$

$$\frac{\partial x_\perp}{\partial\lambda_0} = \frac{\sin\theta}{Q}(\mathbf{V}_0\cdot\mathbf{U})+\frac{1-\cos\theta}{Q}((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{U}) + \frac{\theta-\sin\theta}{Q}(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{U}),$$

$$\frac{\partial x_\perp}{\partial\phi_0} = \cos\lambda_0\left\{\frac{\sin\theta}{Q}(\mathbf{U}_0\cdot\mathbf{U})+\frac{1-\cos\theta}{Q}((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{U}) + \frac{\theta-\sin\theta}{Q}(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{U})\right\}, \quad (63)$$

$$\frac{\partial x_\perp}{\partial x_{\perp 0}} = \mathbf{U}_0\cdot\mathbf{U}, \quad (64)$$

$$\frac{\partial x_\perp}{\partial y_{\perp 0}} = \mathbf{V}_0\cdot\mathbf{U}, \quad (65)$$

$$\frac{\partial y_\perp}{\partial(q/p_0)} = \left(\frac{q}{p}\right)^{-1}[\mathbf{V}\cdot(\mathbf{M}_0-\mathbf{M})], \quad (66)$$

$$\frac{\partial y_\perp}{\partial\lambda_0} = \frac{\sin\theta}{Q}(\mathbf{V}_0\cdot\mathbf{V})+\frac{1-\cos\theta}{Q}((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{V}) + \frac{\theta-\sin\theta}{Q}(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{V}), \quad (67)$$

$$\frac{\partial y_\perp}{\partial\phi_0} = \cos\lambda_0\left\{\frac{\sin\theta}{Q}(\mathbf{U}_0\cdot\mathbf{V})+\frac{1-\cos\theta}{Q}((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{V}) + \frac{\theta-\sin\theta}{Q}(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{V})\right\}, \quad (68)$$

$$\frac{\partial y_\perp}{\partial x_{\perp 0}} = \mathbf{U}_0\cdot\mathbf{V}, \quad (69)$$

$$\frac{\partial y_\perp}{\partial y_{\perp 0}} = \mathbf{V}_0\cdot\mathbf{V}, \quad (70)$$

ns numerically stable at small values of θ are given by

$$\frac{\partial x_\perp}{\partial(q/p_0)} = -\frac{1}{2}|\mathbf{B}|\,s^2\cdot(\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{U}+\frac{1}{3}|\mathbf{B}|^2 s^3\cdot\frac{q}{p}\cdot(\gamma\mathbf{H}-\mathbf{T}_0)\cdot\mathbf{U} + \frac{1}{8}|\mathbf{B}|^3 s^4\cdot\left(\frac{q}{p}\right)^2\cdot(\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{U}, \quad (71)$$

$$\frac{\partial y_\perp}{\partial(q/p_0)} = -\frac{1}{2}|\mathbf{B}|\,s^2\cdot(\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{V}+\frac{1}{3}|\mathbf{B}|^2 s^3\cdot\frac{q}{p}\cdot(\gamma\mathbf{H}-\mathbf{T}_0)\cdot\mathbf{V} + \frac{1}{8}|\mathbf{B}|^3 s^4\cdot\left(\frac{q}{p}\right)^2\cdot(\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{V}. \quad (72)$$

$$\boldsymbol{\sigma}^2(l_2) = \boldsymbol{T}(l_2, l_1)\,\boldsymbol{\sigma}^2(l_1)\,\boldsymbol{T}^T(l_2, l_1) + \boldsymbol{W}^{-1}(l_1)$$

The jacobian transports the erros from one step to another

Here, at each step, multiple scattering and energy loss effects have to be added

On the quantitative modelling of core and tails of multiple scattering by Gaussian mixtures

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Multiple scattering

$$\frac{d\sigma}{d\theta} = 2\pi\left(\frac{2Ze^2}{pv}\right)^2 \frac{\theta}{(\theta^2 + \theta_{\min}^2)^2}. \quad \rightarrow \quad f(\theta) = \frac{k\theta}{(\theta^2 + \theta_{\min}^2)^2} I_{[0,\theta_{\max}]}(\theta)$$

There is no simple closed form for the cumulative distribution function of the projected scattering angle. For the simulation one therefore has to go back to the scattering angle θ in space, which can be generated by inverting its cumulative distribution function:

$$\theta = ab\sqrt{\frac{1-u}{ub^2 + a^2}} \tag{26}$$

where u is a stochastic variable with a uniform distribution in the interval [0,1]. If φ is uniform in

Multiple scattering

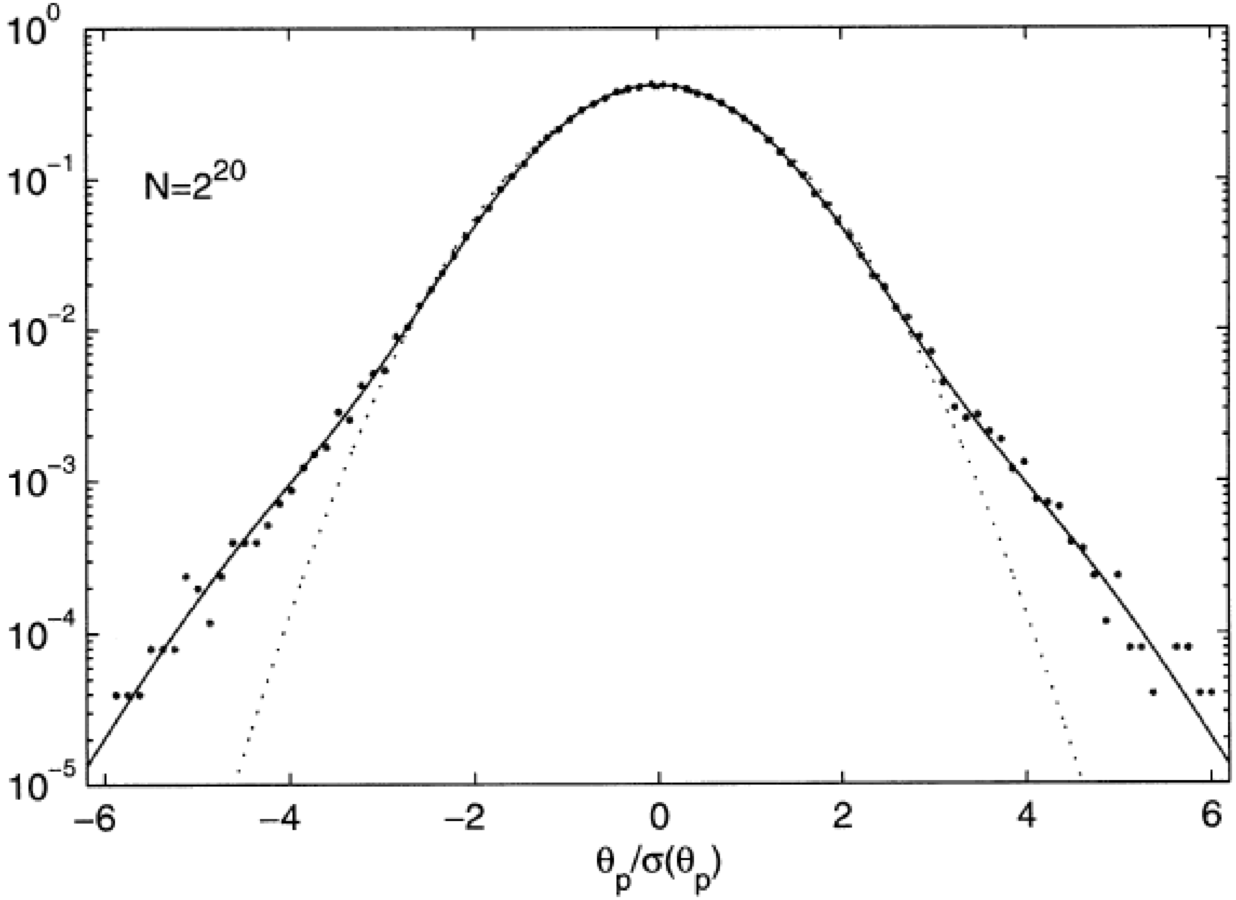


Fig. 3. The density of the projected multiple scattering angle in carbon, in standard measure, for $N = 2^{10}$ (top) and $N = 2^{20}$ (bottom). The dots are the frequencies of a simulated sample obtained by summing over single scatters. The dotted line is the density of a standard Gaussian.

Multiple scattering

Molière's final solution $f_M(\theta)\theta\, d\theta$ of the transport equation is given in space, using the transformation

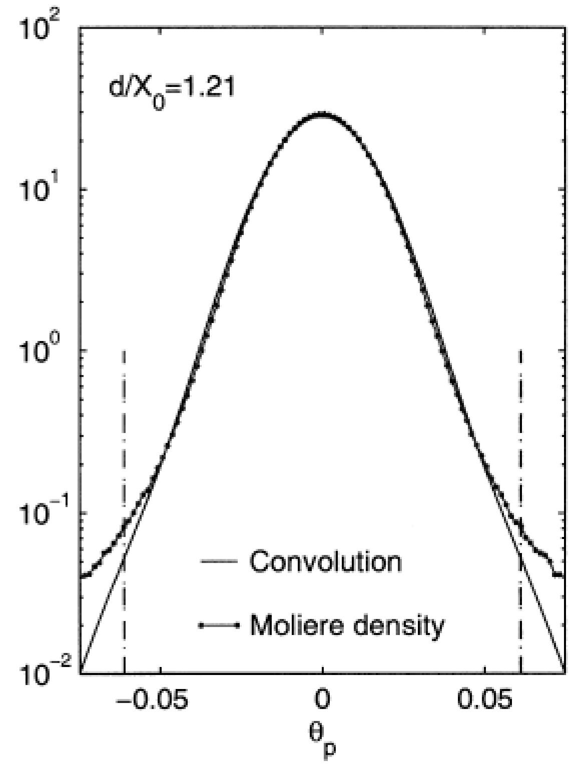
$$f(\theta)d\theta = f_M(\theta)\, d(\cos\theta)\, d\varphi/2\pi \qquad (49)$$

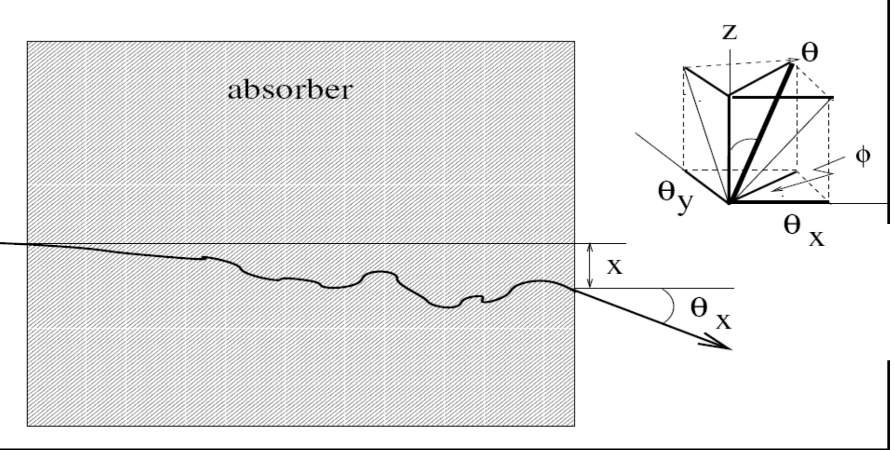
and the approximation $|d(\cos\theta)| = \sin\theta\, d\theta \approx \theta\, d\theta$. In his solution the function $f_M(\theta)$ is approximated by

$$f_M(\theta) \approx \frac{1}{2\,\theta_M^2}\left[f^{(0)}(\theta') + \frac{f^{(1)}(\theta')}{B} + \frac{f^{(2)}(\theta')}{B^2}\right] \qquad (50)$$

where θ_M is the characteristic multiple scattering angle of the target, $\theta' = \theta/(\sqrt{2}\theta_M)$ is the reduced angle, and B is related to the logarithm of the effective number of collisions in the target. The functions $f^{(k)}$ are given by

$$f^{(k)}(\theta') = \frac{1}{n!}\int_0^\infty y\, J_0(\theta' y)\, e^{-y^2/4}\left(\frac{y^2}{4}\ln\frac{y^2}{4}\right)^k dy \qquad (51)$$





Multiple scattering

$$<\theta_p^2>\,, \quad <x^2>=\frac{<\theta_p^2>\,d^2}{3}\,, \quad <x,\theta_p>=\frac{<\theta_p^2>\,d}{2}$$

$$p(x,\theta_p;d)=\frac{2\sqrt{3}}{\pi}\,\frac{1}{<\theta_p^2>\,d^2}\,\exp\left[-\frac{4}{<\theta_p^2>}\left(\frac{\theta_p^2}{d}-\frac{3x\theta_p}{d^2}+\frac{3x^2}{d^3}\right)\right]$$

$$<\theta_p^2>=\frac{(0.0136)^2\,d}{p^2\,\beta^2 X_0}\left[1+0.038\,\ln\left(\frac{d}{X_0}\right)\right]^2$$

PDG: wrong

$$<\theta_p^2>=\frac{184.96\cdot 10^{-6}}{p^2}\,\frac{d}{\beta^2 X_0}\,.$$

GEANE

$$<\theta_p^2>=\frac{225\cdot 10^{-6}}{p^2}\,\frac{d}{\beta^2 X_s}\,, \qquad X_s=X_0\,\frac{Z+1}{Z}\,\frac{\ln(287\,Z^{-1/2})}{\ln(159\,Z^{-1/3})}$$

R. Frühwirth and M. Regler, Nucl. Instr. and Meth. A456(2001)369

$$\lambda \equiv -\theta_z \ , \quad \phi \equiv \frac{\theta_y}{\cos\lambda} \ , \quad y_\perp \equiv y \ , \quad z_\perp \equiv z$$

To transport the errors from the multiple scattering reference to SC, we use the standard error propagation [13]:

$$< t_i, t_j >= \sum_{lm} \frac{\partial t_i}{\partial s_l} \frac{\partial t_j}{\partial s_m} < s_l, s_m > \quad . \tag{52}$$

Taking into account (48, 49-51) and writing only non-zero terms, we easily obtain the elements of the multiple scattering covariance matrix in the SC system [1]:

$$< \lambda^2 > \;=\; < \theta_z^2 >=< \theta_p^2 > , \tag{53}$$

$$< \lambda, z > \;=\; - < \theta_p, z >= -\frac{< \theta_p^2 > \ d}{2} , \tag{54}$$

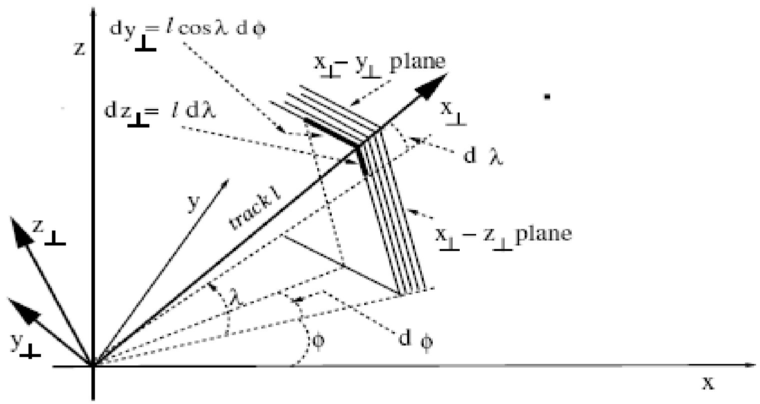
$$< \phi^2 > \;=\; \frac{< \theta_p^2 >}{\cos^2\lambda} , \tag{55}$$

$$< y, \phi > \;=\; \frac{\partial y}{\partial y} \frac{\partial \phi}{\partial \theta_y} < y, \theta_y >= \frac{1}{\cos\lambda} \frac{< \theta_p^2 > \ d}{2} \tag{56}$$

$$< y^2 > \;=\; \frac{< \theta_p^2 > \ d^2}{3} \tag{57}$$

$$< z^2 > \;=\; \frac{< \theta_p^2 > \ d^2}{3} \tag{58}$$

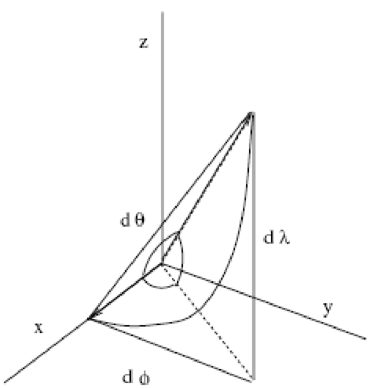
$$\boldsymbol{\sigma}^2(l_2) = \boldsymbol{T}(l_2, l_1)\,\boldsymbol{\sigma}^2(l_1)\,\boldsymbol{T}^T(l_2, l_1) + \boldsymbol{W}^{-1}(l_1)$$



$$dz_\perp = l\,d\lambda\ , \qquad dy_\perp = l\cos\lambda d\phi$$

$$\lambda \equiv -\theta_z\ , \quad \phi \equiv \frac{\theta_y}{\cos\lambda}\ , \quad y_\perp \equiv y\ , \quad z_\perp \equiv z$$

$$< t_i, t_j > = \sum_{lm} \frac{\partial t_i}{\partial s_l}\frac{\partial t_j}{\partial s_m} < s_l, s_m >$$



	1/p	λ	ϕ	$y_\perp$	$z_\perp$
1/p	0	0	0	0	0
λ	0	$<\theta_p^2>$	0	0	$-\frac{<\theta_p^2>\ \mathrm{d}l}{2}$
ϕ	0	0	$\frac{<\theta_p^2>}{\cos^2\lambda}$	$\frac{<\theta_p^2>\ \mathrm{d}l}{(2\cos\lambda)}$	0
$y_\perp$	0	0	$\frac{<\theta_p^2>\ \mathrm{d}l}{(2\cos\lambda)}$	$\frac{<\theta_p^2>\ (\mathrm{d}l)^2}{3}$	
$z_\perp$	0	$-\frac{<\theta_p^2>\ \mathrm{d}l}{2}$	0	0	$\frac{<\theta_p^2>\ (\mathrm{d}l)^2}{3}$

Energy loss

The fluctuations in ionization for one particle of charge z, mass m, velocity β, are characterized by the parameter κ,

$$\kappa = \frac{\xi}{E_{\max}} , \tag{60}$$

which is proportional to the ratio of mean energy loss to the maximum allowed energy transfer $E_{\max}$ in a single collision with an atomic electron:

$$E_{\max} = \frac{2m_e\beta^2\gamma^2}{1 + 2\gamma m_e/m + (m_e/m)^2} , \tag{61}$$

where $\gamma = 1/\sqrt{1-\beta^2} = E/m$ and m_e is the electron mass. The parameter ξ comes from the Rutherford scattering cross section and is defined as [11]:

$$\xi = 153.4 \frac{z^2 Z}{\beta^2 A} \rho d \quad (\text{keV}) , \tag{62}$$

where ρ, d, Z and A are the density (g/cm^3), thickness, atomic and mass number of the medium.

Average energy loss

$$\left\langle \frac{dE}{dx} \right\rangle = -4\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \ln \frac{2 m_e c^2 \gamma^2 \beta^2}{I^2} T^{max} - \beta^2 - \frac{\delta}{2} \right]$$

BETHE-BLOCH

Fluctuations in energy loss

$$k = \frac{\xi}{E_{max}} = \frac{\textbf{average energy loss}}{\textbf{max energy loss in a single collision}}$$

$k > 10$ Gaussian

$0.01 < k < 10$ Vavilov

$$\sigma^2(E) = \xi E_{max}\left(1 - \frac{\beta^2}{2}\right) \Rightarrow \sigma^2\left(\frac{1}{p}\right) \Rightarrow \sigma_{11}^2$$

$k < 0.01;\ N_c > 50$ Landau — μ, σ are infinite !!

$k < 0.01;\ N_c < 50$ Sub-Landau — σ is too large!!

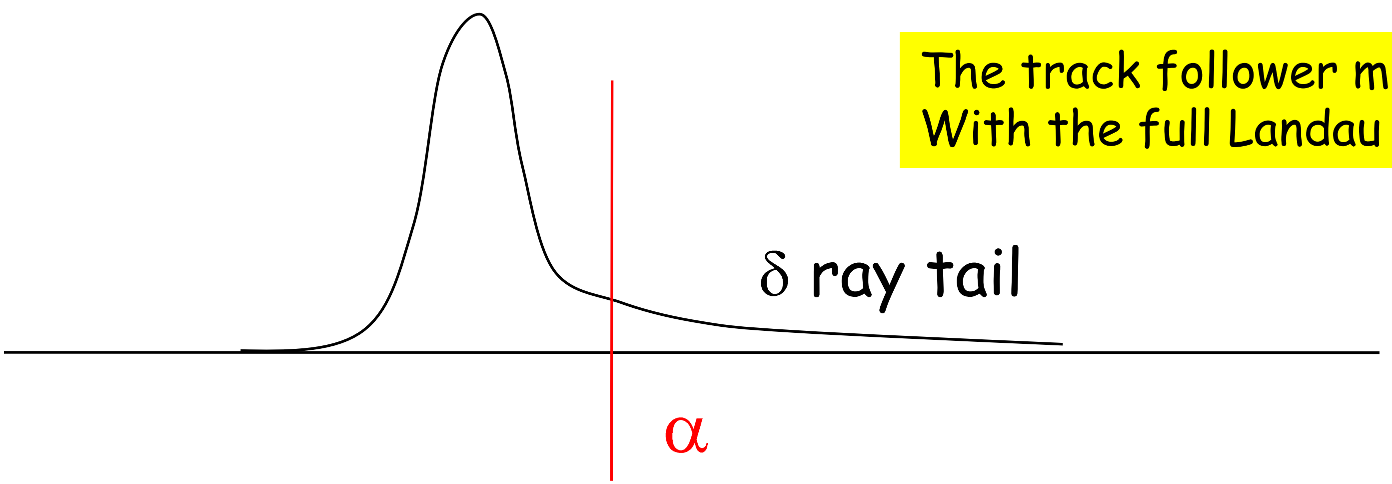
Gauss and Vavilov: no problems for the track follower

GEANE

$$\sigma^2(E) = \xi E_{max}\left(1 - \frac{\beta^2}{2}\right) \Rightarrow \sigma^2\left(\frac{1}{p}\right) \Rightarrow \sigma^2_{11}$$

Landau: problematic distribution

The track follower must be compare
With the full Landau sampling



Sub-Landau: what distribution?

Gauss and Vavilov: no problems for the track follower

$$\sigma^2(E) = \frac{\xi^2}{\kappa}(1 - \beta^2/2) = \xi\, E_{\max}(1 - \beta^2/2) \ . \qquad (63)$$

Taking into account the energy-momentum equation

$$E^2 = p^2 + m^2 \quad \rightarrow \quad \frac{\mathrm{dp}}{\mathrm{dE}} = \frac{E}{p} = \frac{1}{\beta} \ ,$$

and the error transformation

$$\sigma^2(1/p) = \left[\frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{1}{p}\right)\right]^2 \sigma^2(p) = \frac{1}{p^4}\,\sigma^2(p) = \frac{E^2}{p^6}\,\sigma^2(E)$$

GEANE and GEANT4E contain only this

Improvements

- New error calculation in energy loss for heavy particles
- New error calculation for bremsstrahlung

Truncated Landau:

λ_{max}	α	Mean	σ_α
11.1	0.90	1.61	2.83
22.4	0.95	2.40	4.23
110.0	0.99	4.19	10.16
200.0	0.995	4.82	13.88
256.0	0.996	5.08	15.76
339.0	0.997	5.37	18.19
507.0	0.998	5.78	22.33
1007.0	0.999	6.48	31.59

Table 1: Result of the integration $\alpha = \int_{\lambda_{min}}^{\lambda_{max}} f(\lambda)\,\mathrm{d}\lambda$ of the Landau distribution from $\lambda_{min} \simeq -3.5$ to λ_{max} of the table. The mean and the standard deviation of the truncated distribution are also shown. For this distribution, the full mean and the variance are infinite, only the cumulative can be calculated

Solution (GEANT3 & GEANT4): truncation of the distribution tail to have as a mean the average dE/dx

$$\lambda_{max} = 0.60715 + 1.1934\,\langle\lambda\rangle + (0.67794 + 0.052382\,\langle\lambda\rangle)\exp(0.94753 + 0.74442\,\langle\lambda\rangle)$$

Original GEANE

GEANE for PANDA modified with the the α-tail

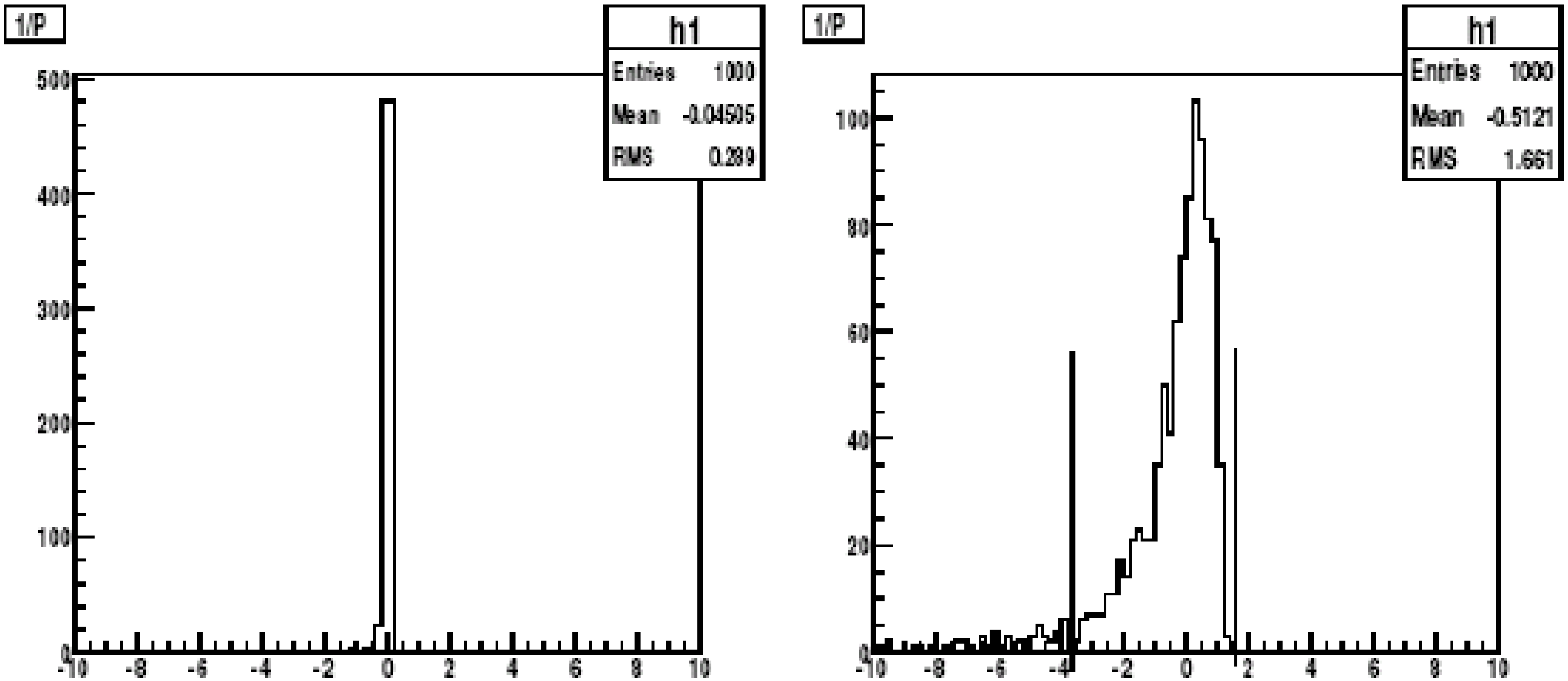


Figure 10: Pull distribution $\Delta(1/p)/\sigma$ for 1 GeV muons after passing through the PANDA straw tube detector. Left: Standard GEANE result (RMS$\simeq$ 0.3 in the displayed window); right: result after the modification with $\alpha = 0.995$ (see the text). The region between the vertical lines has RMS= 1.03.

Urban model works well

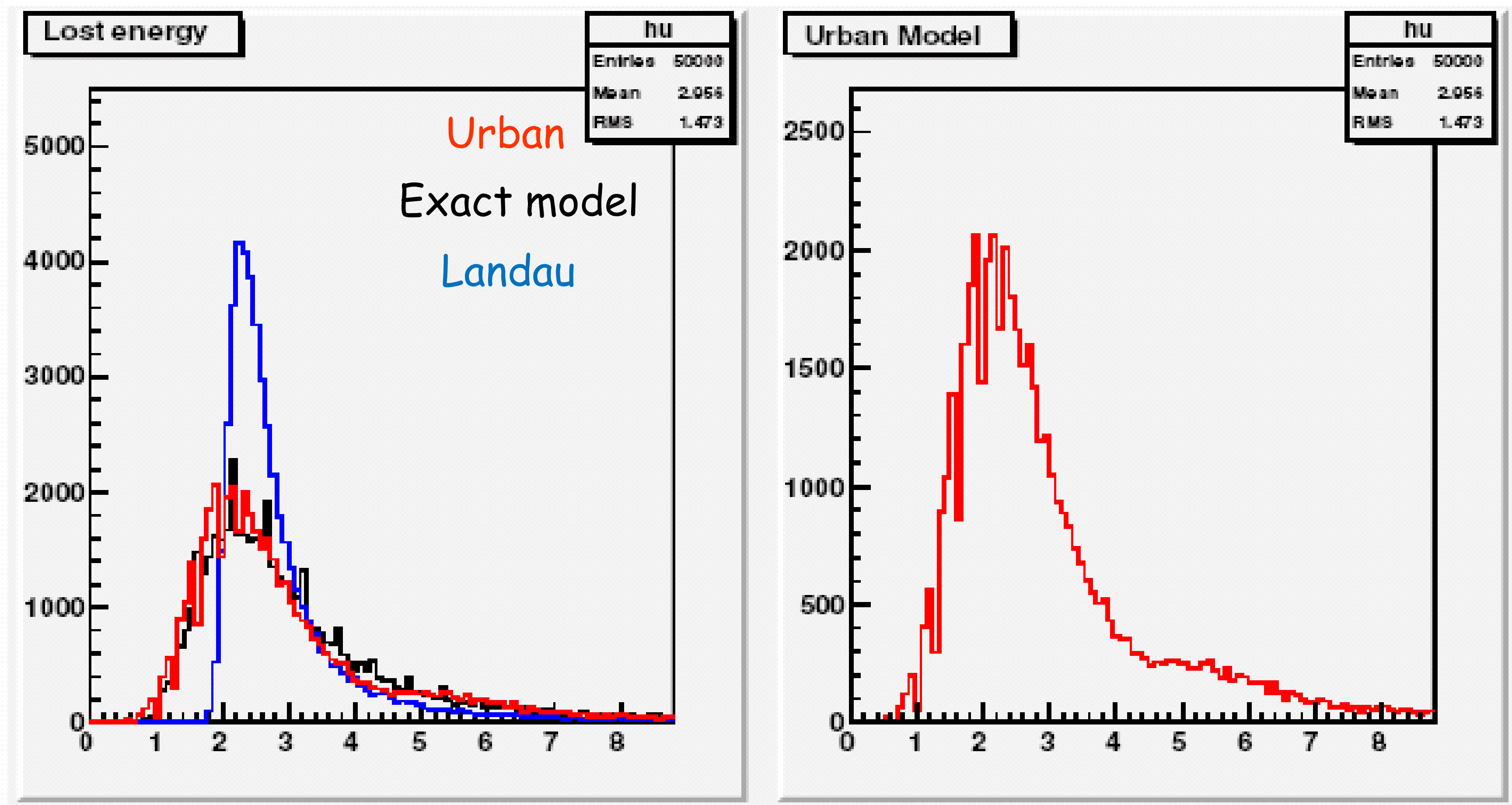


Figure 2: Urban and simulated distribution

1.5 cm of Ar/CO2 90/10 1.2 GeV pions

In summary, our method calculates the $1/p$ variance of eq. (5) with a variance $\sigma^2(E)$ due to the ionization energy loss calculated as follows:

a) for big and moderate absorbers when $\kappa > 0.01$, the variance $\sigma^2(E)$ is given by eq. (4) (old GEANE method);

b) for thin absorbers, $\kappa < 0.01$, when the number of collisions from eq. (10) is $N_c > 50$, $\sigma^2(E)$ is given by eq. (9);

c) for very thin absorbers, when $\kappa < 0.01$ and $N_c < 50$, the variance $\sigma^2(E)$ is given by eq. (17).

The matching between Urban and Landau is obtained for $\delta = 0.9999$

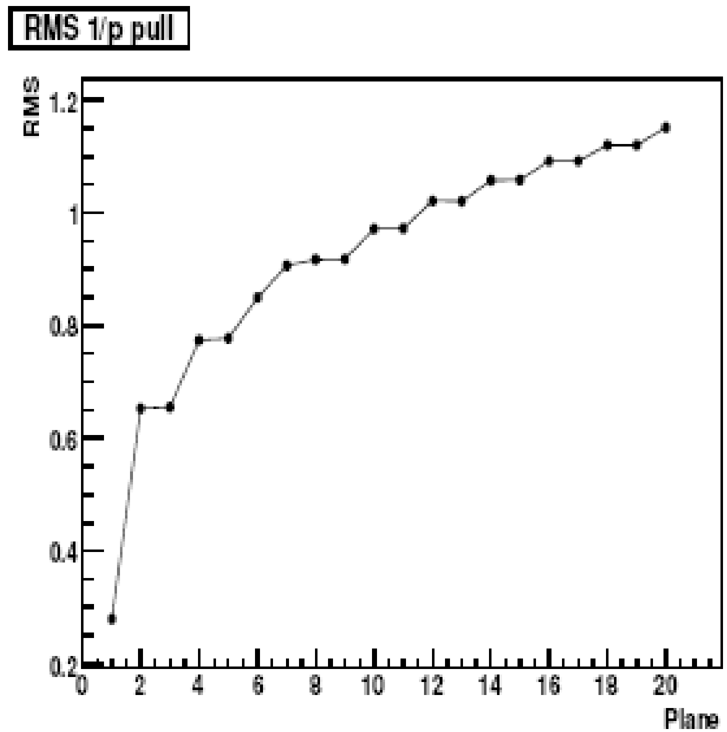


Figure 4: Values of the standard deviations of the $1/p$ pull variable with truncation parameter $\delta = 0.9999$ from eq. (15), as a function of the number of the traversed layers. The data refer to 1 Gev pions traversing layers formed by a 1 mm thick Al (Landau distribution) and a 1 cm thick Ar gas (Urban distribution) absorbers at NTP.

Bremsstrahlung

The radiative energy loss straggling distribution for the energy E of a particle of incident energy E_0 on an absorber of thickness x, was first deduced by Heitler [28], using an approximate expression for the bremsstrahlung cross section:

$$f(E) = \frac{1}{E_0 \Gamma(l)} \left(\ln \frac{E_0}{E} \right)^{l-1} \quad , \quad l = \frac{x}{X_0 \ln 2} \ , \tag{18}$$

where X_0 is the radiation length of the absorber and Γ is the gamma function.

$$\begin{aligned} \langle E \rangle &= E_0 \frac{1}{2^l} \ , \qquad \left\langle E^2 \right\rangle = E_0^2 \frac{1}{3^l} \\ \sigma^2[E] &= \left\langle E^2 \right\rangle - \langle E \rangle^2 = E_0^2 \left(\frac{1}{3^l} - \frac{1}{4^l} \right) \ . \end{aligned}$$

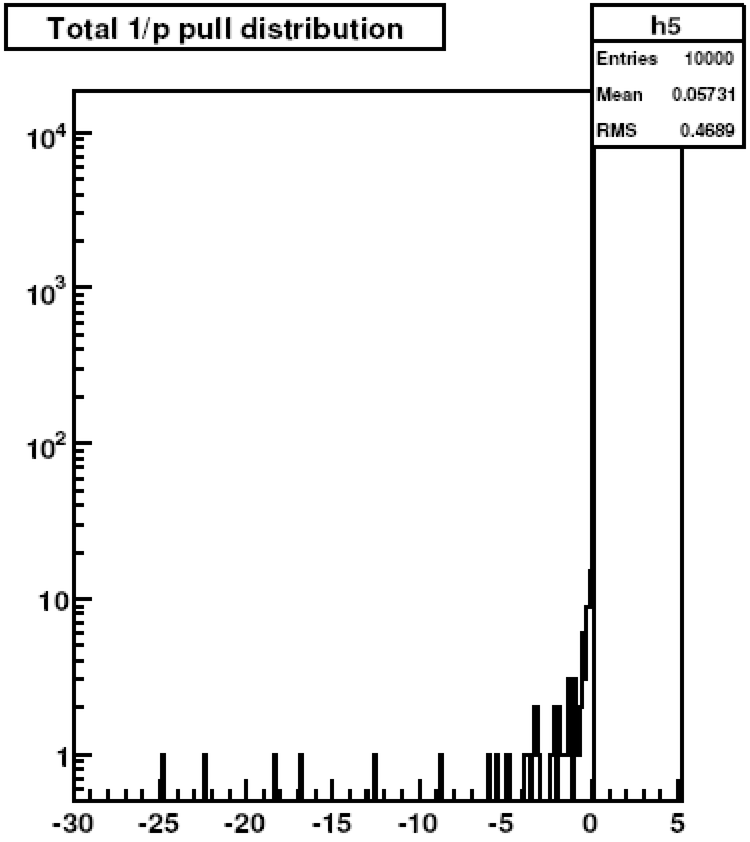
Bremsstrahlung

absorber	energy (GeV)	Heitler equation μ	Heitler equation σ	GEANT3 μ	GEANT3 σ	GEANT4 μ	GEANT4 σ
10 cm Ar	0.5	0.4995	0.0097	0.4995	0.0097	0.4995	0.0105
10 cm Ar	1.0	0.9991	0.0194	0.9991	0.0198	0.9991	0.0203
1 cm Al	0.5	0.447	0.098	0.444	0.100	0.444	0.098
1 cm Al	1.0	0.894	0.195	0.891	0.203	0.891	0.201
1 cm Al	10	9.01	1.95	8.96	2.04	8.95	2.06

Table 2: comparison between the mean energy μ and standard deviation σ (MeV) from the the GEANT3 and GEANT4 simulated distributions relative to 10^5 electrons and from the Heitler formula after passing some absorbers.

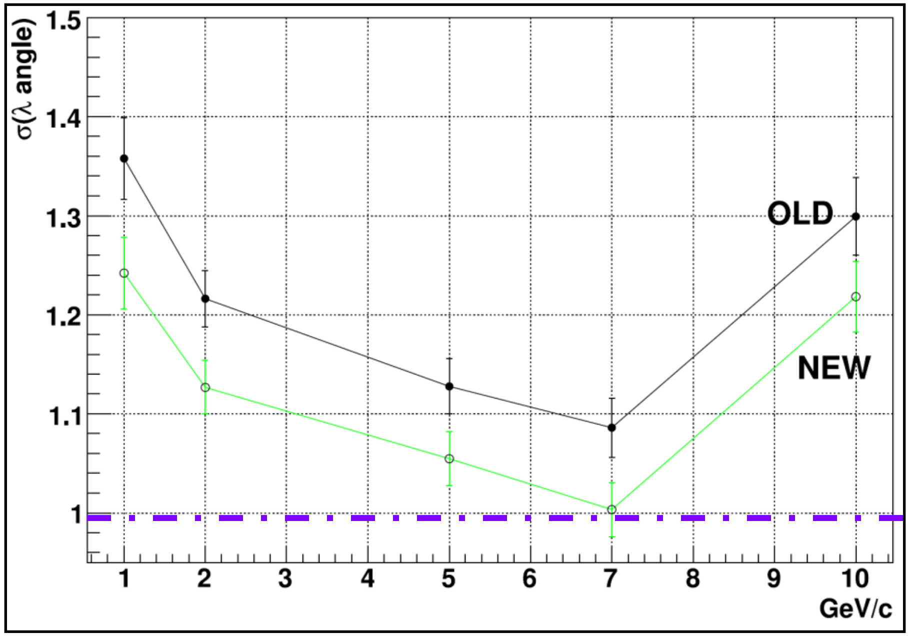
Bremsstrahlung

$$\sigma[1/E] = 0.5\,[1/E_2, 1/E_1]\;, \qquad \text{where}$$
$$E_2 = \mathrm{Min}(E_0, \langle E\rangle + \sigma[E])\;,$$
$$E_1 = \begin{cases} \langle E\rangle - \sigma[E] & \text{if} \quad E_2 = \langle E\rangle + \sigma[E] \\ E_0 - 2\sigma[E] & \text{if} \quad E_2 = E_0 \end{cases} \quad .$$

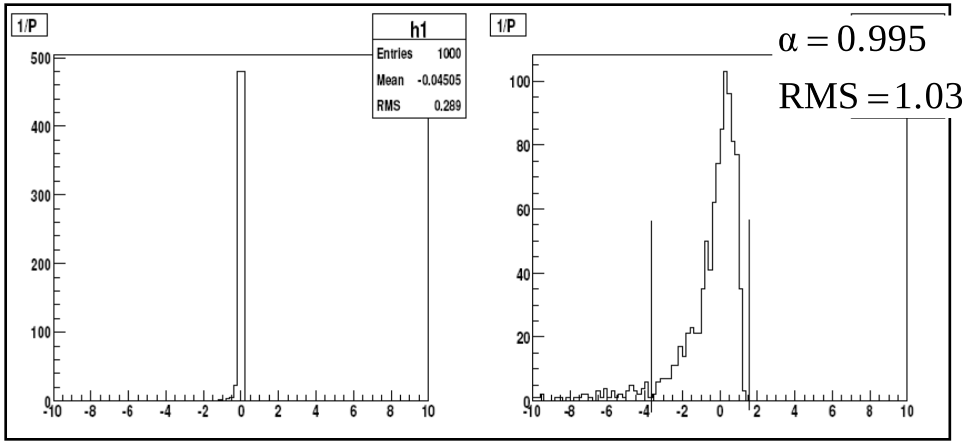


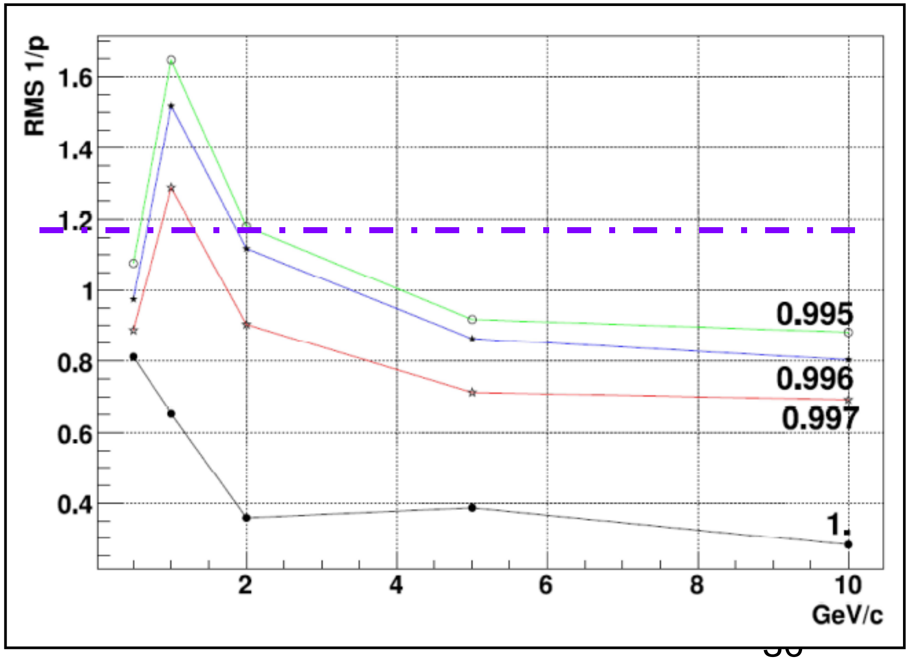
Track propagation: physical effects

Multiple scattering



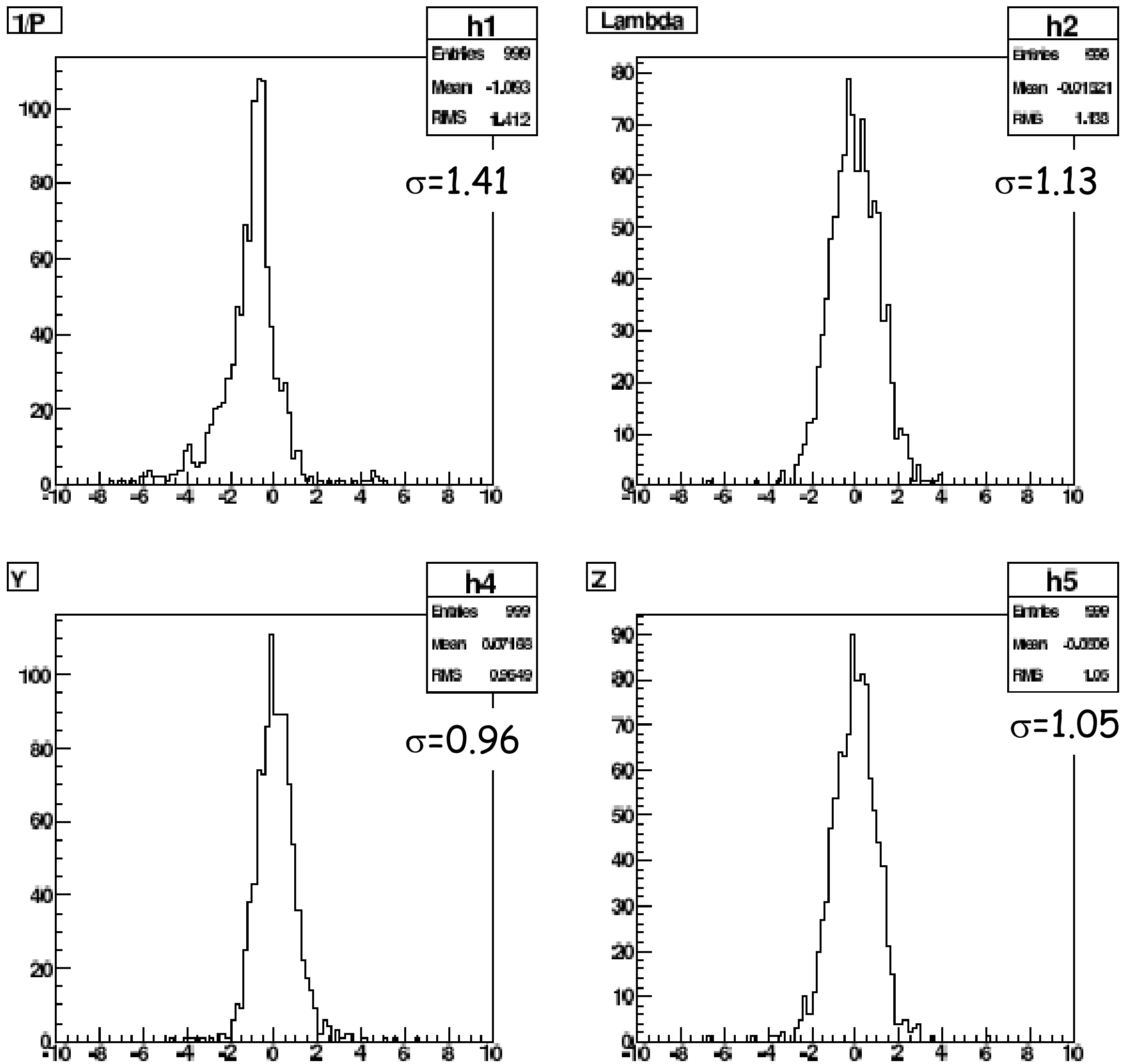
Energy loss straggling





Pull

$$\frac{(x)_{MC} - (x)_{GEANE}}{\sigma_{GEANE}}$$

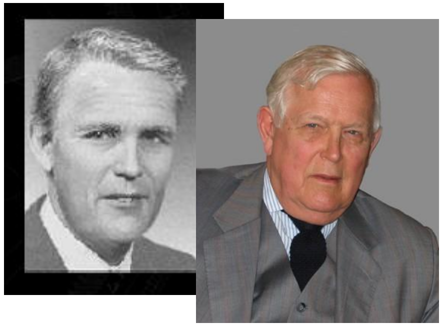


Pull

$$\frac{(x)_{MC} - (x)_{GEANE}}{\sigma_{GEANE}}$$

Figure 11: Pull distributions of the 5 track parameters in the case of 2 GeV muons that have passed through the whole detector, just before the PANDA

A Bayesian technique:The KALMAN filter



Consider the well-known weighted mean:

$$\chi^2(\mu)=\frac{(x_1-\mu)^2}{\sigma_1^2}+\frac{(x_2-\mu)^2}{\sigma_2^2}, \quad \frac{\partial\chi^2(\mu)}{\partial\mu}=0 \Rightarrow \mu=\frac{\dfrac{x_1}{\sigma_1^2}+\dfrac{x_2}{\sigma_2^2}}{\dfrac{1}{\sigma_1^2}+\dfrac{1}{\sigma_2^2}}$$

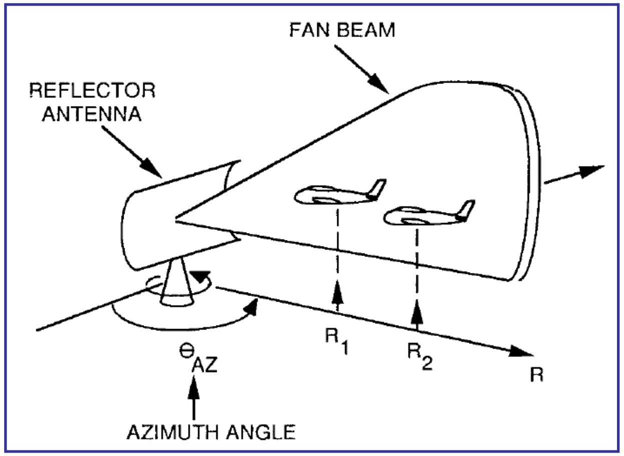
A simple algebraic manipulation gives the recursive form:

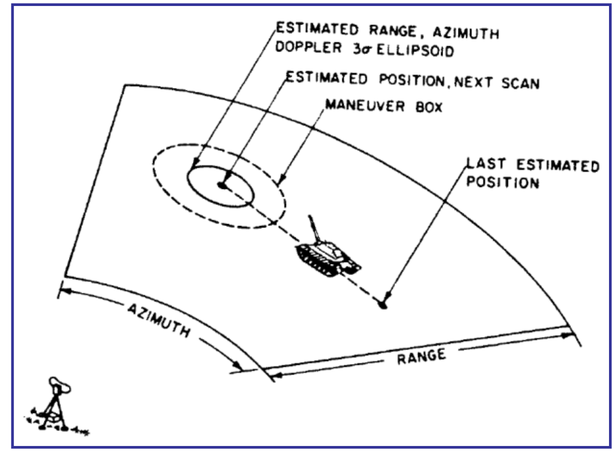
$$\mu=\frac{\dfrac{x_1}{\sigma_1^2}+\dfrac{x_2}{\sigma_2^2}}{\dfrac{1}{\sigma_1^2}+\dfrac{1}{\sigma_2^2}}=\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\left(\frac{x_1}{\sigma_1^2}+\frac{x_2}{\sigma_2^2}\right)=\frac{x_1\sigma_2^2+x_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}=x_1+\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\left(x_2-x_1\right)$$

Kalman= the measurement is weighted with a model prediction (track following)

prediction

Example: Radar Applications





In a radar application, where one is interested in following a target, information about the location, speed, and acceleration of the target is measured at different moments in time with corruption by noise.

State vector

$$r = \{ x, y, z, v_x, v_y, v_z \}$$

position (x, y, z) — velocity (v_x, v_y, v_z)

error of x

Covariance matrix

$$C = \begin{Bmatrix} \sigma^2_x & & & \dots & & \\ & \sigma^2_y & & & & \\ & & \sigma^2_z & & & \\ & & & \sigma^2_{vx} & & \\ & \dots & & & \sigma^2_{vy} & \\ & & & & & \sigma^2_{vz} \end{Bmatrix}$$

Apollo Flight Journal

December 21, 1968. The Apollo 8 spacecraft has just been sent on its way to the Moon.
003:46:31 Collins: Roger. At your convenience, would you please go P00 and Accept? We're going to update to your W-matrix.

The original idea is very simple

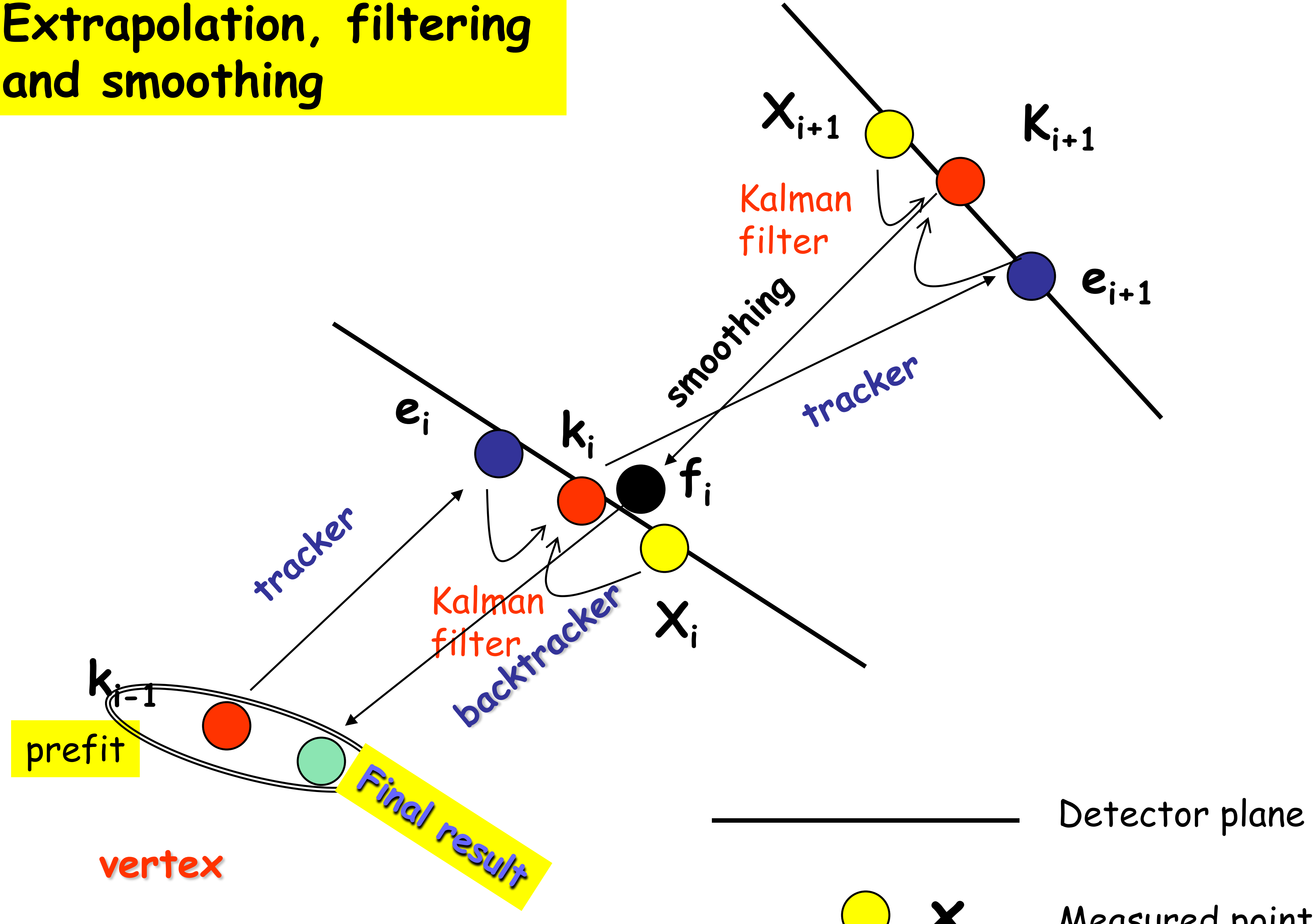
When m is measured at t_2 and $x(t_1,t_2)$ is the **prediction** from t_1 to t_2, the best evaluation of x at t_2 is

This is called the Kalman filter recursive form

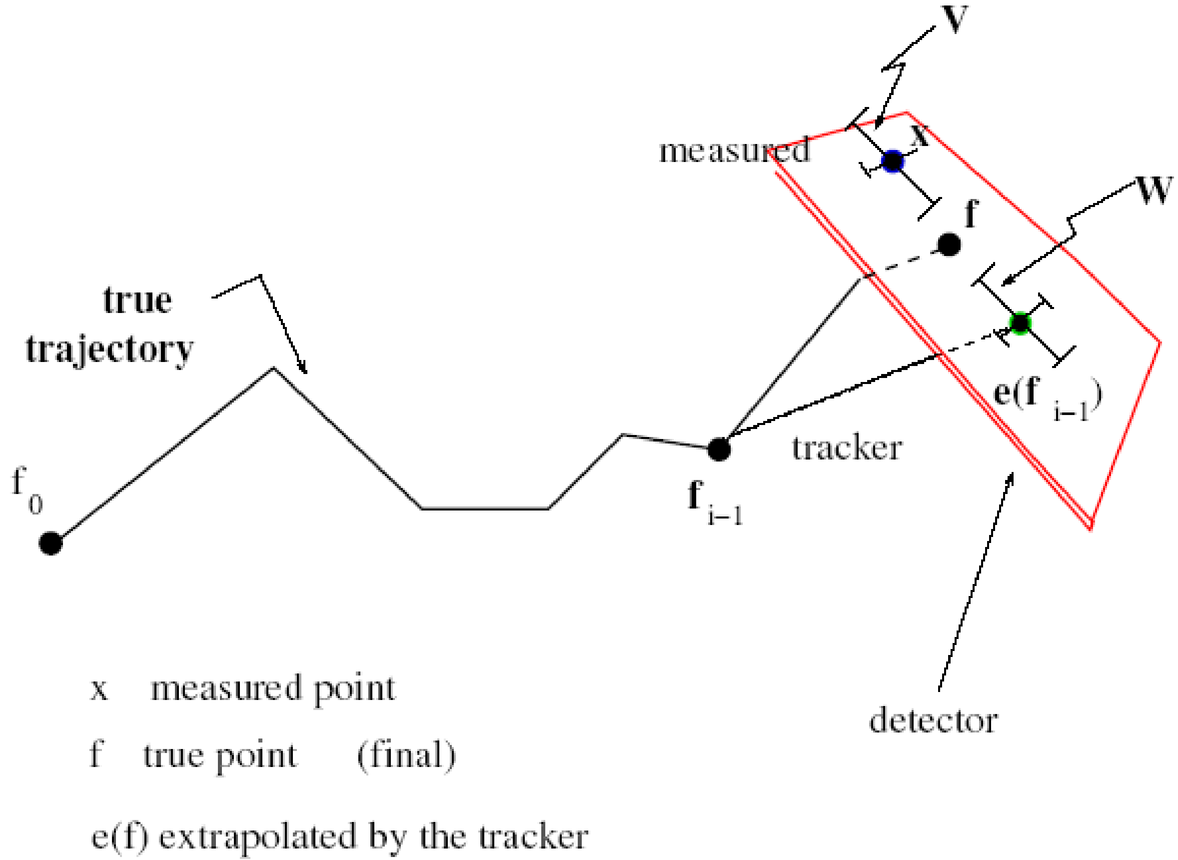
$$x(t_2) = m + \frac{\sigma_m^2}{\sigma_m^2 + \sigma_2^2}\left(x(t_1,t_2) - m\right)$$

Kalman= the measurement is weighted with a model prediction (track following)

Extrapolation, filtering and smoothing



Tracking with Kalman



The best estimate of the track is given by minimizing w.r.t the f variables:

$$\chi^2(f) = \sum_i [(e_i[f_{i-1}]-f_i)\boldsymbol{W}_{i-1}(e_i[f_{i-1}]-f_i)]+(x_i-f_i)\boldsymbol{V}_i(x_i-f_i) \tag{1}$$

Note the $\boldsymbol{W}$ matrix associated to e_i because the extrapolation start from the true point.

The minimization gives:

$$\frac{\partial \chi^2}{\partial f_i} = \boldsymbol{W}_{i-1,i}(e_i[f_{i-1}] - f_i) + \boldsymbol{V}(x_i - f_i) + \boldsymbol{T}_{i,i+1}\boldsymbol{W}_{i,i+1}(e_{i+1}[f_i] - f_{i+1}) = 0 \qquad (2)$$

where the last (*extra*) term comes from the extrapolation procedure (tracker).

The best way to solve eq (2) is the Kalman algorithm (Kalman, 1961). It is based on three steps:

V. Innocente and E. Nagy, NIM A324(1993)297 (see their sect. 4.3. Correct their eq. below the (34) one with our eq.(9)).

- **EXTRAPOLATION**: calculation of e_i and $\boldsymbol{W}$. Deterministic step made by the tracker.

$$e_i = \boldsymbol{G}_{i-1,i}[k_{i-1}] \quad (3)$$

$$\boldsymbol{\sigma}^2[e_i] = \boldsymbol{T}_{i-1,i}\,\boldsymbol{\sigma}^2[k_{i-1}]\,\boldsymbol{T}^T_{i-1,i} + \boldsymbol{W}^{-1}_{i-1,i} \quad (4)$$

Square brackets mean function argument

e_i = EXTRAP. extrapolation

k_i = result of the Kalman filter

$\boldsymbol{T}_{i-1,i}$ = EXTRAP. transport matrix

$\boldsymbol{\sigma}(k)^2$ = Kalman error matrix

$\boldsymbol{\sigma}^2[e_i]$ = EXTRAP. error matrix

$\boldsymbol{W}_{i-1,i}$ = EXTRAP. energy loss and multiple scattering weight matrix

$\boldsymbol{W}^{-1}_{i-1,i}$ = covariance matrix inverse of W

- **FILTERING: minimizes the first two terms of eq. (2). It is simply the weighted mean;**

$$k_i = \boldsymbol{\sigma}^2[k_i]\left(\boldsymbol{\sigma}^{-2}[e_i]\,e_i + \boldsymbol{V}_i x_i\right) \quad (5)$$

$$\boldsymbol{\sigma}^{-2}[k_i] = \boldsymbol{\sigma}^{-2}[e_i] + \boldsymbol{V}_i \quad (6)$$

x_i = measured points

k_i = Kalman average value

$\boldsymbol{\sigma}(k)^2$ = Kalman error matrix

$\boldsymbol{\sigma}^2(e)$ = EXTRAP. error matrix

$\boldsymbol{V}$ = original weight matrix of the measured points

$$\mu = \frac{\dfrac{x_1}{\sigma_1^2} + \dfrac{x_2}{\sigma_2^2}}{\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}}$$

- **SMOOTHING**: necessary to minimize a χ^2 in the presence of the extrapolation term (last term in eq. (2)).

$$f_i = k_i + \boldsymbol{A}_i \left(f_{i+1} - e_{i+1}\right) \quad (7)$$

$$\boldsymbol{\sigma}^2[f_i] = \boldsymbol{\sigma}^2[k_i] + \boldsymbol{A}_i \left(\boldsymbol{\sigma}^2[f_{i+1}] - \boldsymbol{\sigma}^2[e_{i+1}]\right) \boldsymbol{A}_i^T \quad (8)$$

$$\boldsymbol{A}_i = \boldsymbol{\sigma}^2[k_i]\, \boldsymbol{T}_{i,i+1}^T\, \boldsymbol{\sigma}^{-2}[e_{i+1}] \quad (9)$$

f_i = final average value

$\boldsymbol{\sigma}(k)^2$ = Kalman error matrix

$\boldsymbol{\sigma}^2(e)$ = EXTRAP. error matrix

Track fitting tools

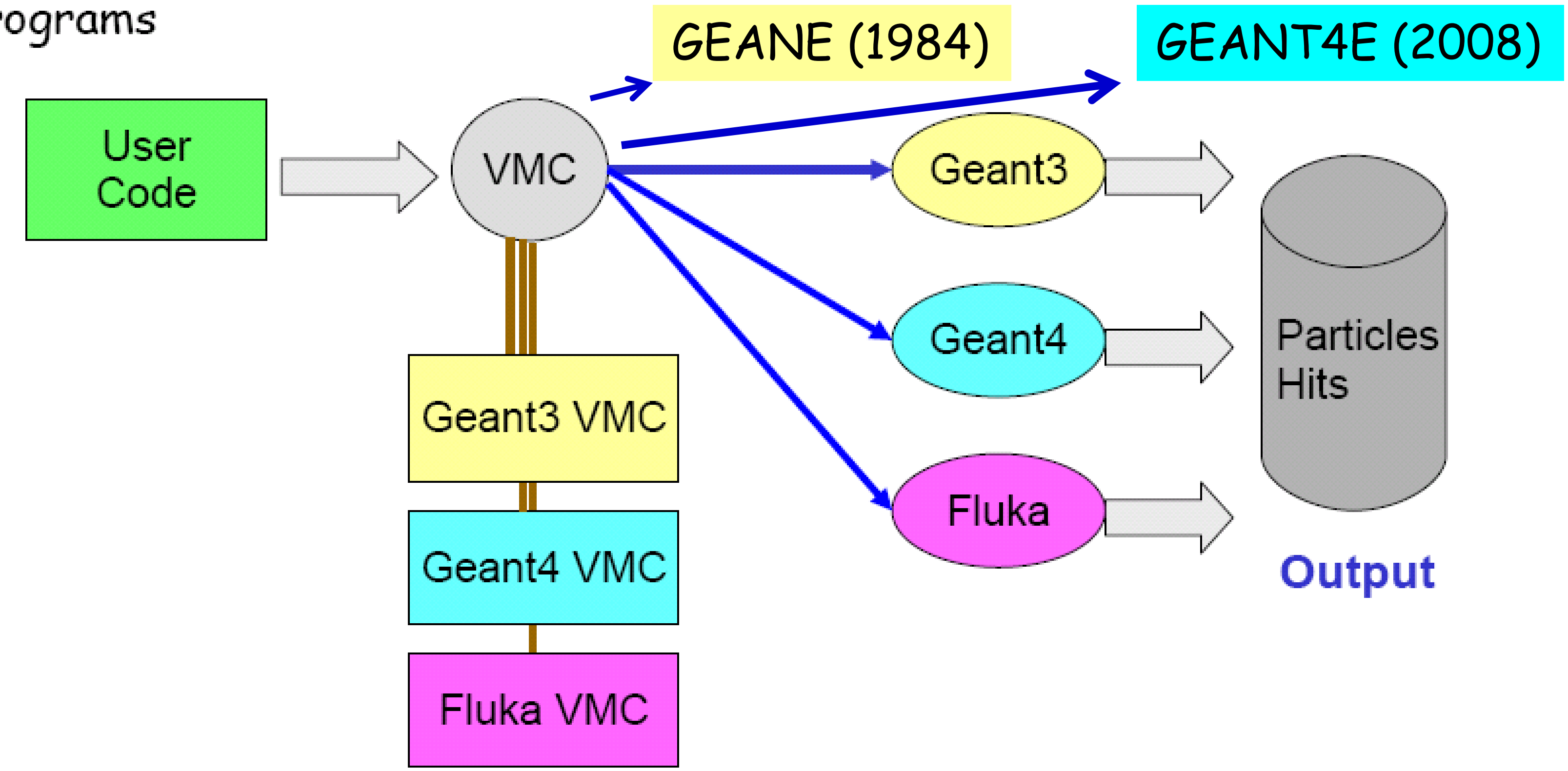
1. the GEANT3-GEANE old chain:
The mathematics is that of Wittek (EMC Collaboration)
The tracking banks and routines are the same as in MC.
The user gives the starting and ending planes or volumes and the **tracking is done automatically**.
It works very well (see the CERN Report W5013 GEANE, 1991).

2. "Modern" experiments:
in the software are implemented some tracking classes:
input: x_i, T_i , σ_i, step, medium, magnetic field
output: new x_i, T_i, σ_i
the user has to manage geometry, medium and detector interface

3. A GEANT4-GEANT4E chain there exists in the new GEANT Root framework.
It is used by CMS but is **not included into the official releases**
(see Pedro Arce's talks in the Web)

The Virtual Monte Carlo (VMC)

Thanks to an abstract VMC layer to Monte Carlo transport codes, the same user application code can be run with different simulation programs

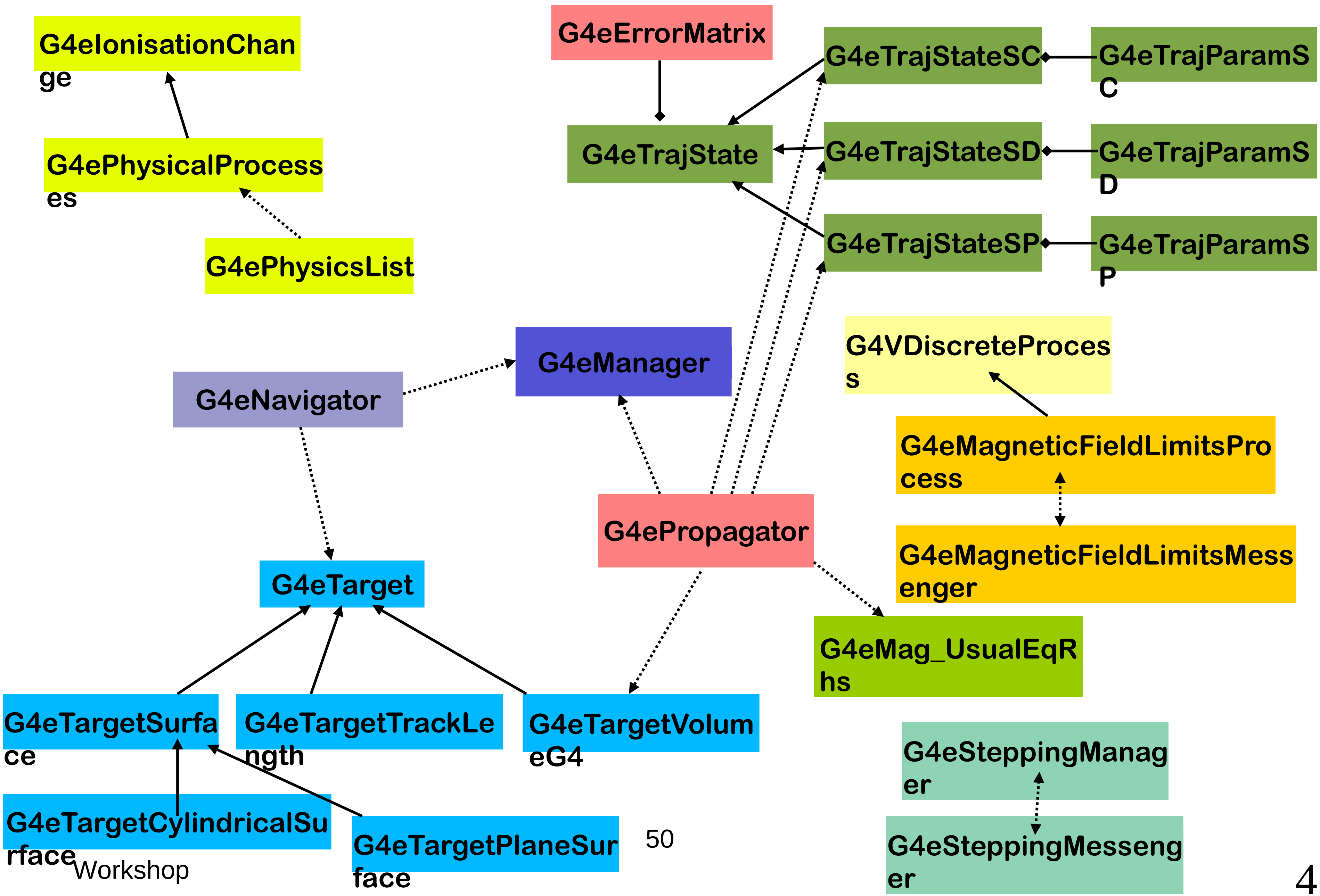


What is GEANT4E

- Track reconstruction needs to match signals in two detector parts
  - Propagate tracks from one detector to the other and compares with real measurement there
  - Make the average between the prediction and the real measurement $\Rightarrow$ it needs the track parameter errors

- Traditionally experiments have used GEANE (based on GEANT3) or their ‘ad hoc’ solution

GEANT4e provides this functionality for the reconstruction software in the context of GEANT4


G4eIonisationChange
G4ePhysicalProcesses
G4ePhysicsList
G4eErrorMatrix
G4eTrajState
G4eTrajStateSC
G4eTrajStateSD
G4eTrajStateSP
G4eTrajParamSC
G4eTrajParamSD
G4eTrajParamSP
G4VDiscreteProcess
G4eMagneticFieldLimitsProcess
G4eMagneticFieldLimitsMessenger
G4eManager
G4eNavigator
G4ePropagator
G4eMag_UsualEqRhs
G4eTarget
G4eTargetSurface
G4eTargetTrackLength
G4eTargetVolumeG4
G4eTargetCylindricalSurface
G4eTargetPlaneSurface
G4eSteppingManager
G4eSteppingMessenger

- User defines the initial track parameters in a given point of the trajectory: **G4eTrajState**
  - Particle type
  - Position
  - Momentum
  - Track errors (5x5 HepSymMatrix)
  - Initial surface where parameters are defined
- Three distinct coordinate systems are supported, as in GEANE, inspired by the EMC collaboration
- (user just needs to give pos. & mom., transformation is done by GEANT4E)

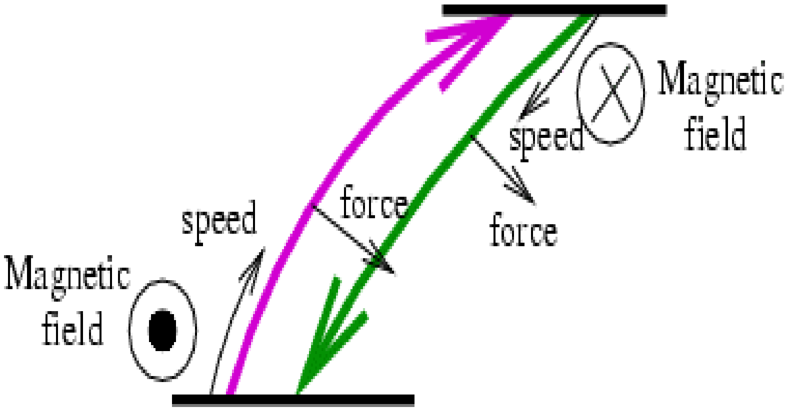
- SC: parameters in the global reference frame
  - $1/p$, λ, ϕ, y_perp, z_perp (p_x = p cos(λ) cos(ϕ), p_y = p cos(λ) sin(ϕ), p_z = p sin(λ), x_perp || trajectory, y_perp parallel to x-y plane)
- SP: paramaters on a plane perpendicular to X
  - $1/p$, y', z', y, z ($y' = dy/dx$, $z' = dz/dx$)
- SD: parameters on a plane in an arbitrary direction
  - $1/p$, v', w', v, w (u,v,w is any orthonormal coordiante system, v, w on the plane)

Magnetic field: G4eMagneticField

- User defines the magnetic field in the standard GEANT4 way

➢ But GEANT4e has to handle the backwards propagation

  ⇒ Magnetic field has to be reversed



Track error propagation

- ❖ Based on the equations of the European Muon Collaboration (same as GEANE)
  - ✓ Error from curved trajectory in magnetic field
  - ✓ Error from multiple scattering
  - ✓ Error from ionisation

- ➢ Formulas assume propagation along an helix
  - Need to make small steps to assure magnetic field constantness and not too big energy loss $\Rightarrow$ makes it slower

- ❑ Another approach to be studied: propagate the error together with the solving of the Runge-Kutta equations
  - Probably slower per step but would not need so many steps

Backwards tracking

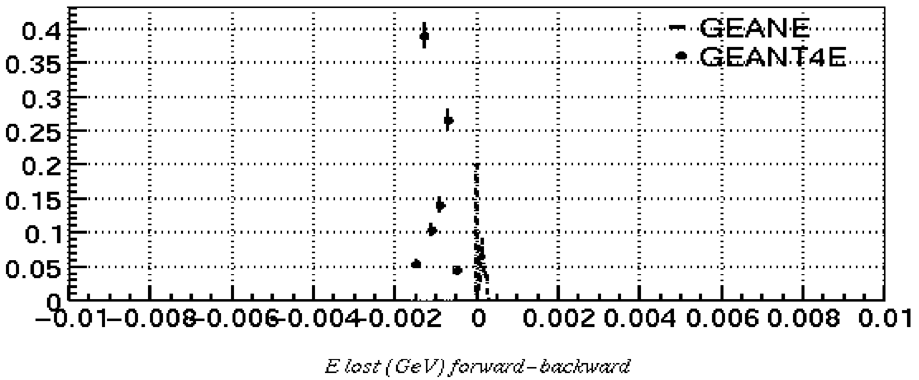
- When reconstruction software wants to know the trajectory that a track has described from a detector part to another, often the track has to be propagated **backwards**
  - ✓The track has to gain instead of losing energy
  - ✓The value of the magnetic field has to be reversed

- But the energy lost (gained) in one step is calculated
  - Forward tracking: using the energy at the beginning of the step
  - Backward tracking: using **the energy at the end of the step**
- And similarly for the curvature in magnetic field

Backward tracking (2)

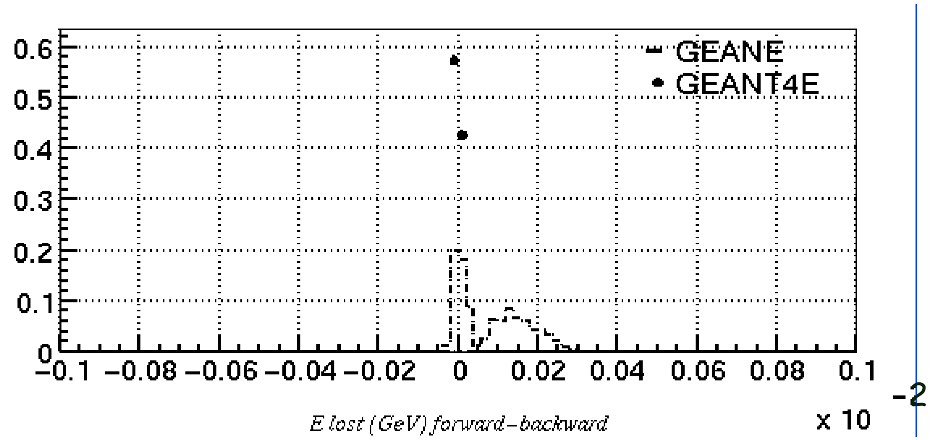
Difference in energy when a 20 GeV track is propagated forwards and then backwards

NO CORRECTION



Difference in energy when a 20 GeV track is propagated forwards and then backwards

CORRECTED



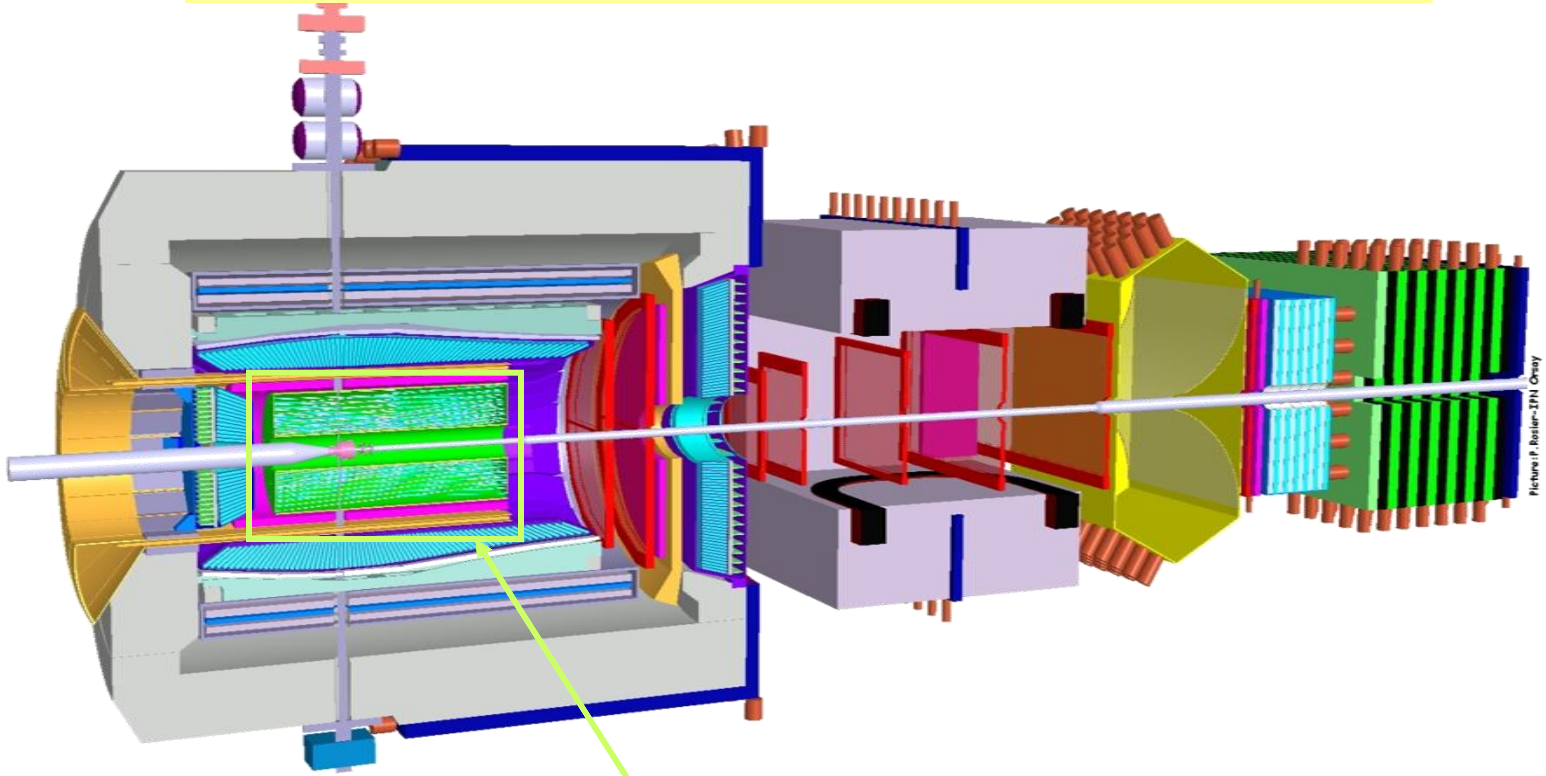
Timing GEANE vs GEANT4E

GEANT3		GEANT4	
GEANT3	0.205	GEANT4	0.61
GEANE: Forward or backward	**0.244**	**GEANT4E: Forward or backward**	**1.08**
GEANE: no error Forward or backward	0.114	GEANT4E: no error Forward or backward	0.81

- GEANT4 is 3 times slower than GEANT3
- GEANT4E is 4 times slower than GEANE
- Most of the time is taken by GEANT4
- Error propagation is 30 % of total time (55 % in GEANE)

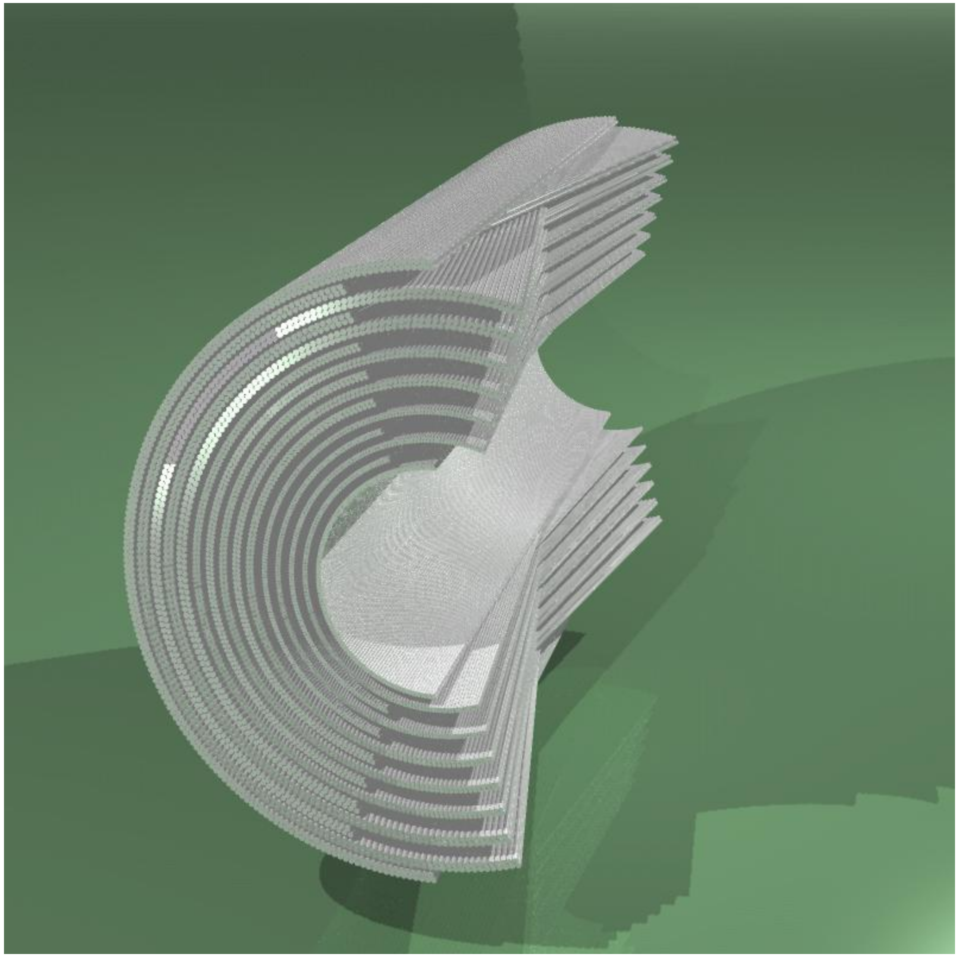
☺ Results have been checked by profiling

PANDA The detector



Central Straw Tube Tracker

Straw tube detector



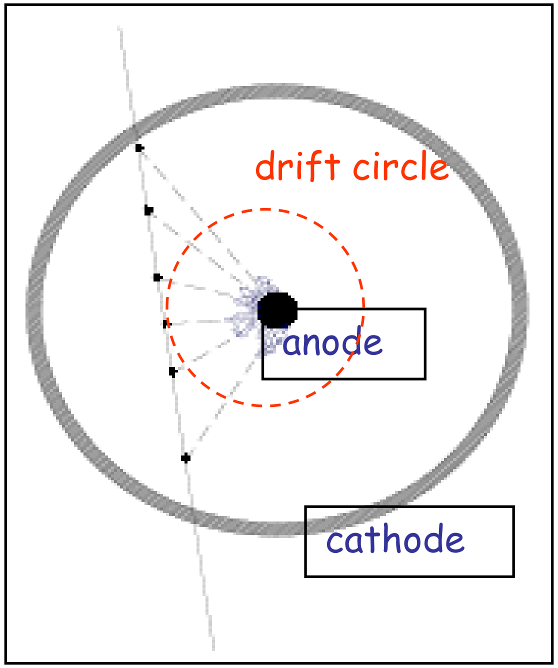
Straw Tube Tracker

Drift tubes for the central tracker

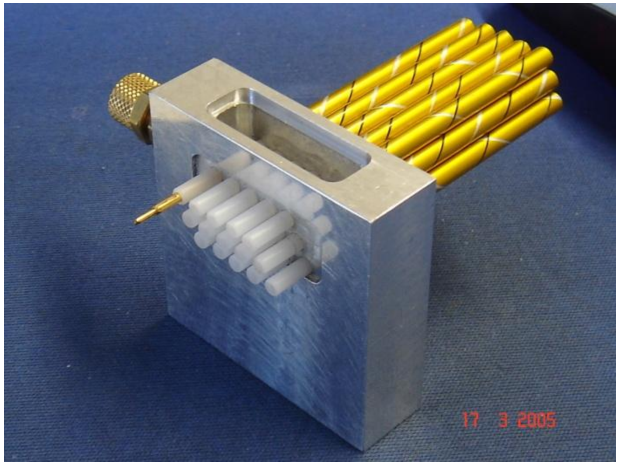
e^- and ions drift on the anode (wire) and on the negative catode (wall)

$$R_{equidrift} = \int_{t_0}^{t_1} w\,dt = w(t_1 - t_0)$$ <u>approximation</u>

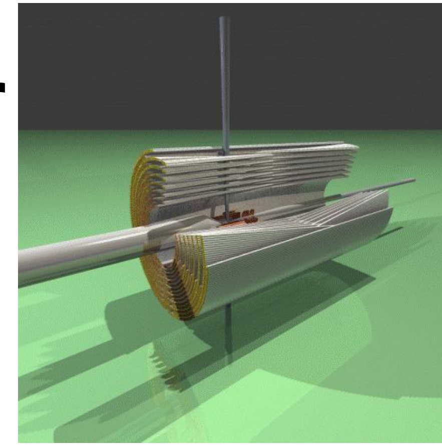
<u>Typical</u> e^- drift velocity: 5 cm/μs



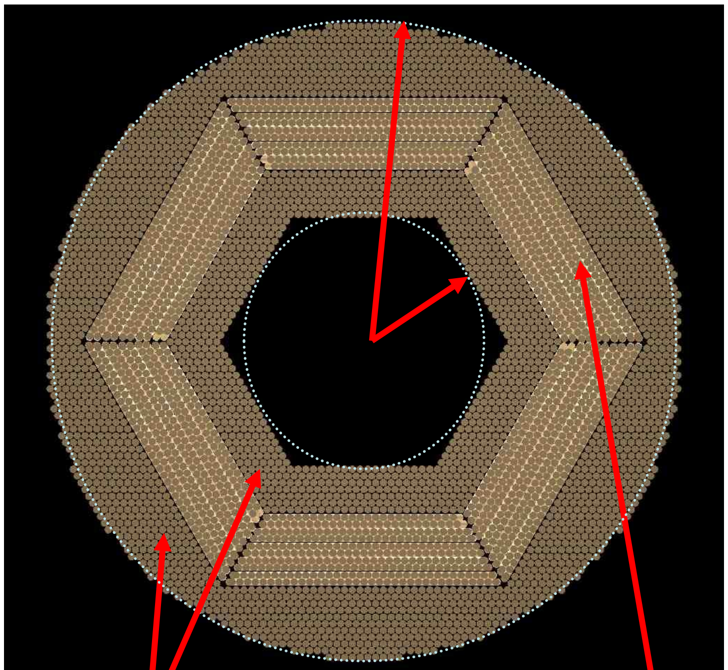
- mechanical stability
- high efficiency
- spatial resolution ~ 150 μ
- thickness X/X_0 ~ 1%
- high rate performances



Straw Tube Tracker



~ 5000 tubi (simulazioni)



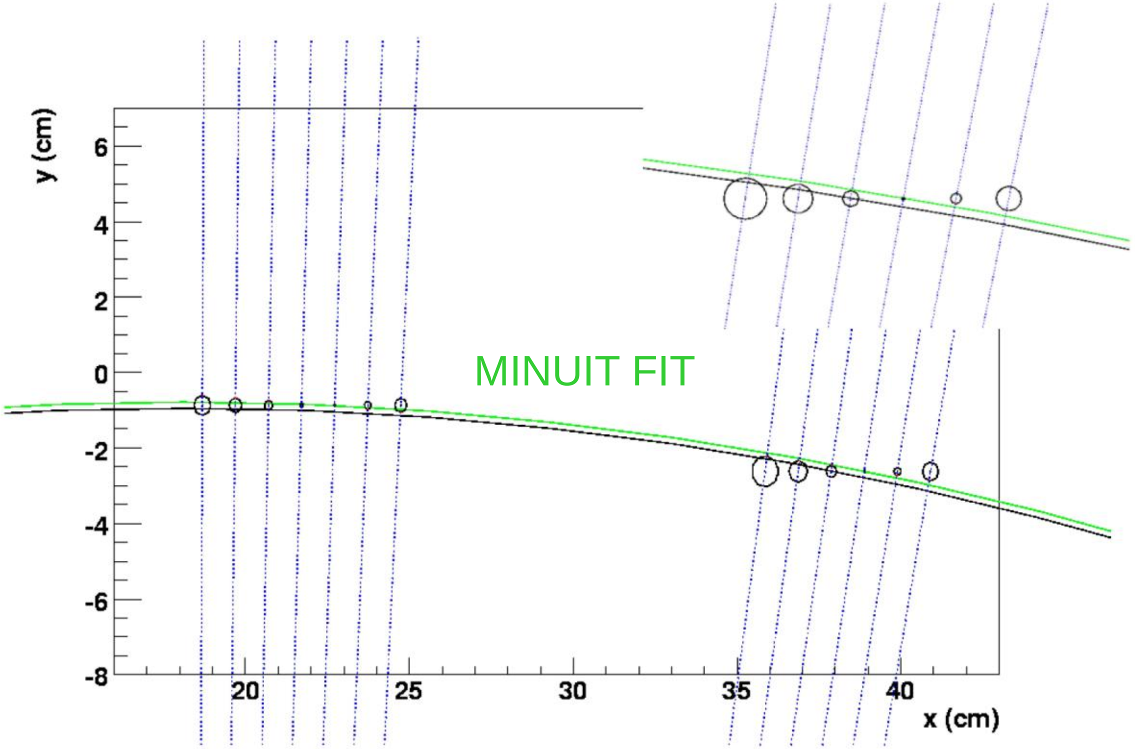
~ 3000 paralleli

stereo

STT characteristics	
z offset	0 cm
internal radius	15 cm
external radius	42 cm
skew angle	3^o
tube wall thickness	30 μm
tube $\varnothing$	1.006 cm
tube standard length	150 cm
wire $\varnothing$	20 μm
wall material	mylar
wire material	copper
gas mixture	argon ($Ar(90\%)/CO_2(10\%)$)

Fit in the straw tube x-y plane

The first fit (MINUIT) is made on the wire centers

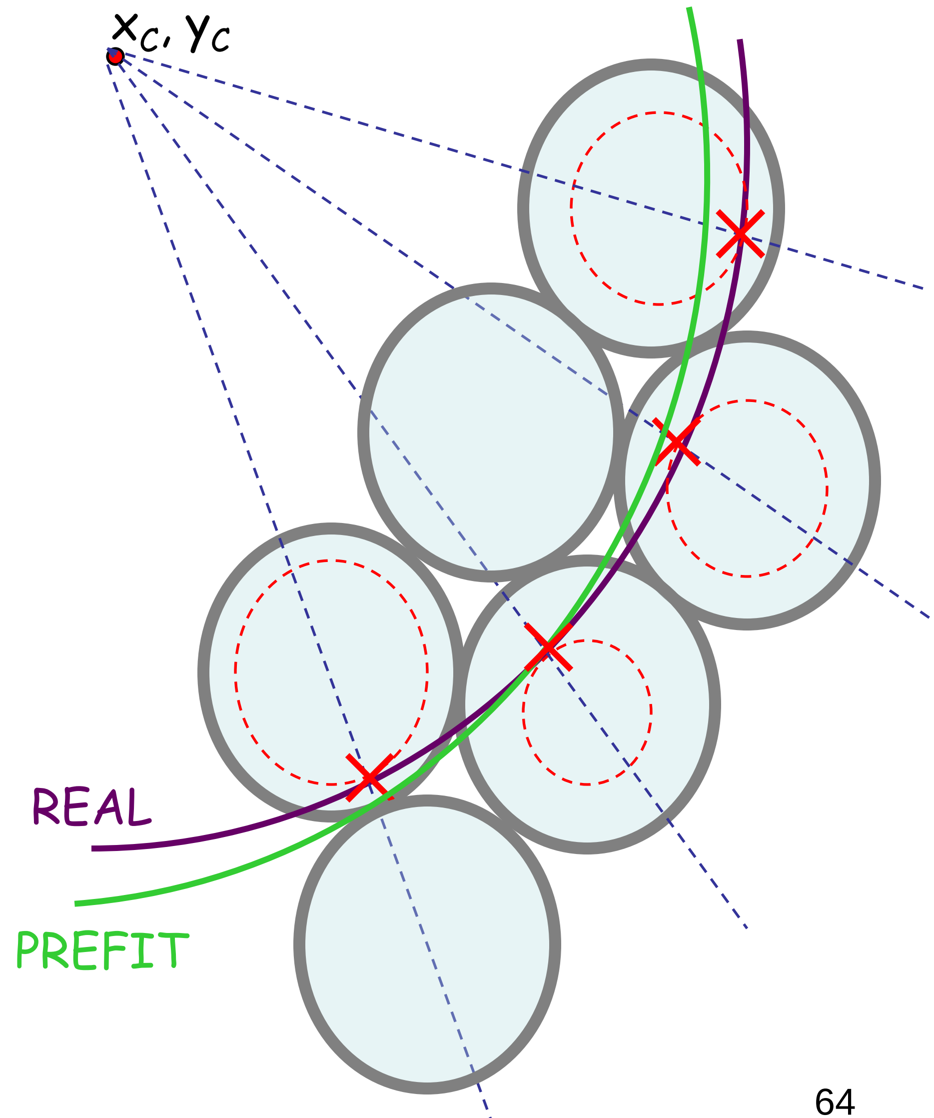


Fit in the straw tube x-y plane

x_C, y_C ... is the curvature center of the reconstructed track from Minuit prefit

Blu lines join the centers of the wires…

The points are the intersections between the blu lines and the drift radii



Helix fit with conformal mapping

M. Hansroul et al., NIM A 270 (1988), 498-501

- (x, y) points are mapped in the (u, v) plane to pass from one circonference to one parabola

$$u = \frac{x}{x^2 + y^2} \qquad v = \frac{y}{x^2 + y^2}$$

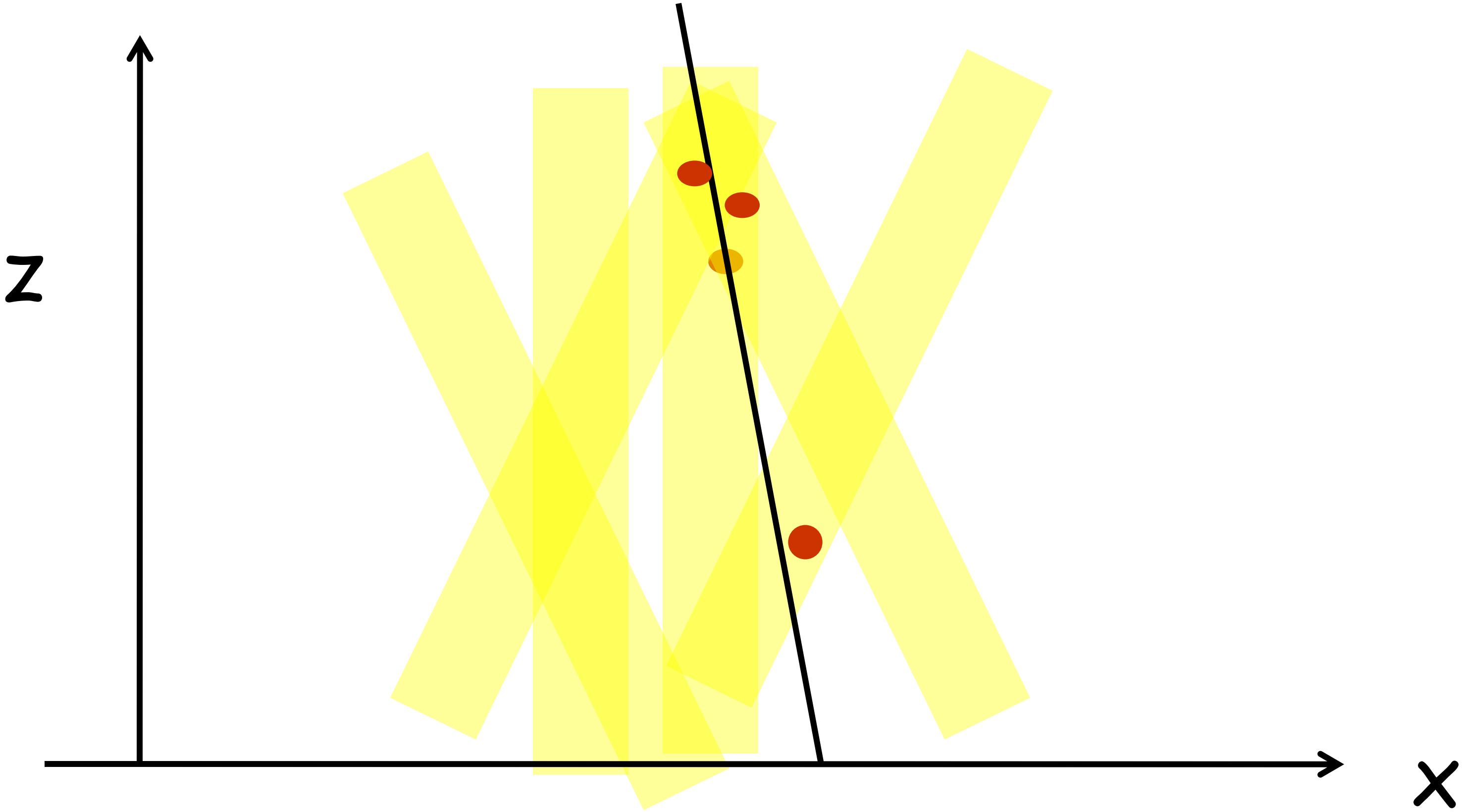
$$(x - a)^2 + (y - b)^2 = R^2 \quad \rightarrow \quad v = \frac{1}{2b} - u\frac{a}{b} - u^2 \varepsilon \left(\frac{R}{b}\right)^3$$

$$\varepsilon = R - \sqrt{a^2 + b^2}$$

The fit with one parabola is of a polynomial type and is much easier

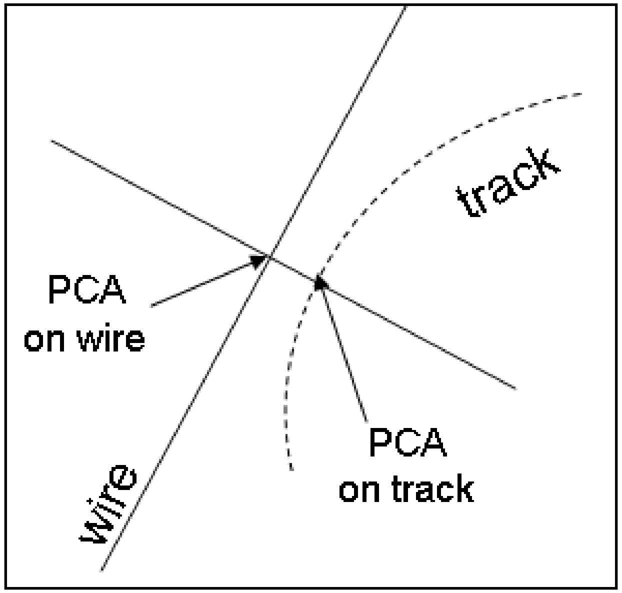
Fit of the Z coordinate

The z coordinate is reconstructed by means of the skewed tubes



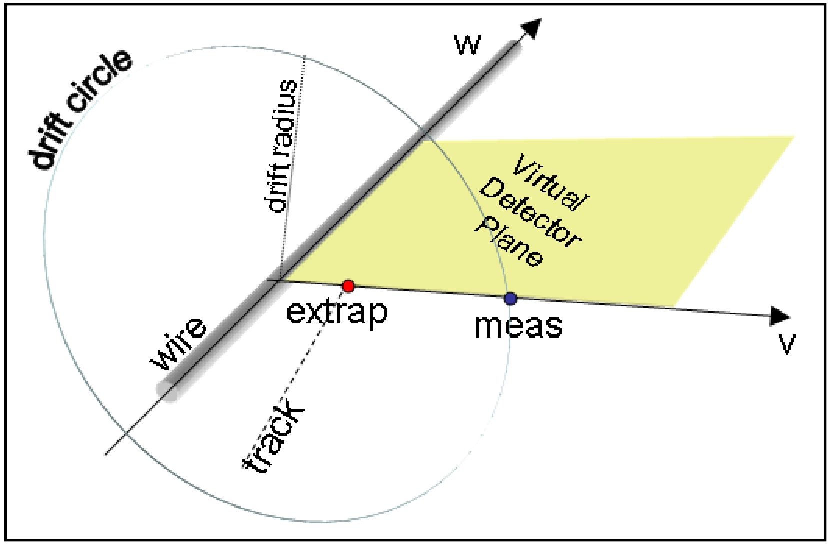
$$z(s) = z_0 + s \cdot \sin\lambda$$

Refit: Kalman fit for a straw tube I



EXTRAPOLATION

Propagation to the point of closest approach (PCA) to the wire

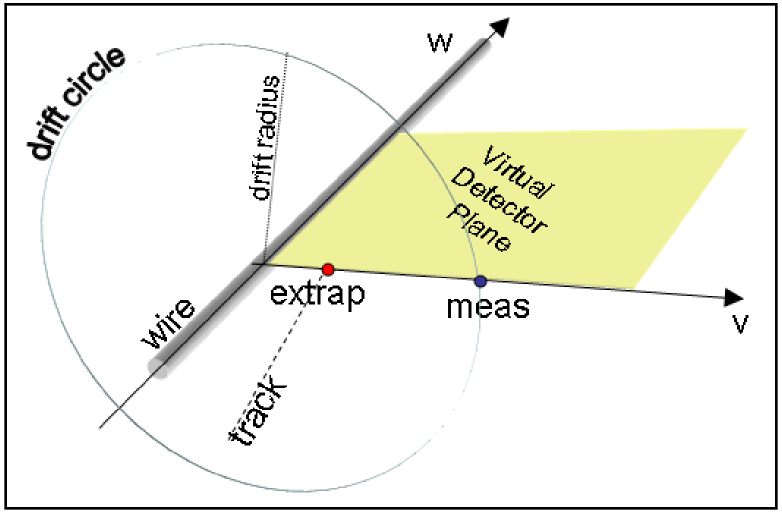


Virtual detector plane $\frac{1}{p}, v', w', v, w$

- v-axis: from PCA on wire
  to PCA on track
- w-axis: wire
- origin: PCA on wire

Refit: Kalman fit for a straw tube II

FILTER



The Kalman weighted average is made between the PCA extrapolated point and the mesured point, given by the intersection of the drift radius with the (v-w) plane

SMOOTHING

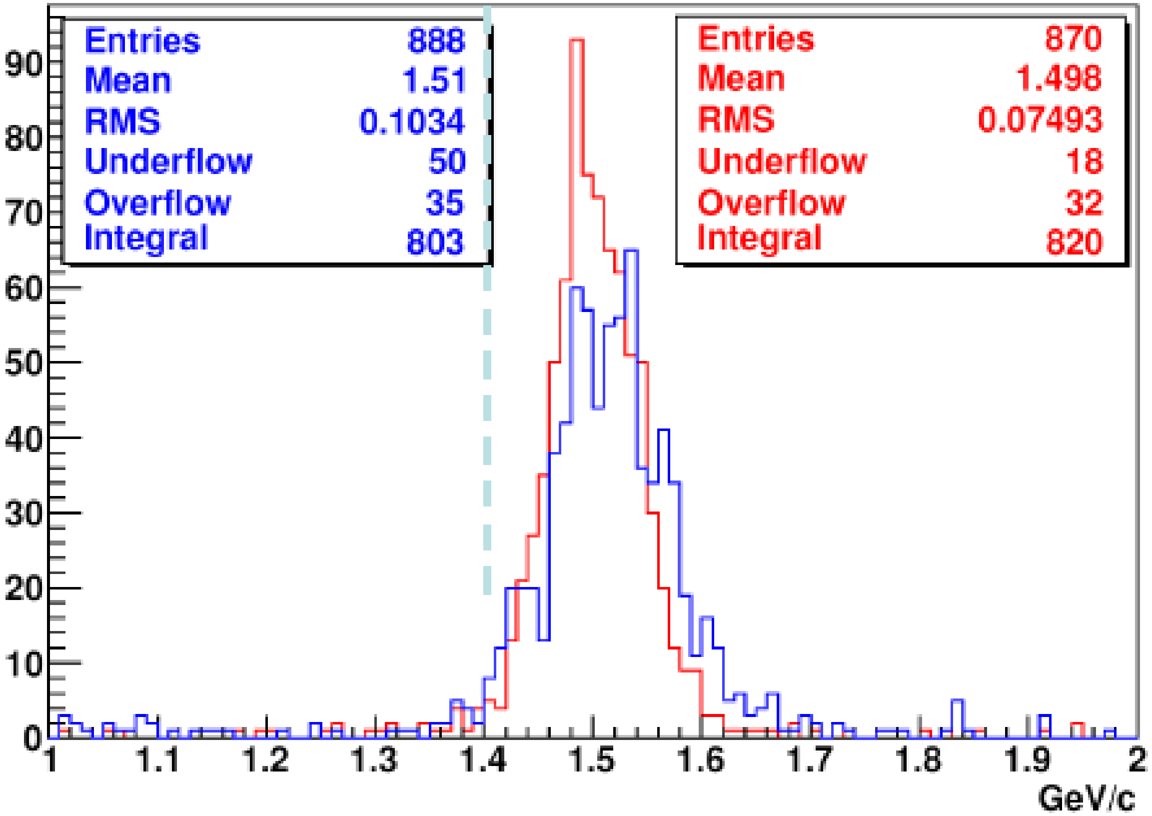
The Kalman smoothing is made is made simply with a backward FILTER (See below)

After three iterations the procedure is stopped

Improvement in momentum resolution

1000 μ^- events, 1.5 GeV/c, isotropy

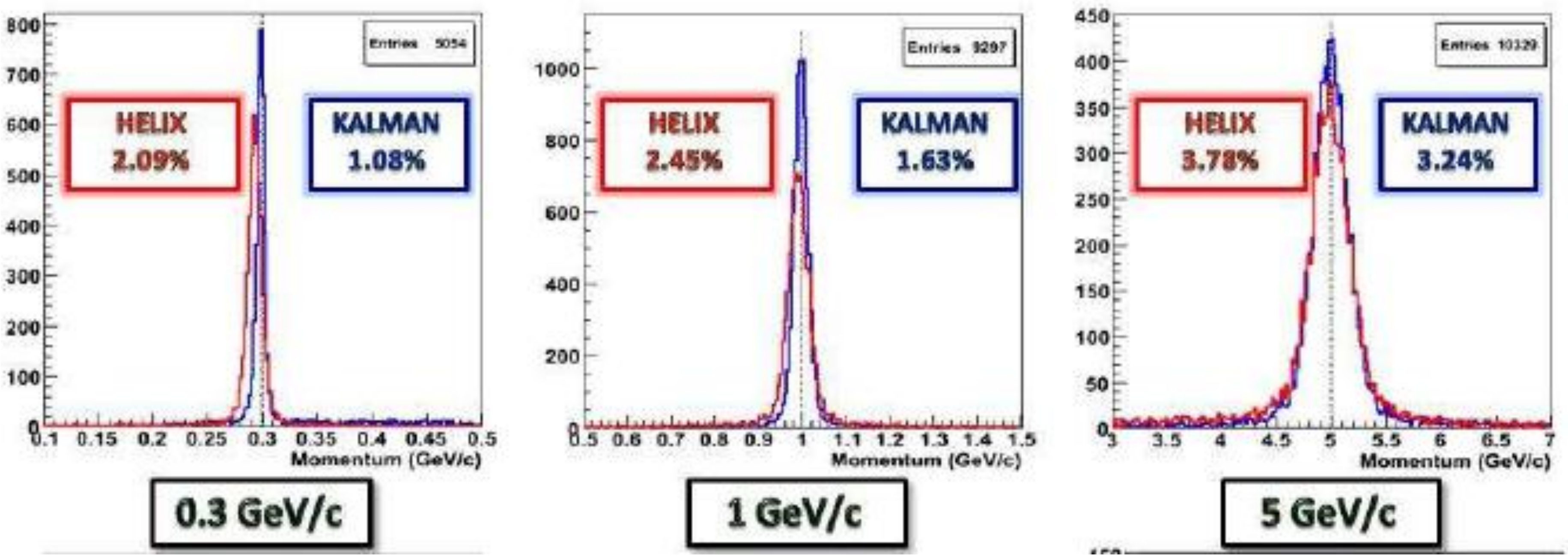
Helix KALMAN



EFFICIENCY

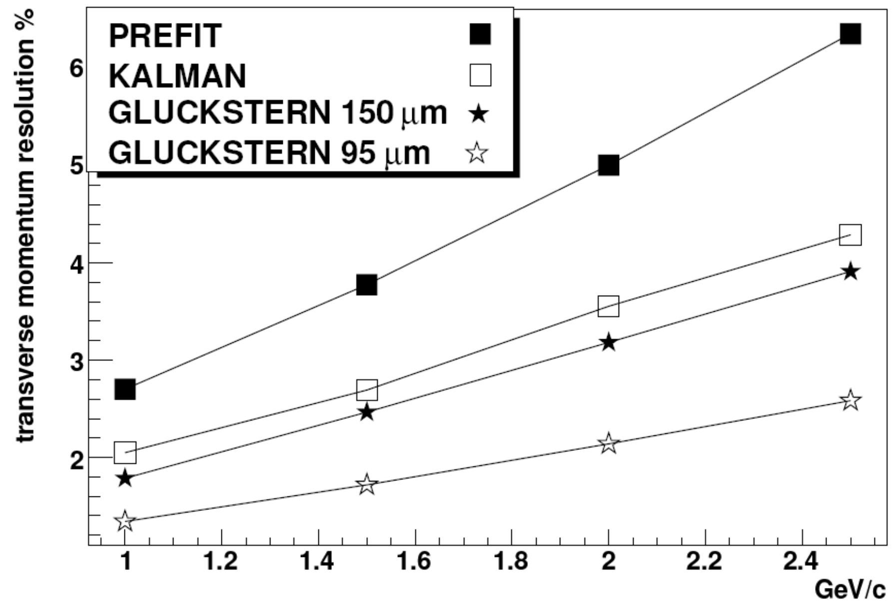
803 820

Improvement in momentum resolution



Improvement in the resolution

Momentum	Helix	Kalman
0.5	2.6 %	2.2 %
1.	3. %	2.2 %
1.5	3.6 %	2.7 %
2.	4.3 %	3.2 %
5.	10.6 %	8.8 %
10.	19.5 %	17.4 %



- 1000 muons in the transverse plane
- L~30 cm
- n ~ 20
- B = 2T
- β = 1
- L/X_0 = 1.1%

$$\delta(k)^2 = \delta(k_{MS})^2 + \delta(k_{meas})^2$$

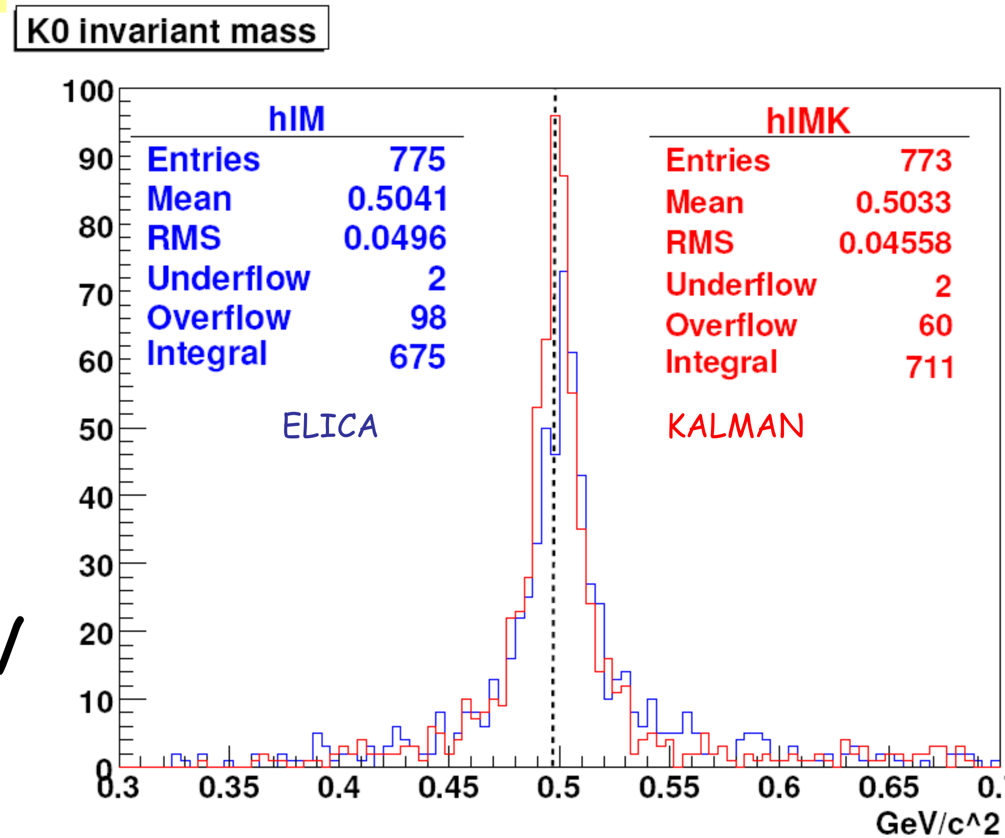
$$\delta(k_{meas})^2 = \frac{\varepsilon}{L'^2}\sqrt{\frac{720}{n+4}}$$

$$\delta(k_{MS})^2 = \frac{0.016(\mathrm{GeV/c})z}{Lp\beta\cos^2(\lambda)}\sqrt{\frac{L}{X_0}}$$

K^0_s invariant mass

GENERATION

- 1000 events
- vertex (0, 0, 0)
- $p_{x,y,z}$ [0, 1.5] GeV/c
- $K^0_s \rightarrow \pi^+\pi^-$ decay
- $p_\pi > 0.3$ GeV/c
- $m(K^0_s) = 0.49767$ GeV



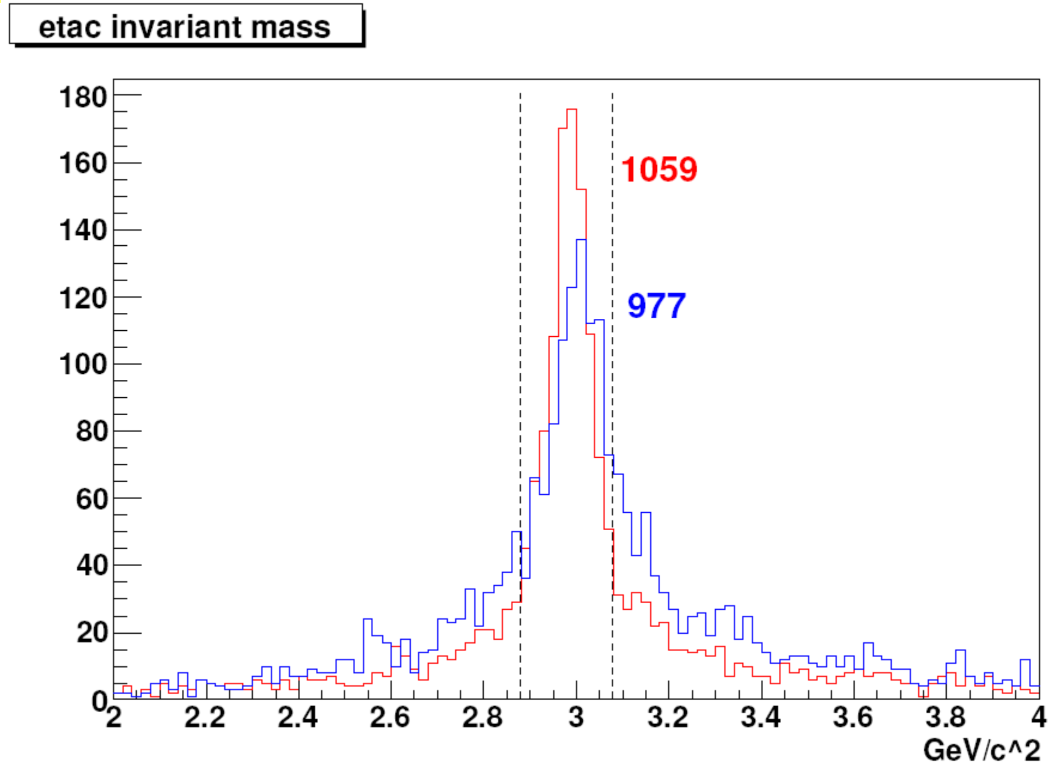
$$IM^2 = 2\left(m_\pi^2 + E_{\pi^+}E_{\pi^-} - \vec{p}_{\pi^+}\cdot\vec{p}_{\pi^-}\right)$$

$$\mu_{\text{PREFIT}} = (0.5004 \pm 0.0007)\ \text{GeV}/\text{c}^2$$
$$\sigma_{\text{PREFIT}} = (0.01266 \pm 0.00072)\ \text{GeV}/\text{c}^2$$
$$\mu_{\text{KALMAN}} = (0.4988 \pm 0.0005)\ \text{GeV}/\text{c}^2$$
$$\sigma_{\text{KALMAN}} = (0.00956 \pm 0.00053)\ \text{GeV}/\text{c}^2$$

η_C invariant mass

GENERATION

- 50000 events
- vertex (0, 0, 0)
- p(anti-p) = 3.68 GeV/c
- $\eta_C \rightarrow K^+\pi^-K^0_s$ decay
- $K^0_s \rightarrow \pi^+\pi^-$ 100% decay
- $m(\eta_C)$ = 2.9798 GeV
- $\Gamma(\eta_C)$ = 0.0270 GeV
- $m(K^0_s)$ = 0.49767 GeV



APPROXIMATION

- no slow tracks particle
- identification from Monte Carlo truth
- secondary vertex from Monte Carlo truth

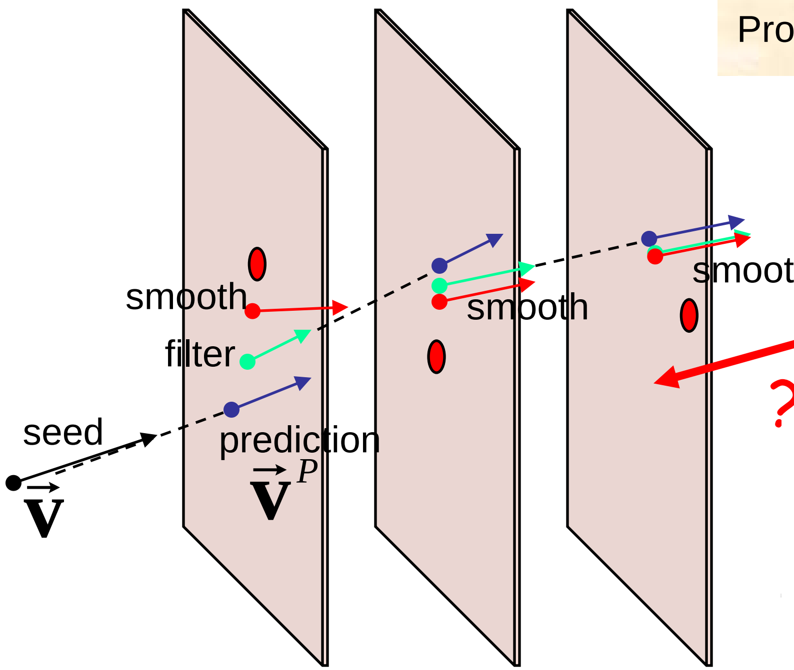
$$\mu_{\text{PREFIT}} = (3.008 \pm 0.003)\ \text{GeV}/c^2$$
$$\sigma_{\text{PREFIT}} = (0.0760 \pm 0.0048)\ \text{GeV}/c^2$$
$$\mu_{\text{KALMAN}} = (2.986 \pm 0.002)\ \text{GeV}/c^2$$
$$\sigma_{\text{KALMAN}} = (0.0555 \pm 0.0022)\ \text{GeV}/c^2$$

Invariant mass resolution = 1.8% against 2.2 %

Kalman filter: summary

- Kalman Filter:
  - Used for track fitting by most of HEP experiments
  - Easy to include random noise processes (ms) and systematic effects (eloss)
  - It is a local and incremental fit (dynamic states)

    We can do simultaneously fitting & patter recognition



Propagator (GEANE)

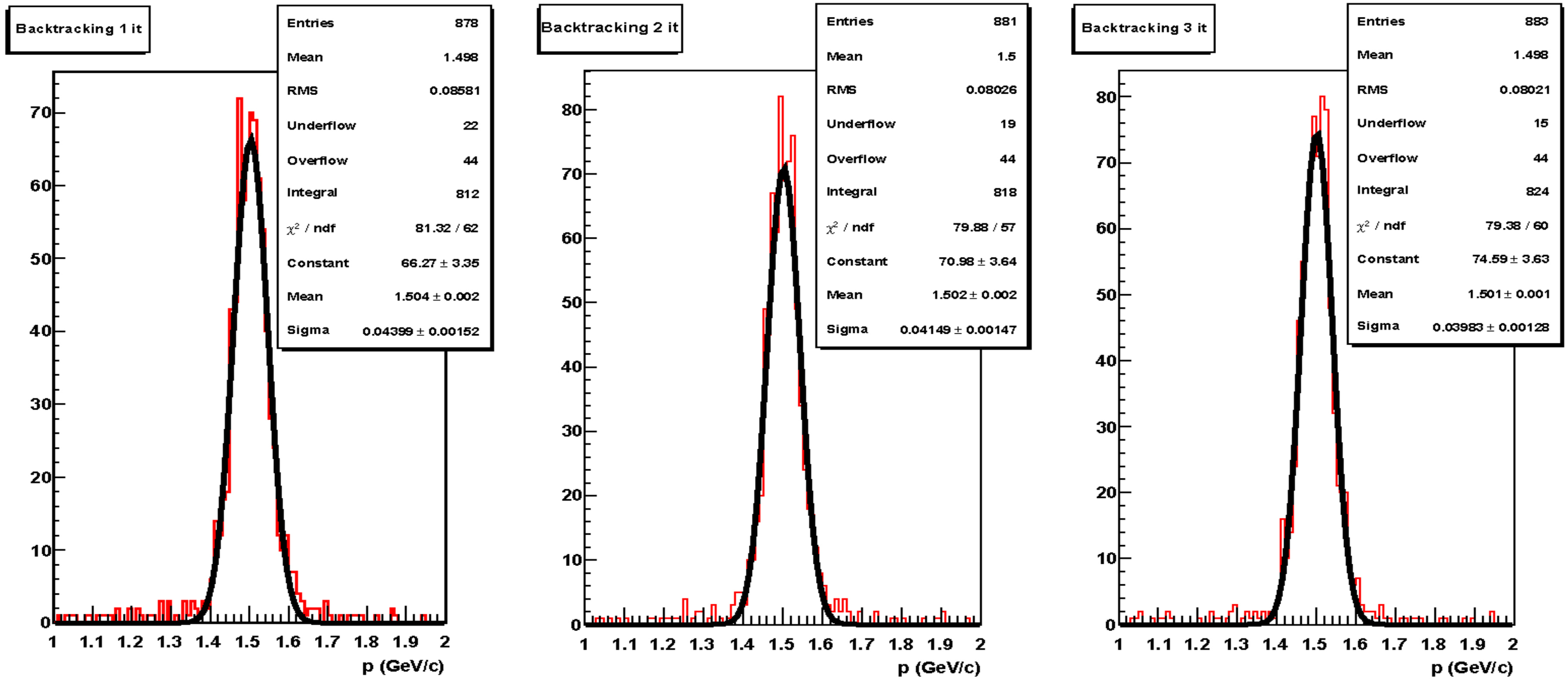
transport matrix

$$e_i = \boldsymbol{G}_{i-1,i}[k_{i-1}] \quad (3)$$

- **Method № 1**, "smoothing": we use the previously shown formulas
- **Method № 2**, "backtracking": we use option "b" of geane
- **Method № 3**, "double Kalman filter": we use option "b" of geane **and** we make a weighted average between the last step and the previous iteration.

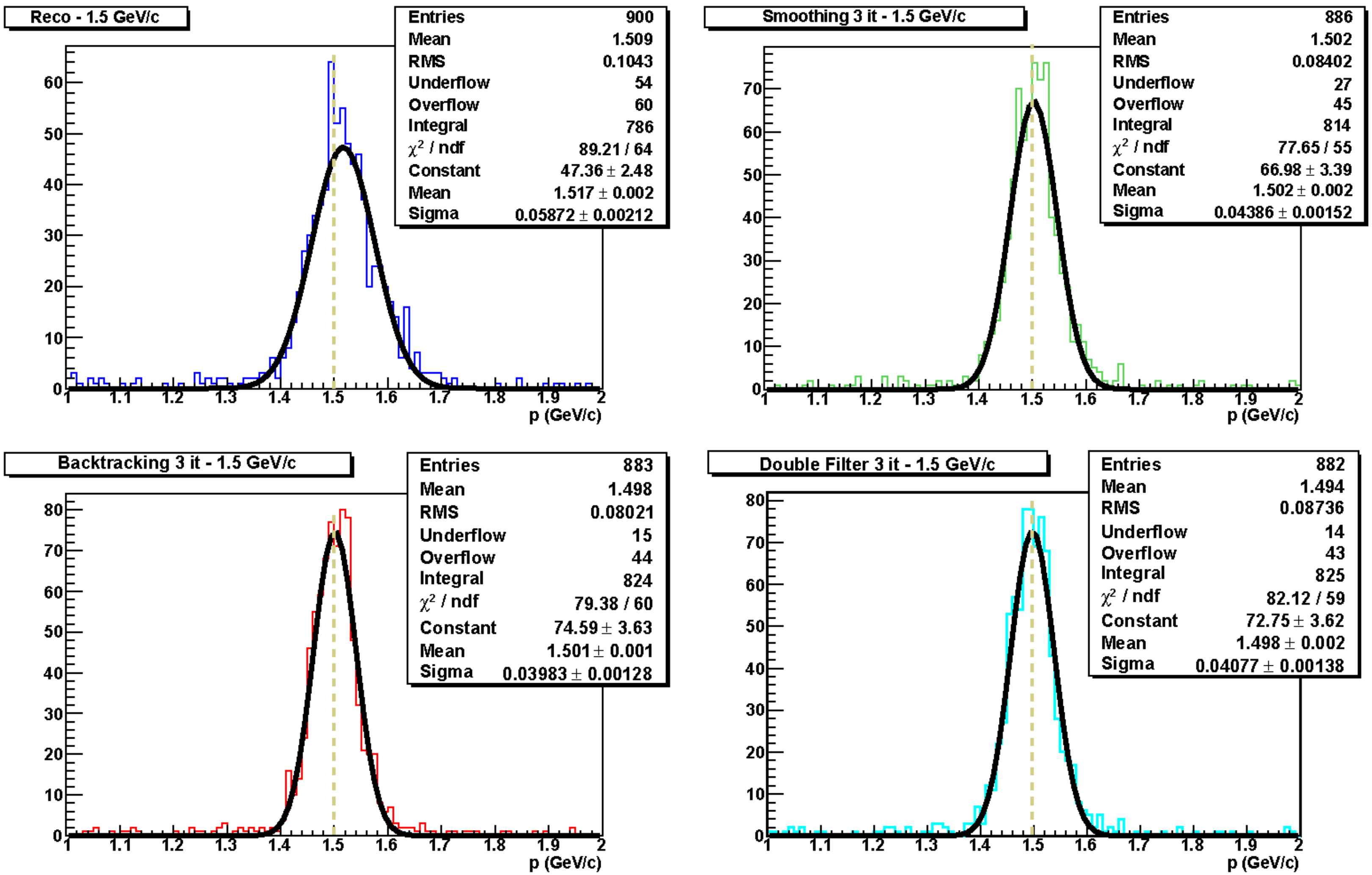
Three different ways to go backward

Three iterations tested in STT



Results with the three methods (stt ONLY)

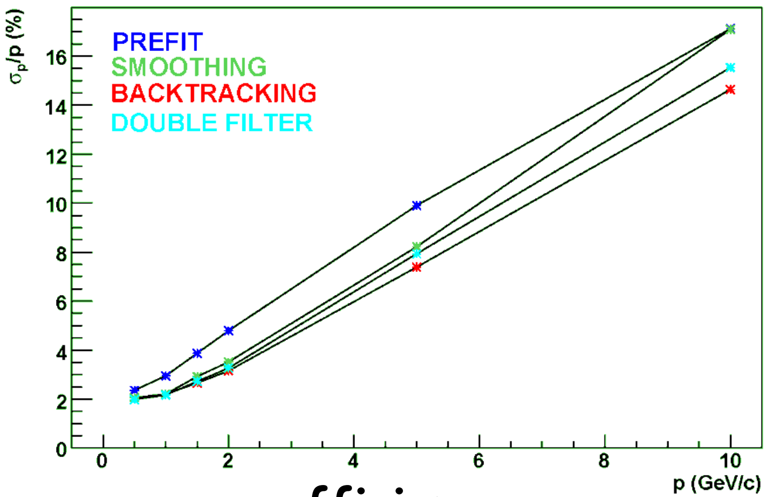
- Method № 1, "smoothing": we use the previously shown formulas
- Method № 2, "backtracking": we use option "b" of geane
- Method № 3, "double Kalman filter": we use option "b" of geane **and** we make a weighted average between the last step and the previous iteration.



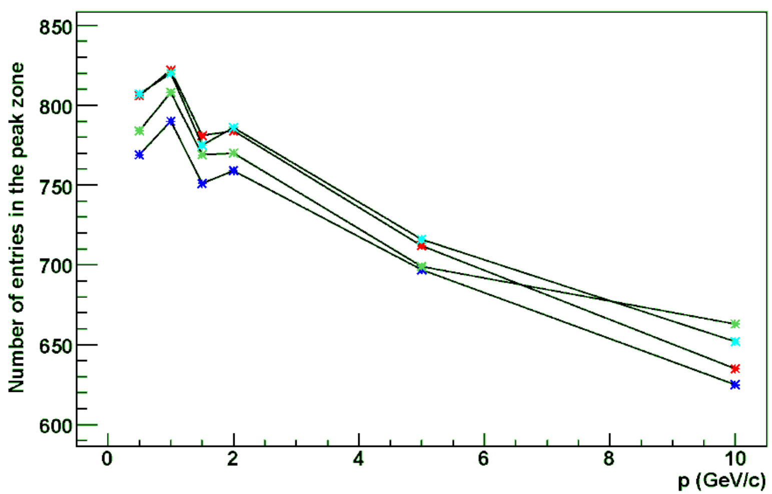
Backtracking and double filter look better than smoothing!

Results at different momenta with the three methods

resolution



efficiency



resolution table

p (GeV/c)	σ(p)/p percentage			
	prefit	smooth.	btrack.	double filt.
0.5	2.36	2.06	1.99	1.98
1.	2.95	2.21	2.20	2.17
1.5	3.87	2.91	2.6	2.72
2.	4.79	3.52	3.15	3.27
5.	9.90	8.22	7.39	7.94
10.	17.12	17.10	14.64	15.54

Beyond Kalman: not gauusian models (electrons)

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Computer Physics Communications 154 (2003) 131–142

Computer Physics Communications

www.elsevier.com/locate/cpc

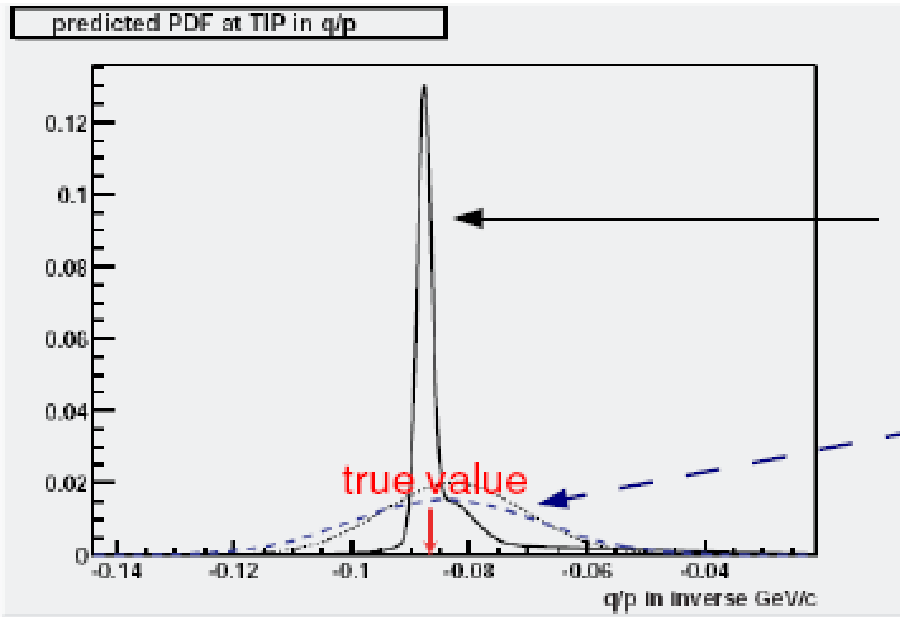
A Gaussian-mixture approximation of the Bethe–Heitler model of electron energy loss by bremsstrahlung

R. Frühwirth

Institut für Hochenergiephysik der Österreichischen Akademie der Wissenschaften, Vienna, Austria

Received 23 April 2003

- idea: describe P(z) as a weighted sum of several Gaussian distributions
- split fit in different component, one for each Gauss. add up the final results.



From Adam, Fruhwirth,Strandlie,Todorov. This plot shows the result from a single fit.

PDF of sum of components from Gaussian Sum Filter

PDF of result from normal Kalman Filter, describing e-loss with single gaus

- but
  - number of components can become very large ... must run many fits to fit a single track

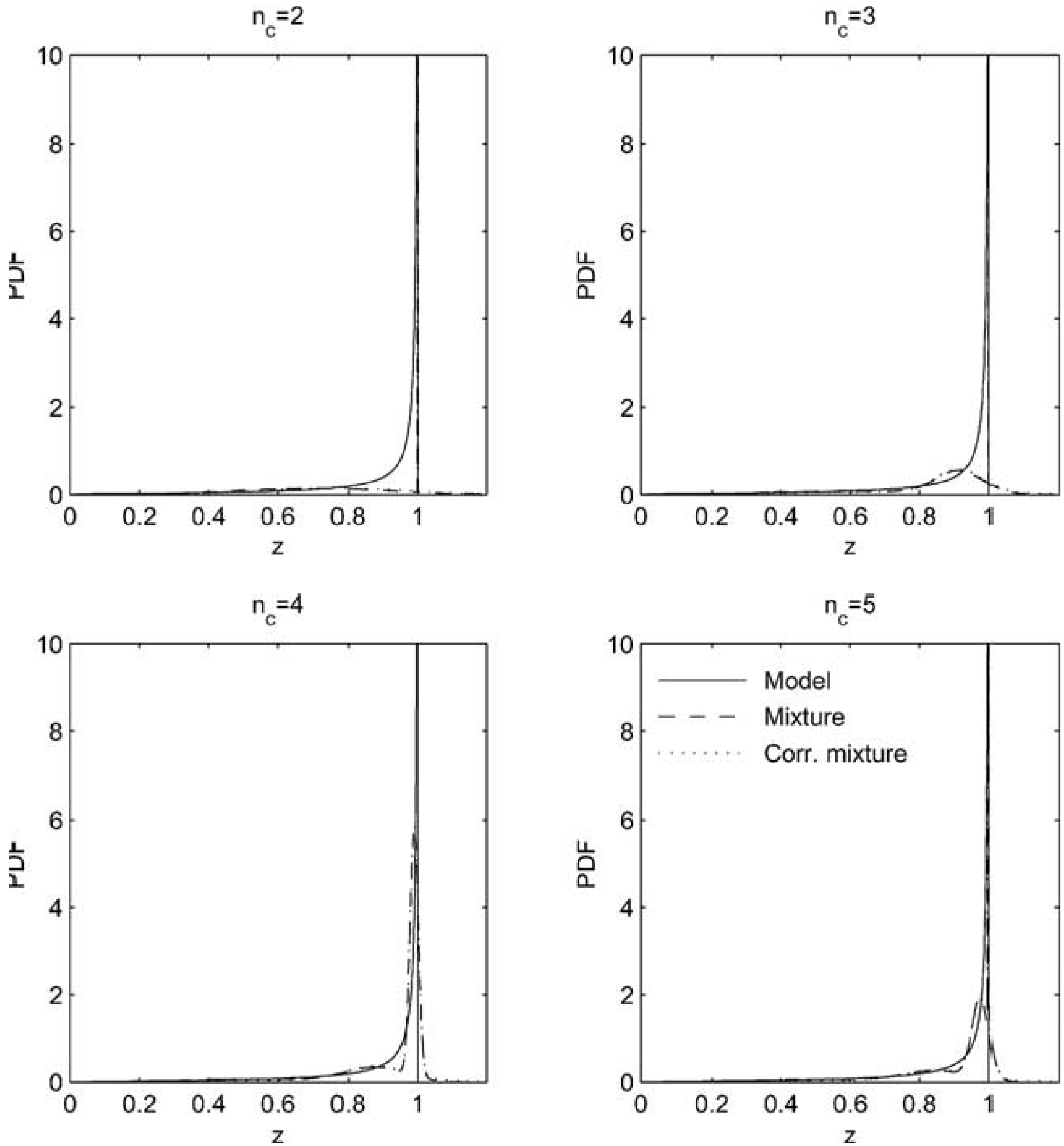


Fig. 9. PDFs of the model and of CDF-mixtures for $t = 0.02$.

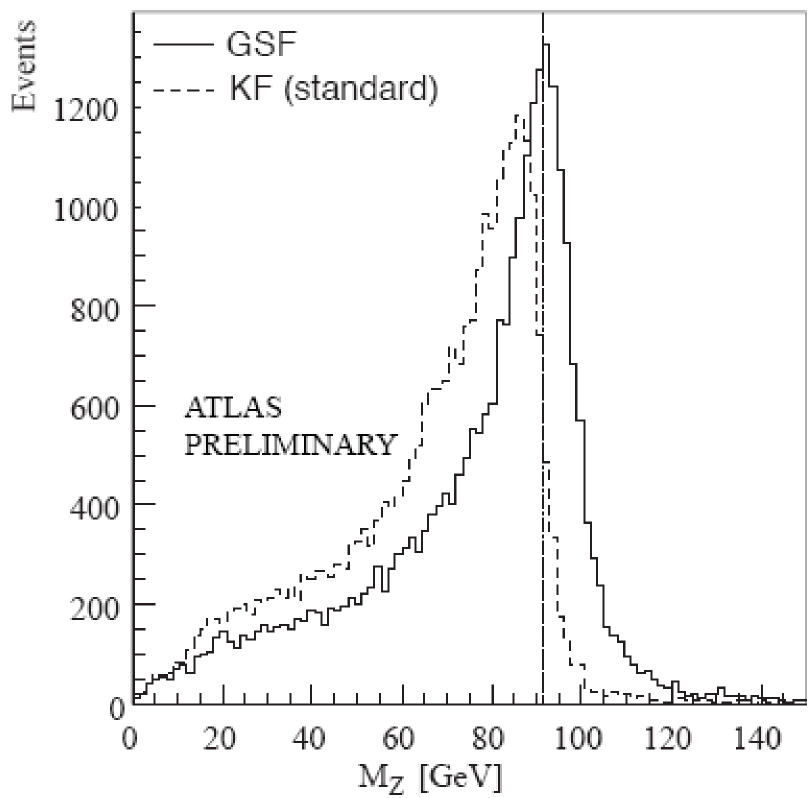


Figure 6. Improvement of the Z mass distribution for $Z \to e^+e^-$, when the electron tracks are (re-)fitted with the Gaussian sum filter instead of the standard ATLAS Kalman filter. In the standard Kalman filter, energy loss is applied only as mean ionisation loss and even approximated as being Gaussian distributed.

Conclusions

Progressive fits takes into account magnetic field , energy loss and multiple scattering effects

The tracker is an essential part of this technique

Virtual MC interfaces are useful to assure the use of the necessary software.

Old tools can be useful!

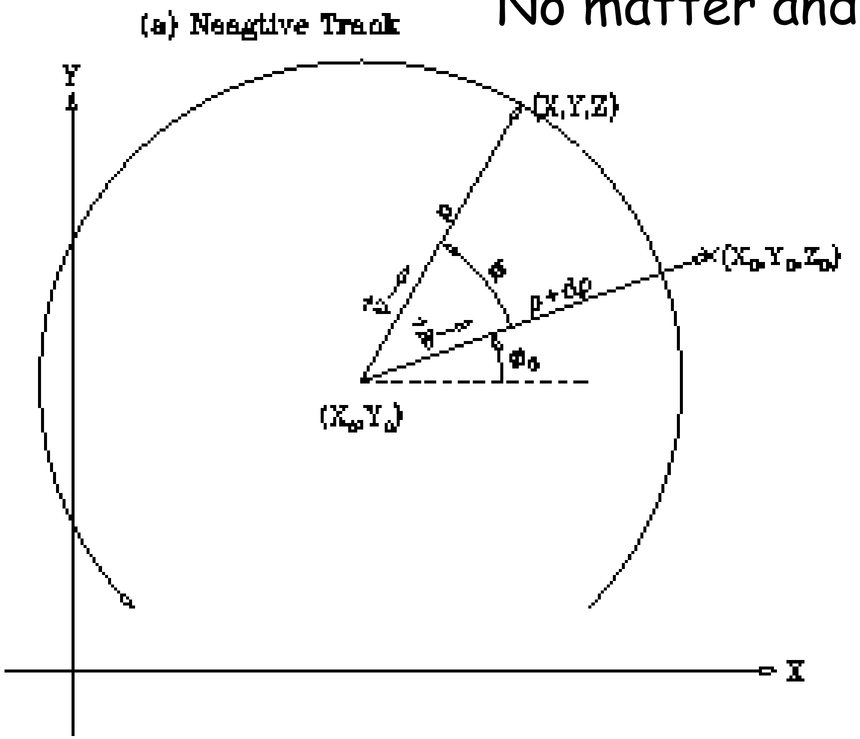
END

GEANE: a short story

To my knowledge (CERN biased), here is a brief tracker story:

The Helix

No matter and uniform magnetic field



$$x(s) = x_0 + R_H\left[\cos\left(\Phi_0 + h\,s\,\frac{\cos\lambda}{R_H}\right) - \cos\Phi_0\right]$$

$$y(s) = y_0 + R_H\left[\sin\left(\Phi_0 + h\,s\,\frac{\cos\lambda}{R_H}\right) - \sin\Phi_0\right]$$

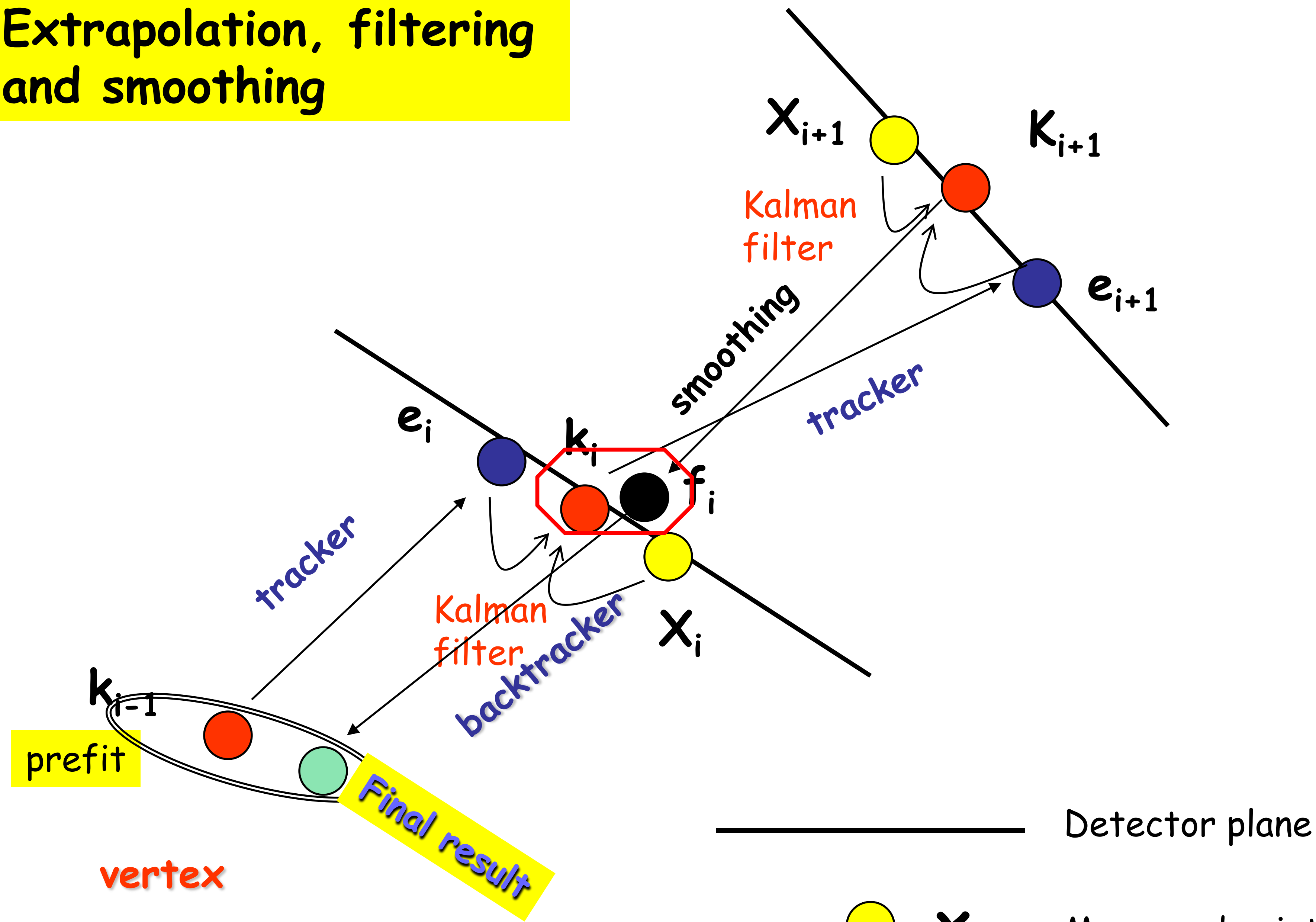
$$z(s) = z_0 + s\sin\lambda$$

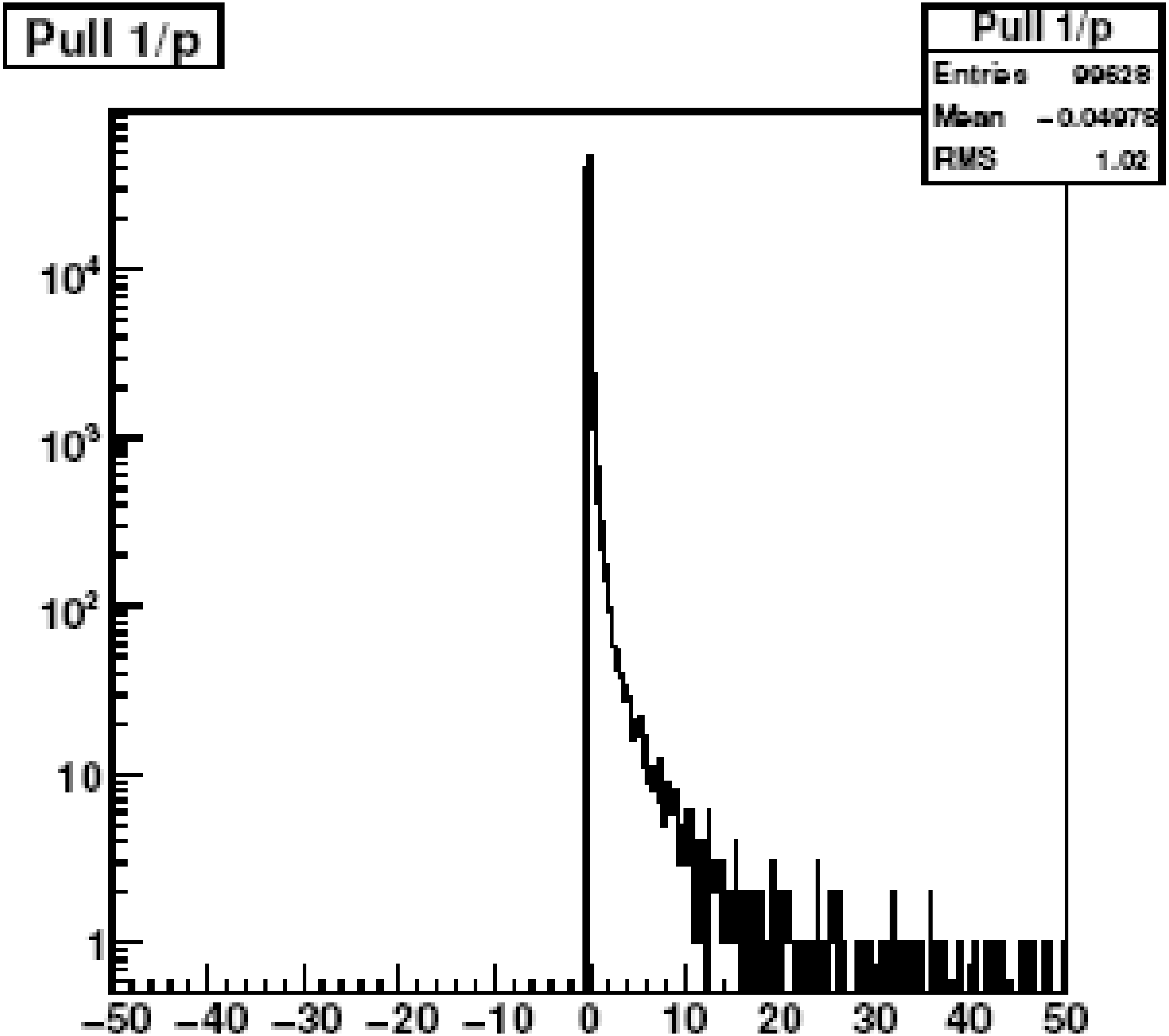
s = track length
$P(x_0, y_0, z_0)$ = starting point
λ = dip angle
Φ_0 = azimuthal angle
R_H = radius
$h = -sign(qB_z)$

B // z → two planes:
xy : circle
z - s : straight line

$$s = \frac{R}{h\cos\lambda}\left[\arctan\left(\frac{R\sin\Phi_0 + (y - y_0)}{R\cos\Phi_0 + (x - x_0)}\right) - \Phi_0\right]$$

Extrapolation, filtering and smoothing




Pull 1/p
Pull 1/p
Entries 99628
Mean −0.04978
RMS 1.02
10^4
10^3
10^2
10
1
−50 −40 −30 −20 −10 0 10 20 30 40 50

Thin gaseous absorbers: The Urban distribution

- excitation macroscopic cross sections Σ_1 and Σ_2:

$$\Sigma_i = C \frac{f_i}{E_i} \frac{\ln(2m\beta^2\gamma^2/e_i) - \beta^2}{\ln(2m\beta^2\gamma^2/I) - \beta^2}(1-r) \ , \quad i = 1,2$$

$$I = 16Z^{0.9} \text{ (eV)} \ , \quad f_2 = \begin{cases} 0 \text{ if } Z \le 2 \\ 2/Z \text{ if } Z > 2 \end{cases} \ , \quad f_1 = 1 - f_2$$

$$e_2 = 10Z^2 \text{ (eV)} \ , \quad e_1 = \left(\frac{I}{e_2^{f_2}}\right)^{1/f_1} \ , \quad r = 0.4 \ , \quad C = \frac{E_{\text{med}}}{\Delta x} \ ,$$

and $E_{\text{med}} \equiv (\mathrm{d}E/\mathrm{d}x) \cdot \Delta x$ is the energy lost in the absorber of thickness Δx;

- ionization macroscopic cross section Σ_3:

$$\Sigma_3 = C \frac{E_{\max}}{I(E_{\max} + I)\ln((E_{\max} + I)/I)} r$$

- number of total collisions N_c:

$$N_c = (\Sigma_1 + \Sigma_2 + \Sigma_3)\Delta x = N_1 + N_2 + N_3 \ . \qquad (8)$$

$$E = (\Sigma_1 e_1 + \Sigma_2 e_2 + \Sigma_3 E_3)\Delta x = N_1 e_1 + N_2 e_2 + N_3 E_3 \ , \qquad (9)$$

where e_1 and e_2 are the two fixed excitation energies of the model and E_3 is the energy lost by δ-electron emission. This is a stochastic quantity that follows approximately the distribution [?]:

δ-ray tail

$$E_3 \sim g(E) \text{ where } g(E) = \frac{I(E_{\text{max}} + I)}{E_{\text{max}}} \frac{1}{E^2} \ , \quad I < E < E_{\text{max}} + I \ . \qquad (10)$$

In GEANT3 and GEANT4 the energy E is obtained by eq. (9) by sampling N_1, N_2 and N_3 from the Poisson distribution and E_3 from $g(E)$.

Therefore, the sampling of the excitation energy is

$$E_e = N_1 e_1 + N_2 e_2 \ , \qquad (11)$$

with E_1 and E_2 are consant and N_1, N_2 are sample from the Poisson distribution, whereas the delta ray ionization energy is sampled as:

$$E_i = \sum_{j=1}^{N_3} \frac{I}{1 - u(E_{\text{max}}/(E_{\text{max}} + I))} \ . \qquad (12)$$

Truncation of the Urban tail of distribution

$$\frac{I(E_{max}+I)}{E\max}\int_I^{E_\alpha}\frac{1}{E^2}\,\mathrm{d}E=\frac{(E_{max}+I)}{E_{max}}\frac{E_\alpha-I}{E_\alpha}=\alpha$$

$$\to E_\alpha=\frac{I}{1-\alpha\,E_{max}/(E_{max}+I)}$$

The mean and variance of the truncated distribution are:

$$\begin{aligned}\langle E_3\rangle &= \frac{I(E_{max}+I)}{E_{max}}\int_I^{E_\alpha}\frac{1}{E}\,\mathrm{d}E=\frac{I(E_{max}+I)}{E_{max}}\ln\left(\frac{E_\alpha}{I}\right)\ ,\\ \langle E_3^2\rangle &= \frac{I(E_{max}+I)}{E_{max}}\int_I^{E_\alpha}\,\mathrm{d}E=\frac{I(E_{max}+I)}{E_{max}}(E_\alpha-I)\ ,\\ \sigma_\alpha^2(E_3) &= <E_3^2>-<E_3>^2\ . \end{aligned}\qquad(13)$$

Then, the error propagation applied to eq. (9), where a random sum is present, where N_1, N_2, N_3 and E_3 are random variables, gives:

$$\sigma^2(E)=\langle N_1\rangle\,e_1^2+\langle N_2\rangle\,e_2^2+\langle N_3\rangle\,\langle E_3\rangle^2+\sigma^2[E_3]\,\langle N_3\rangle\qquad(14)$$

Is the Urban distribution a good model?

Comparison with an “exact” model
in the case of a thin gas layer

SECONDARY AND TOTAL IONIZATION

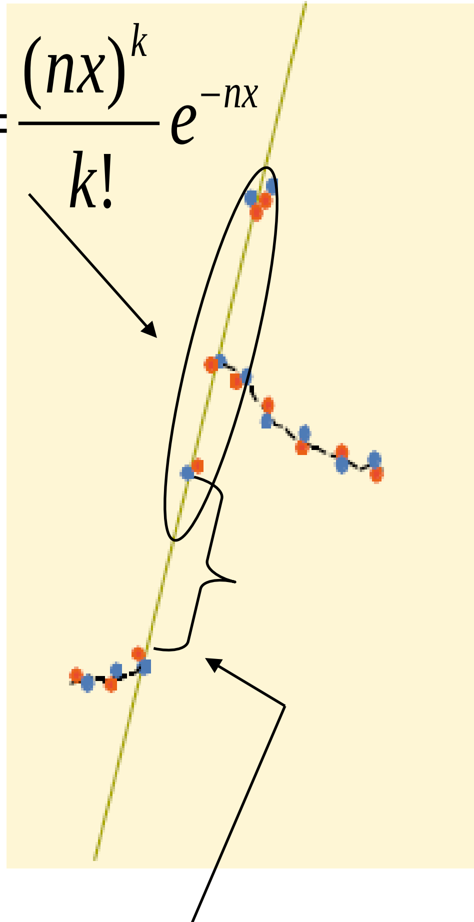
CLUSTERS AND DELTA ELECTRONS:

MIP particle:

	Argon	CO2
Cluster/cm	26	35

Effective cluster/cm: $n_{MIP}\, dE/dx / (dE/dx)_{min}$

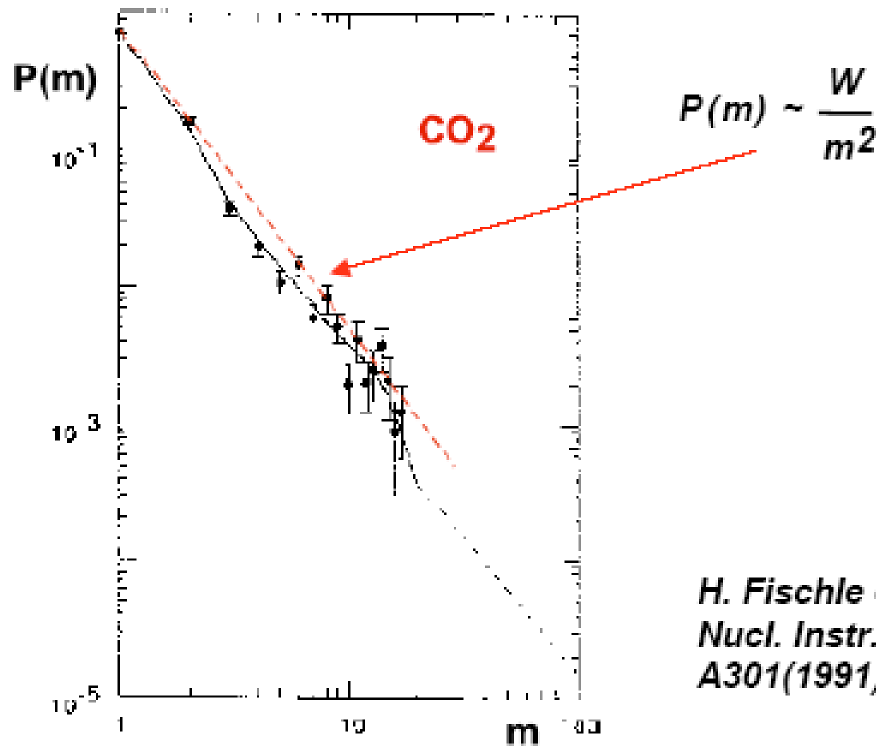
$$p(n) = \frac{(nx)^k}{k!} e^{-nx}$$



$$l(x) = n\, e^{-nx}$$

N: total ion-electron pairs **N/n ~ 2.8**

CLUSTER SIZE DISTRIBUTION:



H. Fischle et al, Nucl. Instr. and Meth. A301(1991)202