Convergence of correlations in the n.n. 2D Ising model: primary fields
[and the stress-energy tensor]

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[Sample of a critical 2D Ising configuration (with two disorders), © Clément Hongler (EPFL)]

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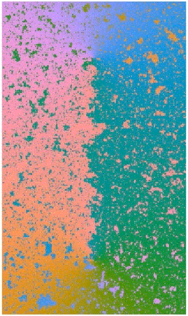
2D Ising model: convergence of correlations at criticality

[see also arXiv:1605.09035]



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- N.n. 2D Ising model: combinatorics
  - ◦ dimers and fermionic observables
  - ◦ discrete holomorphicity at criticality
  - ◦ spinor observables and spin correlations
  - ◦ spin-disorder formalism

- Spin correlations at criticality
  - ◦ Riemann boundary value problems for holomorphic spinors in continuum
  - ◦ Convergence [Ch.–Hongler–Izyurov]

- Other primary fields: $\sigma, \mu, \varepsilon, \psi, \overline{\psi}$
  - ◦ Convergence and fusion rules
  - ◦ Construction of mixed correlations via Riemann boundary value problems

- [Stress-energy tensor]
  - ◦ (Some) discrete version of T and $\overline{T}$
  - ◦ Convergence [Ch.–Glazman–Smirnov]

Nearest-neighbor Ising (or Lenz-Ising) model in 2D

Definition: *Lenz-Ising model* on a planar graph G^* (dual to G) is a random assignment of $+/-$ spins to vertices of G^* (faces of G)

Q: I heard this is called a (site) percolation?
A: .. according to the following probabilities:

$$\begin{aligned} \mathbb{P}\left[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}\right] &\propto \exp\left[\beta \sum\nolimits_{e=\langle uv \rangle} J_{uv}\sigma_u\sigma_v\right] \\ &\propto \prod\nolimits_{e=\langle uv \rangle : \sigma_u \neq \sigma_v} x_{uv}\,, \end{aligned}$$

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv \rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

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Disclaimer: 2D, nearest-neighbor, no external magnetic field.

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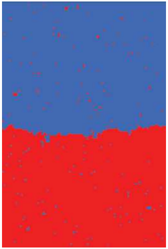
where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv\rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

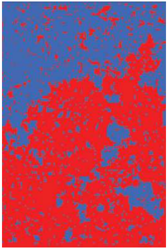
- It is also convenient to use the parametrization $x_{uv} = \tan(\frac{1}{2}\theta_{uv})$.
- Working with subgraphs of *regular lattices*, one can consider the *homogeneous model* in which all x_{uv} are equal to each other.

Phase transition (e.g., on $\mathbb{Z}^2$)

E.g., Dobrushin boundary conditions: $+1$ on (ab) and -1 on (ba):



$x < x_{\mathrm{crit}}$



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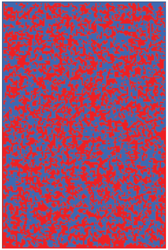
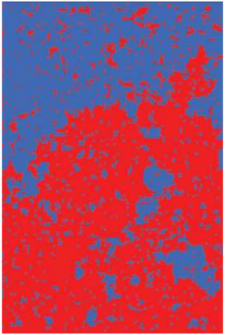


$x > x_{\mathrm{crit}}$

- Ising (1925): no phase transition in 1D $\rightsquigarrow$ doubts about 2+D;
- Peierls (1936): existence of the phase transition in 2D;
- Kramers-Wannier (1941): $x_{\text{self-dual}} = \sqrt{2} - 1 = \tan(\frac{1}{2} \cdot \frac{\pi}{4})$;
- Onsager (1944): sharp phase transition at $x_{\text{crit}} = \sqrt{2} - 1$.

At criticality (e.g., on $\mathbb{Z}^2$):

- Kaufman-Onsager(1948-49), Yang(1952): scaling exponent $\frac{1}{8}$ for the magnetization. [via spin-spin correlations in $\mathbb{Z}^2$ at $x \uparrow x_{\mathrm{crit}}$]
- At criticality, for $\Omega_\delta \to \Omega$ and $u_\delta \to u \in \Omega$, it should be $\mathbb{E}_{\Omega_\delta}[\sigma_{u_\delta}] \asymp \delta^{\frac{1}{8}}$ as $\delta \to 0$.



$x = x_{\mathrm{crit}}$

• **Question:** Convergence of (rescaled) spin correlations and conformal covariance of their **scaling limits in arbitrary planar domains:**

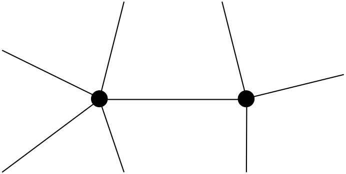
$$\begin{aligned} \delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}[\sigma_{u_{1,\delta}} \ldots \sigma_{u_{n,\delta}}] \quad &\to \quad \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_\Omega \\ &= \quad \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle_{\varphi(\Omega)} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}} \end{aligned}$$

• In the infinite-volume setup other techniques are available, notably "exact bosonization" approach due to J. Dubédat.

2D Ising model as a dimer model (on a non-bipartite graph) [Fisher, Kasteleyn ('60s+),..., Kenyon, Dubédat ('00s+),...]

- **Partition function** $\mathcal{Z} = \sum_{\sigma \in \{\pm 1\}^{V(G^*)}} \prod_{e=\langle uv \rangle : \sigma_u \neq \sigma_v} x_{uv}$

- There exist various representations of the 2D Ising model via dimers on an auxiliary graph



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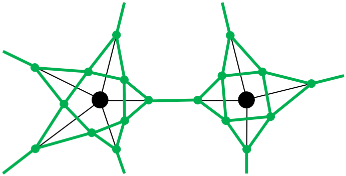

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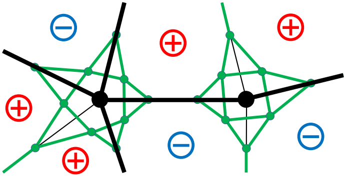
- There exist various representations of the 2D Ising model via dimers on an auxiliary graph: e.g. 1-to-$2^{|V(G)|}$ correspondence of $\{\pm 1\}^{V(G^*)}$ with dimers on **this G_F**



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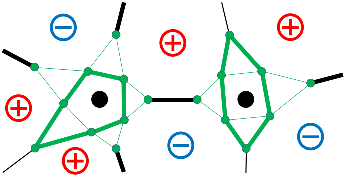

- **Kasteleyn's theory:** $\mathcal{Z} = \mathbf{Pf}[\mathbf{K}]$ [$\mathbf{K} = -\mathbf{K}^\top$ is a weighted adjacency matrix of G_F]

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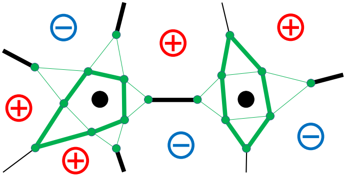

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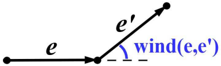
- **Kac–Ward formula (1952–..., 1999–...):** $\mathcal{Z}^2 = \det[\mathrm{Id} - \mathrm{T}]$,

$$\mathrm{T}_{e,e'} = \begin{cases} \exp[\frac{i}{2}\mathbf{wind}(\boldsymbol{e},\boldsymbol{e'})]\cdot(x_e x_{e'})^{1/2} \\ 0 \end{cases}$$



[**is equivalent** to the **Kasteleyn theorem for dimers on G_F**]

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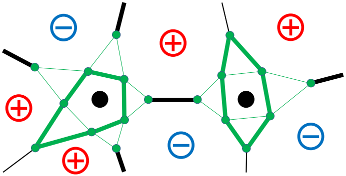


- **Kasteleyn's theory:** $\mathcal{Z} = \mathrm{Pf}[\mathbf{K}]$ [$\mathbf{K} = -\mathbf{K}^\top$ is a weighted adjacency matrix of G_F]

- Note that $V(G_F) \cong \{$*oriented edges and corners of G*$\}$

- **Local relations** for the entries $\mathbf{K}^{-1}_{a,e}$ and $\mathbf{K}^{-1}_{a,c}$ of the inverse Kasteleyn (or the inverse Kac–Ward) matrix:

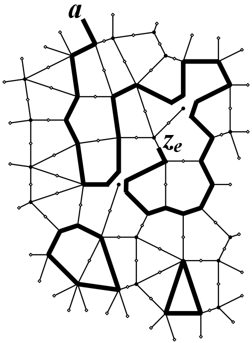
(an equivalent form of) the identity $\mathbf{K}\cdot\mathbf{K}^{-1} = \mathrm{Id}$

Fermionic observables: combinatorial definition [Smirnov'00s]

For an oriented edge a and a midedge z_e (similarly, for a corner c),

$$F_G(a, z_e) \;:=\; \overline{\eta}_a \sum_{\omega \in \mathrm{Conf}_G(a, z_e)} \Big[e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)} \prod_{\langle uv \rangle \in \omega} x_{uv} \Big]$$

where η_a denotes the (once and forever fixed) square root of the direction of a.



- The factor $e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)}$ does not depend on the way how ω is split into non-intersecting loops and a path $a \rightsquigarrow z_e$.

- **Via dimers on G_F:** $F_G(a, c) = \overline{\eta}_c \mathrm{K}^{-1}_{c,a}$

$$F_G(a, z_e) = \overline{\eta}_e \mathrm{K}^{-1}_{e,a} + \overline{\eta}_{\overline{e}} \mathrm{K}^{-1}_{\overline{e},a}$$

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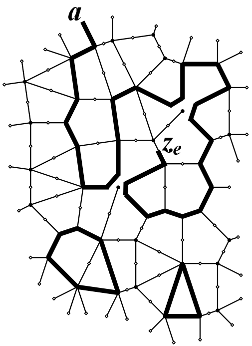

• **Local relations: at criticality,** can be thought of as some (strong) form of **discrete Cauchy–Riemann equations.**

• **Boundary conditions $F(a,z_e)\in\overline{\eta}_{\overline{e}}\mathbb{R}$** ($\overline{e}$ is oriented outwards) uniquely determine F as a solution to an appropriate

discrete Riemann-type boundary value problem.

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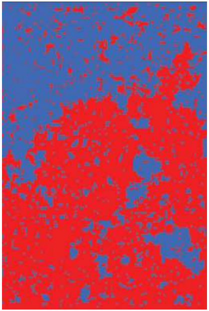
Fermionic observables *per se* can be used

- to construct (discrete) martingales for growing **interfaces** and then to study their convergence to SLE curves *[Smirnov(2006), ..., Ch.–Duminil-Copin –Hongler–Kemppainen–Smirnov(2013)]*

- to analyze the **energy density field** *[Hongler–Smirnov, Hongler (2010)]*

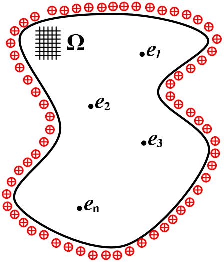
$$\varepsilon_e \ := \ \delta^{-1} \cdot [\, \sigma_{e^-} \sigma_{e^+} - \varepsilon_e^\infty \,]$$



where $e^{\pm}$ are the two neighboring faces separated by an edge e

- but **more involved** ones are needed to study **spin correlations**

Energy density: convergence and conformal covariance



- Three local primary fields: $\mathbf{1}$, $\boldsymbol{\sigma}$ (spin), $\boldsymbol{\varepsilon}$ (energy density); Scaling exponents: $\mathbf{0}$, $\frac{\mathbf{1}}{\mathbf{8}}$, $\mathbf{1}$.

- **Theorem:** *[Hongler–Smirnov, Hongler (2010)]* If $\Omega_\delta \to \Omega$ and $e_{k,\delta} \to z_k$ as $\delta \to 0$, then

$$\boldsymbol{\delta^{-n}} \cdot \mathbb{E}^+_{\Omega_\delta}[\varepsilon_{e_{1,\delta}} \dots \varepsilon_{e_{n,\delta}}] \underset{\delta\to 0}{\to} \mathcal{C}^n_\varepsilon \cdot \langle \boldsymbol{\varepsilon_{z_1} \dots \varepsilon_{z_n}} \rangle^+_{\boldsymbol{\Omega}}$$

where $\mathcal{C}_\varepsilon$ is a lattice-dependent constant,

$$\langle \varepsilon_{z_1} \dots \varepsilon_{z_n} \rangle^+_\Omega = \langle \varepsilon_{\varphi(z_1)} \dots \varepsilon_{\varphi(z_n)} \rangle^+_{\Omega'} \cdot \prod_{s=1}^n |\varphi'(u_s)|$$

for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\langle \boldsymbol{\varepsilon_{z_1} \dots \varepsilon_{z_n}} \rangle^+_{\mathbb{H}} = i^n \cdot \mathrm{Pf}\left[(z_s - z_m)^{-1}\right]^{2n}_{s,m=1}, \quad z_s = \overline{z}_{2n+1-s}\,.$$

- **Ingredients:** convergence of basic **fermionic observables** (via Riemann-type b.v.p.) and (built-in) **Pfaffian formalism**

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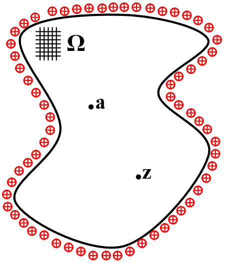

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- **Riemann-type boundary value problem to consider (sketch):**

○ $\boldsymbol{f^{[\eta]}_{\Omega}(a,z)}$ is holomorphic in Ω except at a given point $a \in \Omega$;

○ $\mathrm{Im}\big[\, f^{[\eta]}_{\Omega}(a,\zeta)\sqrt{\tau(\zeta)}\,\big] = 0$, where $\tau(\zeta)$ is the counterclockwise (clockwise for free boundary conditions) tangent vector at $\zeta \in \partial\Omega$;

○ $f^{[\eta]}_{\Omega}(a,z) = \frac{(2i)^{-1/2}\eta}{z-a} + \ldots$ as $z \to a$, where η should be thought of as a square root of the direction of the edge $a_\delta \to a$.

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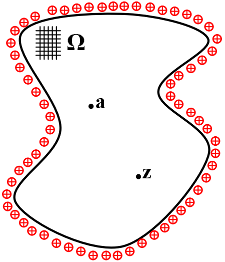

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∘ $f^{[\eta]}_{\Omega}(a,z) = \frac{(2i)^{-1/2}\eta}{z-a} + \ldots = 2^{-\frac{1}{2}}[e^{-i\frac{\pi}{4}}\eta \cdot f_{\Omega}(a,z) + e^{i\frac{\pi}{4}}\overline{\eta} \cdot f^{\dagger}_{\Omega}(a,z)]$

- $\langle \psi_z \psi_a \rangle^{+}_{\Omega} := f_\Omega(a,z)$, $\langle \psi_z \overline{\psi}_a \rangle^{+}_{\Omega} := f^{\dagger}_{\Omega}(a,z)$ and $\varepsilon_z := i\psi_z\overline{\psi}_z$.

Spin correlations and spinor observables: combinatorics

- spin configurations on G^*
  $\leftsquigarrow\!\!\!\rightsquigarrow$ domain walls on G
  $\leftsquigarrow\!\!\!\rightsquigarrow$ dimers on G_F
- **Kasteleyn's theory:** $\mathcal{Z} = \mathrm{Pf}[\mathbf{K}]$

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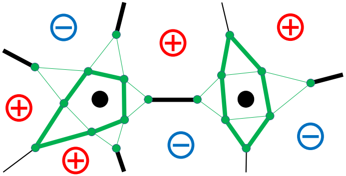

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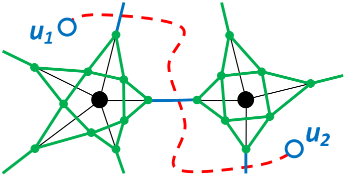

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[$\mathbf{K} = -\mathbf{K}^\top$ is a weighted adjacency matrix of G_F]

- **Claim**:

$$\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] \;=\; \frac{\mathrm{Pf}[\,\mathbf{K}_{[u_1,\ldots,u_n]}\,]}{\mathrm{Pf}[\,K\,]},$$

where $\mathbf{K}_{[u_1,\ldots,u_n]}$ is obtained from $\mathbf{K}$ by changing the sign of its entries on **slits linking** $u_1, \ldots, u_n$ (and, possibly, u_{out}) pairwise.

- **More invariant way** to think about entries of $\mathbf{K}^{-1}_{[u_1,\ldots,u_n]}$:

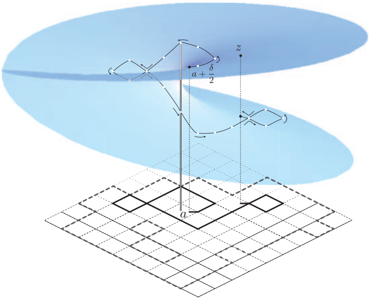
double-covers of G branching over $u_1, \ldots, u_n$

Spin correlations and spinor observables: combinatorics

Main tool: spinors on the double cover $[\Omega_\delta; u_1, \ldots, u_n]$.

$$F_{\Omega_\delta}(z) := \left[\mathcal{Z}^+_{\Omega_\delta}[\sigma_{u_1}\ldots\sigma_{u_n}]\right]^{-1} \cdot \sum_{\omega \in \mathrm{Conf}_{\Omega_\delta}(u_1^{\to}, z)} \phi_{u_1,\ldots,u_n}(\omega, z) \cdot x_{\mathrm{crit}}^{\#\mathrm{edges}(\omega)},$$

$$\phi_{u_1,\ldots,u_n}(\omega, z) := e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\omega))} \cdot (-1)^{\#\mathrm{loops}(\omega \setminus \mathrm{p}(\omega))} \cdot \mathrm{sheet}(\mathrm{p}(\omega), z).$$



- $\mathrm{wind}(\mathrm{p}(\gamma))$ is the winding of the path $\mathrm{p}(\gamma) : u_1^{\to} = u_1 + \frac{\delta}{2} \rightsquigarrow z$;
- $\#\mathrm{loops}$ – those containing an odd number of $u_1, \ldots, u_n$ inside;
- $\mathrm{sheet}(\mathrm{p}(\gamma), z) = +1$, if $\mathrm{p}(\gamma)$ defines z, and -1 otherwise.
- **Note that $F(z^\sharp) = -F(z^\flat)$** if $z^\sharp, z^\flat$ lie over the same edge of Ω_δ.

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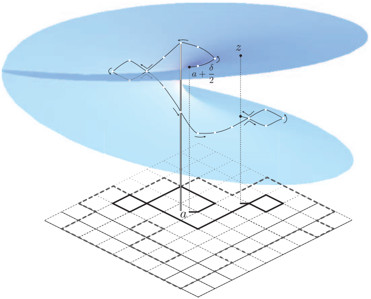


Claim:

$$F_{\Omega_\delta}(u_1 + \tfrac{3\delta}{2}) = \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1+2\delta}\ldots\sigma_{u_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1}\ldots\sigma_{u_n}]}$$

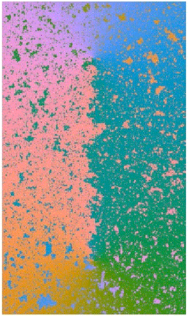
Thus, **spatial derivatives of spin correlations** can be studied via the analysis of spinor observables.

- **Remark:** Both fermionic and spinor observables can be introduced using **spin-disorder formalism of Kadanoff and Ceva.**

Spin-disorder formalism of Kadanoff and Ceva

• Recall that spins σ_u are assigned to the faces of G. Given (an even number of) *vertices* $v_1, \dots, v_m$, link them pairwise by a collection of paths $\varkappa = \varkappa^{[v_1,\dots,v_m]}$ and replace x_e by x_e^{-1} for all $e \in \varkappa$. Denote

$$\langle \boldsymbol{\mu_{v_1} \dots \mu_{v_m}} \rangle_{\boldsymbol{G}} \; := \; \mathcal{Z}_G^{[v_1,\dots,v_m]} / \mathcal{Z}_G \,.$$



[two disorders inserted]

• Equivalently, one may think of the Ising model on a double-cover $G^{[v_1,\dots,v_m]}$ that branches over each of $v_1, \dots, v_m$ with the *spin-flip symmetry* constrain $\sigma_{u^\sharp} = -\sigma_{u^\flat}$ if $u^\sharp$ and $u^\flat$ lie over the same face of G. Let

$$\langle \boldsymbol{\mu_{v_1} \dots \mu_{v_m} \sigma_{u_1} \dots \sigma_{u_n}} \rangle_{\boldsymbol{G}} \; := \; \mathbb{E}_{G^{[v_1,\dots,v_m]}}[\sigma_{u_1} \dots \sigma_{u_n}] \cdot \langle \mu_{v_1} \dots \mu_{v_m} \rangle_G \,.$$

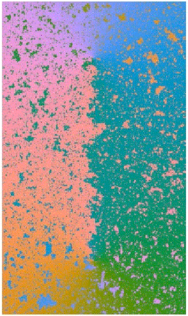
• By definition, $\langle \boldsymbol{\mu_{v_1} \dots \mu_{v_m} \sigma_{u_1} \dots \sigma_{u_n}} \rangle_{\boldsymbol{G}}$ changes the sign when one of the faces u_k goes around of one of the vertices v_s.

Spin-disorder formalism of Kadanoff and Ceva

- By definition, $\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_G$ changes the sign when one of the faces u_k goes around of one of the vertices v_s.
- For a corner c lying in a face $u(c)$ near a vertex $v(c)$, denote $\chi_c := \mu_{v(c)} \sigma_{u(c)}$.
- **Claim:**

$$\langle \chi_{c_1} \ldots \chi_{c_{2k}} \rangle_G = \mathrm{Pf}[\, \langle \chi_{c_p} \chi_{c_q} \rangle_G \,]_{p,q=1}^{2k}$$

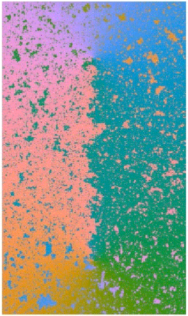
and $\langle \chi_d \chi_c \rangle_G = \mathrm{K}_{c,d}^{-1}$ provided that all the vertices $v(c_q)$ are pairwise distinct.



[two disorders inserted]

- **Remark:** This also works in presence of other spins and disorders. The antisymmetry $\langle \chi_d \chi_c \rangle_G = -\langle \chi_c \chi_d \rangle_G$ is caused by the sign change of the corresponding spin-disorder correlation.

Spin-disorder formalism of Kadanoff and Ceva



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- By definition, $\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_G$ changes the sign when one of the faces u_k goes around of one of the vertices v_s.
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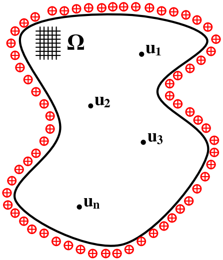
and $\langle \chi_d \chi_c \rangle_G = \mathrm{K}_{c,d}^{-1}$ provided that all the vertices $v(c_q)$ are pairwise distinct.

- The "corner" (resp., "edge") values of the special spinor observable on $[\Omega_\delta; u_1, \ldots, u_n]$ discussed above can be written as

$$\frac{\langle \chi_c \mu_{v(u_1^{\rightarrow})} \sigma_{u_2} \ldots \sigma_{u_n} \rangle_{\Omega_\delta}}{\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_{\Omega_\delta}} \quad \left(\text{resp., } \frac{\langle \psi_z \mu_{v(u_1^{\rightarrow})} \sigma_{u_2} \ldots \sigma_{u_n} \rangle_{\Omega_\delta}}{\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_{\Omega_\delta}} \right),$$

[ψ_z can be thought of as linear combinations of nearby χ_c's]

Spin correlations: convergence and conformal covariance



- Three local primary fields:
  $\mathbf{1}$, $\boldsymbol{\sigma}$ (spin), ε (energy density);
  Scaling exponents: $\mathbf{0}$, $\mathbf{\frac{1}{8}}$, $\mathbf{1}$.

- **Theorem:** *[Ch.–Hongler–Izyurov (2012)]*
  If $\Omega_\delta \to \Omega$ and $u_{k,\delta} \to u_k$ as $\delta \to 0$, then

$$\boldsymbol{\delta}^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_{1,\delta}} \ldots \sigma_{u_{n,\delta}}] \underset{\delta \to 0}{\to} \mathcal{C}^n_\sigma \cdot \langle \boldsymbol{\sigma}_{u_1} \ldots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\boldsymbol{\Omega}}$$

where $\mathcal{C}_\sigma$ is a lattice-dependent constant,

$$\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^{+}_{\Omega} = \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle^{+}_{\Omega'} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}$$

for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\Big[\langle \boldsymbol{\sigma}_{u_1} \ldots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\mathbb{H}} \Big]^2 = \prod_{1 \leqslant s \leqslant n} (2 \operatorname{Im} u_s)^{-\frac{1}{4}} \times \sum_{\beta \in \{\pm 1\}^n} \prod_{s<m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\frac{\beta_s \beta_m}{2}}$$

Spin correlations: convergence and conformal covariance

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  $\mathbf{1}$, $\boldsymbol{\sigma}$ (spin), ε (energy density);
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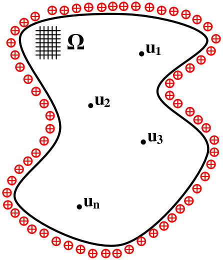

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General strategy: • in discrete: encode **spatial derivatives** as values of discrete holomorphic spinors F^δ that solve some

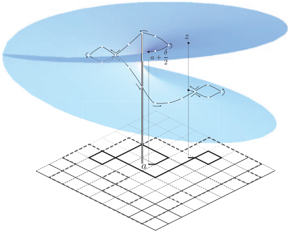
discrete Riemann-type boundary value problems;

• discrete→continuum: prove convergence of F^δ to the solutions f of the similar continuous b.v.p. [non-trivial technicalities];

• continuum→discrete: find the limit of (spatial derivatives of) using the convergence $F^\delta \to f$ [via coefficients at singularities].

Spin correlations: convergence and conformal covariance

Example: to handle $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:



- $g(z^\sharp) \equiv -g(z^\flat)$, branches over u;
- $\mathrm{Im}\big[\, g(\zeta)\sqrt{\tau(\zeta)}\,\big] = 0$ for $\zeta \in \partial\Omega$;
- $g(z) = \frac{(2i)^{-1/2}}{\sqrt{z-u}} + \dots$

Spin correlations: convergence and conformal covariance

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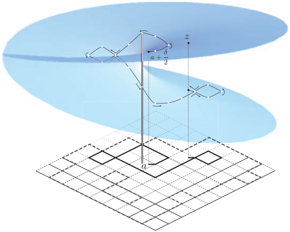

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- $g(z) = \frac{(2i)^{-1/2}}{\sqrt{z-u}}[1+2\mathcal{A}_\Omega(u)(z-u)+\ldots]$

Claim: If Ω_δ converges to Ω as $\delta \to 0$, then

- $(2\delta)^{-1}\log\Big[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2\delta}]\,/\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\Big] \to \mathrm{Re}[\,\mathcal{A}_\Omega(u)\,]\,;$
- $(2\delta)^{-1}\log\Big[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2i\delta}]\,/\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\Big] \to -\,\mathrm{Im}\,[\,\mathcal{A}_\Omega(u)\,]\,.$

Spin correlations: convergence and conformal covariance

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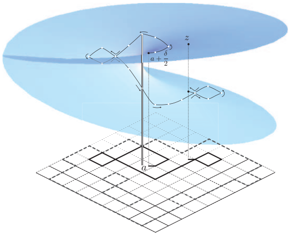

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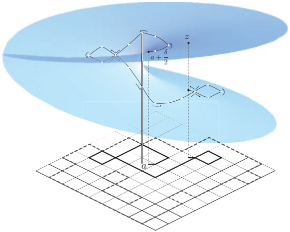
- $(2\delta)^{-1}\log\left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2i\delta}]\,/\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\right] \to -\,\mathrm{Im}\,[\,\mathcal{A}_\Omega(u)\,]\,.$

Conformal covariance $\frac{1}{8}$: for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w)) \cdot (\phi'(w))^{1/2}\,;$
- $\mathcal{A}_\Omega(z) = \mathcal{A}_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \phi''(z)/\phi'(z)\,.$

Spin correlations: convergence and conformal covariance

Example: to handle $\mathbb{E}^+_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:



- $\circ\ g(z^\sharp) \equiv -g(z^\flat)$, branches over u;
- $\circ\ \mathrm{Im}\big[\,g(\zeta)\sqrt{\tau(\zeta)}\,\big] = 0$ for $\zeta \in \partial\Omega$;
- $\circ\ g(z) = \frac{(2i)^{-1/2}}{\sqrt{z-u}}[1+2\mathcal{A}_\Omega(u)(z-u)+\ldots]$

Claim: If Ω_δ converges to Ω as $\delta \to 0$, then

$$\circ \quad (2\delta)^{-1}\log\left[\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_\delta+2\delta}]\,/\,\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_\delta}]\right] \to \mathrm{Re}[\,\mathcal{A}_\Omega(u)\,]\,;$$

$$\circ \quad (2\delta)^{-1}\log\left[\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_\delta+2i\delta}]\,/\,\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_\delta}]\right] \to -\,\mathrm{Im}\,[\,\mathcal{A}_\Omega(u)\,]\,.$$

Quite a lot of technical work is needed, e.g.:

- to handle tricky boundary conditions [Dirichlet for $\int \mathrm{Re}[f^2dz]$];
- to prove convergence, incl. near singularities [complex analysis];
- to recover the **normalization** of $\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1}\ldots\sigma_{u_n}]$ [probability].

Spin correlations: multiplicative normalization

We define $\langle \boldsymbol{\sigma}_{u_1} \ldots \boldsymbol{\sigma}_{u_n} \rangle_{\boldsymbol{\Omega}}^{+} := \exp[\int \mathcal{L}(u_1, \ldots, u_n)]$, where

$$\mathcal{L}_\Omega(u_1, \ldots, u_n) := \sum_{s=1}^{n} \mathrm{Re}\,[\,\mathcal{A}_\Omega(u_s; u_1, ..., \hat{u}_s, ..., u_n) du_s\,]\,,$$

where the coefficients $\mathcal{A}_\Omega(\ldots)$ are defined via solutions to similar Riemann boundary values problems and the normalization satisfies

$$\begin{array}{rcll} \langle \sigma_{u_1} ... \sigma_{u_n} \rangle_\Omega^{+} & \sim & \langle \sigma_{u_1} ... \sigma_{u_{n-1}} \rangle_\Omega^{+} \cdot \langle \sigma_{u_n} \rangle_\Omega^{+} & \text{as} \;\; u_n \to \partial\Omega\,, \\ \langle \sigma_{u_1} \sigma_{u_2} \rangle_\Omega^{+} & \sim & |u_2 - u_1|^{-1/4} & \text{as} \;\; u_2 \to u_1 \in \Omega\,. \end{array}$$

Spin correlations: multiplicative normalization

We define $\langle \sigma_{u_1} \dots \sigma_{u_n} \rangle^{+}_{\Omega} := \exp[\,\int \mathcal{L}(u_1, \dots, u_n)\,]$, where

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- $\circ$ $g(z^{\sharp}) \equiv -g(z^{\flat})$ is a holomorphic spinor on $[\Omega; u_1, ..., u_n]$;
- $\circ$ $\mathrm{Im}\big[\, g(\zeta)(\tau(\zeta))^{\frac{1}{2}}\,\big] = 0$ for $\zeta \in \partial\Omega$;
- $\circ$ $g(z) = e^{i\frac{\pi}{4}} c_s \cdot (z - u_s)^{-\frac{1}{2}} + \dots$ for some (unknown) $c_s \in \mathbb{R}$, $s \geqslant 2$;
- $\circ$ $g(z) = 2^{-\frac{1}{2}} e^{-i\frac{\pi}{4}} (z - u_1)^{-\frac{1}{2}} [1 + 2\mathcal{A}_{\Omega}(u_1; u_2, ..., u_n)(z - u_1) + \dots]$

Spin correlations: **multiplicative normalization**

We define $\langle \boldsymbol{\sigma}_{u_1} \ldots \boldsymbol{\sigma}_{u_n} \rangle_{\Omega}^{+} := \exp[\,\int \mathcal{L}(u_1, \ldots, u_n)\,]$, where

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$$\begin{array}{rcll} \langle \sigma_{u_1} ... \sigma_{u_n} \rangle_{\Omega}^{+} & \sim & \langle \sigma_{u_1} ... \sigma_{u_{n-1}} \rangle_{\Omega}^{+} \cdot \langle \sigma_{u_n} \rangle_{\Omega}^{+} & \text{as} \ \ u_n \to \partial\Omega\,, \\ \langle \sigma_{u_1} \sigma_{u_2} \rangle_{\Omega}^{+} & \sim & |u_2 - u_1|^{-1/4} & \text{as} \ \ u_2 \to u_1 \in \Omega\,. \end{array}$$

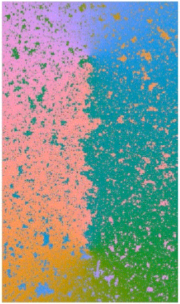
Remarks: • The fact that $\mathcal{L}_{\Omega,n}$ is a closed differential form and the existence of an appropriate multiplicative normalization are not a priori clear but can be deduced **along the proof of convergence.**

• This also works for **mixed fixed/free boundary conditions** and/or in **multiply connected domains.** (No explicit formulae!)

[not published, a part of a larger project in progress...]

Mixed correlations: **convergence**

[Ch.–Hongler–Izyurov (2016, in progress)]



• Convergence of **mixed correlations: spins (σ), disorders (μ), fermions (ψ), energy densities (ε)** (in multiply connected domains Ω, with mixed fixed/free boundary conditions $\mathfrak{b}$) to conformally covariant limits that can be defined via solutions to appropriate Riemann-type boundary value problems in Ω.

• Standard **CFT fusion rules**

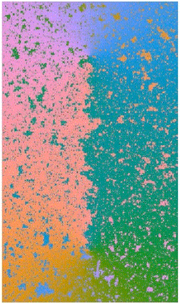
$$\begin{array}{lll} \sigma\mu \rightsquigarrow \eta\psi + \overline{\eta}\overline{\psi}, & \psi\sigma \rightsquigarrow \mu, & \psi\mu \rightsquigarrow \sigma, \\ i\psi\overline{\psi} \rightsquigarrow \varepsilon, & \sigma\sigma \rightsquigarrow 1+\varepsilon, & \mu\mu \rightsquigarrow 1-\varepsilon \end{array}$$

can be deduced from properties of solutions to Riemann-type b.v.p.

• **Stress-energy tensor:** *[Ch.–Glazman–Smirnov (2016)]*

Mixed correlations: convergence

[Ch.–Hongler–Izyurov (2016, in progress)]



- Convergence of **mixed correlations: spins (σ), disorders (μ), fermions (ψ), energy densities (ε)** (in multiply connected domains Ω, with mixed fixed/free boundary conditions $\mathfrak{b}$) to conformally covariant limits that can be defined via solutions to appropriate Riemann-type boundary value problems in Ω.

- Standard **CFT fusion rules**, e.g. $\sigma\sigma \rightsquigarrow 1 + \varepsilon$:

$$\langle \sigma_{u'} \sigma_u \ldots \rangle_\Omega^{\mathfrak{b}} = |u' - u|^{-\frac{1}{4}} \left[\langle \ldots \rangle_\Omega^{\mathfrak{b}} + \tfrac{1}{2} |u' - u| \langle \varepsilon_u \ldots \rangle_\Omega^{\mathfrak{b}} + \ldots \right],$$

can be deduced from properties of solutions to Riemann-type b.v.p.

- **More details:** `arXiv:1605.09035`, `arXiv:1[6]??.?????`

Mixed correlations: properties (fusion rules) and existence

(I) Each $\langle \mu_{v_1} \ldots \mu_{v_n} \sigma_{u_1} \ldots \sigma_{u_m} \rangle_\Omega^{\mathfrak{b}}$ is a spinor defined on the Riemann surface of the function $\left[\prod_{l=1}^{n} \prod_{s=1}^{m} (v_l - u_s)\right]^{\frac{1}{2}}$.
As some of the points $v_1, \ldots, v_n$ approach $u_1, .., u_m$ along the rays $v_s - u_s \in \eta_s^2 \mathbb{R}$, where $|\eta_s| = 1$, there exist limits

$$\langle \psi_{u_1}^{[\eta_1]} \ldots \psi_{u_k}^{[\eta_k]} \mathcal{O}[\mu, \sigma] \rangle_\Omega^{\mathfrak{b}} := \lim_{v_s \to u_s} |(v_1 - u_1) \ldots (v_k - u_k)|^{\frac{1}{4}} \langle \mu_{v_1} \sigma_{u_1} \ldots \mu_{v_k} \sigma_{u_k} \mathcal{O}[\mu, \sigma] \rangle_\Omega^{\mathfrak{b}}.$$

These (real) limits change signs if η_s is replaced by $-\eta_s$ and are anti-symmetric with respect to the order in which ψ's are written.

Mixed correlations: properties (fusion rules) and existence

The spin-disorder correlations $\langle \mu_{v_1} \ldots \mu_{v_n} \sigma_{u_1} \ldots \sigma_{u_m} \rangle_\Omega^{\mathfrak{b}}$ lead to

(I) $\langle \psi_{u_1}^{[\eta_1]} \ldots \psi_{u_k}^{[\eta_k]} \mathcal{O}[\mu,\sigma] \rangle_\Omega^{\mathfrak{b}} :=$

$$\lim_{v_s \to u_s} |(v_1 - u_1) \ldots (v_k - u_k)|^{\frac{1}{4}} \langle \mu_{v_1} \sigma_{u_1} \ldots \mu_{v_k} \sigma_{u_k} \mathcal{O}[\mu,\sigma] \rangle_\Omega^{\mathfrak{b}}.$$

(II) These functions satisfy Pfaffian identites (fermionic Wick rules). Moreover, they depend on η's in a real-linear way:

$$\langle \psi_z^{[\eta]} \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}} = 2^{-\frac{1}{2}} \left[e^{-i\frac{\pi}{4}} \eta \cdot \langle \psi_z \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}} + e^{i\frac{\pi}{4}} \overline{\eta} \cdot \langle \overline{\psi}_z \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}} \right].$$

One has $\overline{\langle \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}}} = \langle \mathcal{O}[\psi^*,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}}$ with $\psi_z^* := \overline{\psi}_z$, $\overline{\psi}_z^* := \psi_z$.

Each of the functions $\langle \psi_z \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}}$ is holomorphic in z and each of $\langle \overline{\psi}_z \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}}$ is anti-holomorphic in z. Moreover,

$$\langle \overline{\psi}_z \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}} = \tau(z) \langle \psi_z \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}} \text{ for } z \in \partial\Omega\,,$$

where $\tau(z)$ denotes the (properly oriented) tangent vector to $\partial\Omega$.

Mixed correlations: properties (fusion rules) and existence

The spin-disorder correlations $\langle \mu_{v_1} \ldots \mu_{v_n} \sigma_{u_1} \ldots \sigma_{u_m} \rangle_\Omega^{\mathfrak{b}}$ lead to

(I) $\langle \psi_{u_1}^{[\eta_1]} \ldots \psi_{u_k}^{[\eta_k]} \mathcal{O}[\mu, \sigma] \rangle_\Omega^{\mathfrak{b}} :=$
$\lim_{v_s \to u_s} |(v_1 - u_1) \ldots (v_k - u_k)|^{\frac{1}{4}} \langle \mu_{v_1} \sigma_{u_1} \ldots \mu_{v_k} \sigma_{u_k} \mathcal{O}[\mu, \sigma] \rangle_\Omega^{\mathfrak{b}}$.

(II) $\langle \psi_z^{[\eta]} \mathcal{O}[\psi, \mu, \sigma] \rangle_\Omega^{\mathfrak{b}} =$
$2^{-\frac{1}{2}} \left[e^{-i\frac{\pi}{4}} \eta \cdot \langle \psi_z \mathcal{O}[\psi, \mu, \sigma] \rangle_\Omega^{\mathfrak{b}} + e^{i\frac{\pi}{4}} \overline{\eta} \cdot \langle \overline{\psi}_z \mathcal{O}[\psi, \mu, \sigma] \rangle_\Omega^{\mathfrak{b}} \right]$.

Moreover, $\langle \overline{\psi}_z \mathcal{O}[\psi, \mu, \sigma] \rangle_\Omega^{\mathfrak{b}} = \tau(z) \langle \psi_z \mathcal{O}[\psi, \mu, \sigma] \rangle_\Omega^{\mathfrak{b}}$ for $z \in \partial\Omega$.

(III) Each of the functions $\langle \psi_z \ldots \rangle_\Omega^{\mathfrak{b}}$ has the following asymptotics (aka operator product expansions) as ψ_z approaches other fields:

$$\langle \psi_z \psi_{z'} \ldots \rangle_\Omega^{\mathfrak{b}} = (z - z')^{-1} \left[\langle \ldots \rangle_\Omega^{\mathfrak{b}} + O(|z - z'|^2) \right], \ \langle \psi_z \overline{\psi}_{z'} \ldots \rangle_\Omega^{\mathfrak{b}} = O(1),$$

$$\langle \psi_z \sigma_u \ldots \rangle_\Omega^{\mathfrak{b}} = 2^{-\frac{1}{2}} e^{\frac{i\pi}{4}} (z - u)^{-\frac{1}{2}} \left[\langle \mu_u \ldots \rangle_\Omega^{\mathfrak{b}} + 4(z - u) \partial_u \langle \mu_u \ldots \rangle_\Omega^{\mathfrak{b}} + \ldots \right],$$

$$\langle \psi_z \mu_v \ldots \rangle_\Omega^{\mathfrak{b}} = 2^{-\frac{1}{2}} e^{\frac{-i\pi}{4}} (z - v)^{-\frac{1}{2}} \left[\langle \sigma_v \ldots \rangle_\Omega^{\mathfrak{b}} + 4(z - v) \partial_v \langle \sigma_v \ldots \rangle_\Omega^{\mathfrak{b}} + \ldots \right],$$

Similar OPEs hold true for the antiholomorphic functions $\langle \overline{\psi}_z \ldots \rangle_\Omega^{\mathfrak{b}}$.

Mixed correlations: properties (fusion rules) and existence

The spin-disorder correlations $\langle \mu_{v_1} \ldots \mu_{v_n} \sigma_{u_1} \ldots \sigma_{u_m} \rangle_\Omega^{\mathfrak{b}}$ lead to

(I) $\langle \psi_{u_1}^{[\eta_1]} \ldots \psi_{u_k}^{[\eta_k]} \mathcal{O}[\mu,\sigma] \rangle_\Omega^{\mathfrak{b}} :=$

$$\lim_{v_s \to u_s} |(v_1 - u_1)\ldots(v_k - u_k)|^{\frac{1}{4}} \langle \mu_{v_1} \sigma_{u_1} \ldots \mu_{v_k} \sigma_{u_k} \mathcal{O}[\mu,\sigma] \rangle_\Omega^{\mathfrak{b}}.$$

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Moreover, $\langle \overline{\psi}_z \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}} = \tau(z) \langle \psi_z \mathcal{O}[\psi,\mu,\sigma] \rangle_\Omega^{\mathfrak{b}}$ for $z \in \partial\Omega$.

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(IV) Denote $\langle \varepsilon_u \mathcal{O}[\varepsilon,\psi,\sigma,\mu] \rangle_\Omega^{\mathfrak{b}} := i \langle \psi_u \overline{\psi}_u \mathcal{O}[\varepsilon,\psi,\sigma,\mu] \rangle_\Omega^{\mathfrak{b}}$. Then

$$\langle \sigma_{u'} \sigma_u \ldots \rangle_\Omega^{\mathfrak{b}} = |u' - u|^{-\frac{1}{4}} \left[\langle \ldots \rangle_\Omega^{\mathfrak{b}} + \tfrac{1}{2} |u' - u| \langle \varepsilon_u \ldots \rangle_\Omega^{\mathfrak{b}} + \ldots \right];$$

$$\langle \mu_{v'} \mu_v \ldots \rangle_\Omega^{\mathfrak{b}} = |v' - v|^{-\frac{1}{4}} \left[\langle \ldots \rangle_\Omega^{\mathfrak{b}} - \tfrac{1}{2} |v' - v| \langle \varepsilon_v \ldots \rangle_\Omega^{\mathfrak{b}} + \ldots \right].$$

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Claim: The set of conditions (I)–(IV) admits a (unique) solution.

Sketch: ∘ $f_{[\Omega; u_1, \ldots, u_n]}^{[\eta]}(a, z) := \langle \psi_z \psi_a^{[\eta]} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_\Omega^{\mathfrak{b}} / \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_\Omega^{\mathfrak{b}}$;

∘ Define all the other correlations starting with these functions;

∘ Prove all other fusion rules **[interplays with convergence(!)].**

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Theorem: *[Ch.-Hongler-Izyurov, 2016]* All mixed correlations of spins, disorders, discrete fermions and energy densities in the Ising model on Ω_δ with boundary conditions $\mathfrak{b}$, after a proper rescaling, converge to their continuous counterparts $\langle \ldots \rangle_\Omega^{\mathfrak{b}}$ as $\delta \to 0$.

Stress-energy tensor *[Ch.–Glazman–Smirnov, arXiv:1604.06339]*

• There exist several ways to introduce a stress-energy tensor as a *local field (function of several nearby spins)* in the 2D Ising model. Presumably, the first was suggested by Kadanoff and Ceva in 1970.

• As $\delta \to 0$, correlations of these *different local fields* should have *the same scaling limits*: CFT correlations of (components of) the holomorphic T_z and anti-holomorphic $\overline{T}_z$ defined ***on a given*** $\boldsymbol{\Omega}$.

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- We would like to have a definition of T_z in discrete, which
  - "geometrically" describes a *perturbation of the metric*,
  - *satisfies (at least, a part of) Cauchy-Riemann equations,*
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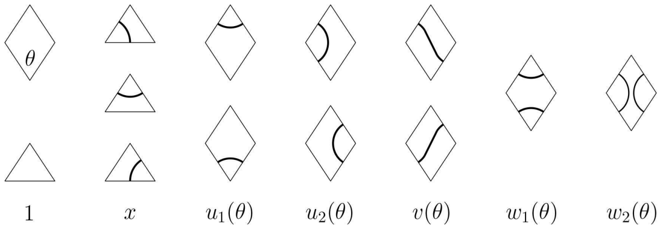
  - ***and hence*** leads to the *correct scaling limits* of correlations.

Remark: in continuum, all the standard properties of T_z (holomorphicity, Schwarzian covariance under conformal maps $\phi : \Omega \to \Omega'$, standard OPEs for TT, $T\sigma$, $T\varepsilon$) can be deduced from the expression of T_z via fermions.

Stress-energy tensor *[Ch.–Glazman–Smirnov, arXiv:1604.06339]*

• Ising model on faces of (a part of) the honeycomb lattice can be equivalently thought of as the loop $O(1)$ model on a discrete domain glued from equilateral triangles $\Longleftrightarrow$ "standard lozenges".

• One can consistently define the loop $O(n)$ model on any (possible, non-flat) discrete domain glued from rhombi and equilateral triangles using the Nienhuis' **"integrable" weights.**



• **Consistency:** $x = u_1(\frac{\pi}{3})$, $x^2 = u_2(\frac{\pi}{3}) = v(\frac{\pi}{3}) = w_1(\frac{\pi}{3})$, $w_2(\frac{\pi}{3}) = 0$.

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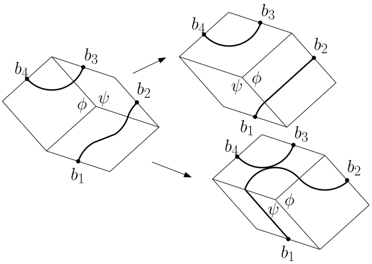
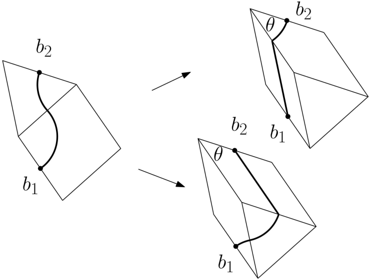
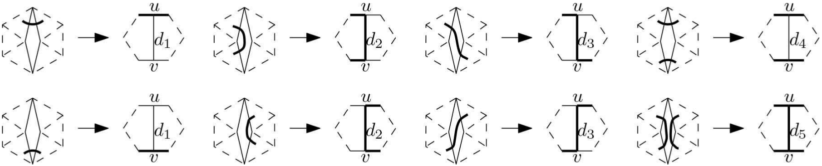
wrt re-gluing of "flat" vertices:





Stress-energy tensor *[Ch.–Glazman–Smirnov, arXiv:1604.06339]*

- **Definition:** Let m be a midline of some hexagon in a discrete domain Ω_δ. We deform the lattice by gluing an additional tiny rhombus of angle $\theta \to 0$ along m, denote the new partition function by $\mathcal{Z}_{\Omega_\delta}(m,\theta)$, and define $\boldsymbol{T_{\Omega_\delta}(m) := cst + \frac{d}{d\theta} \log \mathcal{Z}_{\Omega_\delta}(m,\theta)\big|_{\theta=0}}$
- In fact, one can work with pictures drawn on the original lattice:



weighted by $d_1 := u_1'(0)$, $d_2 := u_2'(0)$, $d_3 := v'(0)$, $d_4 := w_1'(0)$, $d_5 := w_2'(0)$.

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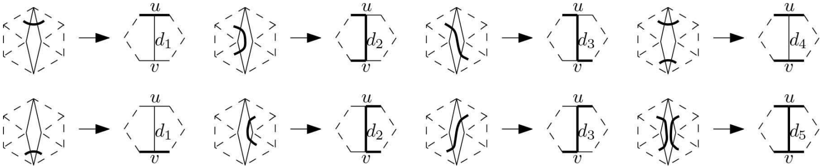

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• For the loop $O(1)$ model, one has $d_4 + d_5 = 2d_1 = -2d_3$. This allows one to *rewrite all these sums via fermions* and leads to the cancelation of main terms in all contributions except of type d_2.

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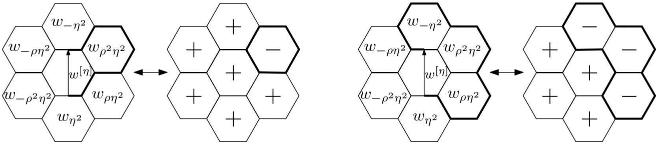
• At the same time, $T(m)$ can be thought of as a *local field:*



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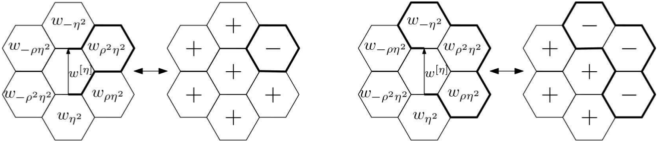


- **Theorem:** Let $\Omega_\delta \to \Omega$ and m_δ be a midline of a hexagon $w_\delta \to w \in \Omega$ oriented in the direction τ. Then

$$\delta^{-2}\mathbb{E}^{+}_{\Omega_\delta}[\boldsymbol{T}(\boldsymbol{m}_\delta)] \to \mathrm{Re}[\boldsymbol{\tau}^2\langle \boldsymbol{T_w}\rangle^{+}_{\Omega}].$$

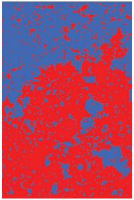
- Since the question is essentially reduced to the convergence of fermions, similar results can be proved for multi-point correlations.

Some research routes and open questions

• Better understanding of "geometric" observables at criticality:
e.g., probability distributions on topological classes of domain walls.

• Near-critical (massive) regime $x - x_{\mathrm{crit}} = m \cdot \delta$: convergence of correlations, massive SLE_3 curves and loop ensembles.

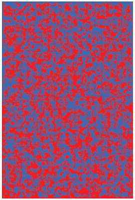
• **Super-critical regime:** e.g., convergence of interfaces to SLE_6 curves for any fixed $x > x_{\mathrm{crit}}$ [known only for $x = 1$ (percolation)]



$x = x_{\mathrm{crit}}$

• **Renormalization**

$$(x - x_{\mathrm{crit}}) \cdot \delta^{-1} \xrightarrow{\text{fixed } x > x_{\mathrm{crit}},\ \delta \to 0} \infty$$



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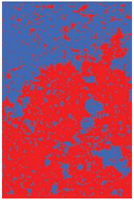

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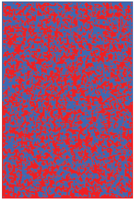

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THANK YOU!