Localization in interacting fermionic chains with quasi-random disorder

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Many Body Localization

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- Numerical evidence of MBL in a huge number of works (starting from Oganesyan, Huse (2007)).
- Experimental evidence of MBL in cold atoms experiments: Bloch et al (2015) by monitoring the time evolution of local observables following a quench (without interaction Inguscio group (2008)).

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- Very few rigorous results. Imbrie (arXiv 2014, PRL 2016) considered a 1d Heisenberg spin chain with random disorder, and showed that MBL rigorous consequence in 1d of an assumption of level attraction.
- A proof of MBL in generality is a challenging problem (single particle description breaks down, full N-particle Schroedinger)

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- Proof of localization of the ground state in Mastropietro CMP2015, PRL2015, CMP2016

The interacting Aubry-Andre' model

- If a_x^+, a_x^-, $x \in \mathbb{Z} \equiv \Lambda$ are spinless creation or annihilation operators on the Fock space verifying $\{a_x^+, a_y^-\} = \delta_{x,y}$, $\{a_x^+, a_y^+\} = \{a_x^-, a_y^-\} = 0$. The Fock space Hamiltonian is

$$H = -\varepsilon(\sum_{x\in\Lambda}(a^+_{x+1}a_x + a^+_{x-1}a^-_x) +$$

$$\sum_{x\in\Lambda} u\cos(2\pi(\omega x + \theta))a^+_x a^-_x + U\sum_{x,y} v(x-y)a^+_x a^-_x a^+_y a^-_y$$

with $v(x-y) = \delta_{y-x,1} + \delta_{x-y,1}$.

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- Spinless version of the model realized in Bloch et al (2015) (here non local interaction).
- Early studies of the extended phase in Mastropietro (1999) and Giamarchi, Mohunna,Vidal (1999)

The Aubry-Andre' model

- In the non interacting case the states are obtained by the antisymmetrization (fermions) of the eigenfunctions of almost Mathieu equation

$$-\varepsilon\psi(x+1) - \varepsilon\psi(x-1) + u\cos(2\pi(\omega x + \theta))\psi(x) = E\psi(x)$$

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- the spectrum is a Cantor set for all irrational ω. For almost every ω, θ the almost Mathieu operator has
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  a)for $\varepsilon/u<\frac{1}{2}$ exponentially decaying eigenfunctions (Anderson localization);
  b)for $\varepsilon/u>\frac{1}{2}$ purely absolutely continuous spectrum (extended quasi-Bloch waves)
- Metal insulator transition (with no interaction) seen experimentally by Inguscio et al (2008)

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- Such remarkable properties are related to a deep connection between the non interacting Aubry-Andre model and the Kolmogorov-Arnold-Moser (KAM) theorem of classical mechanics.
- A crucial assumption of KAM and of the analysis of almost mathieu is that the frequency verify a number theoretical condition called Diophantine condition to deal with small divisors.
- We impose a Diophantine condition on the frequency

$$||\omega x|| \geq C_0 |x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \quad (*)$$

$||.||$ is the norm on the one dimensional torus of period 1.

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- Renormalized expansion around the anti-integrable limit

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- $$H = -\varepsilon\Big(\sum_{x\in\Lambda}(a^+_{x+1}a_x + a^+_{x-1}a^-_x) +$$
$$\sum_{x\in\Lambda} u\cos(2\pi(\omega x+\theta))a^+_x a^-_x + U\sum_{x,y} v(x-y)a^+_x a^-_x a^+_y a^-_y$$

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  with $v(x-y) = \delta_{y-x,1} + \delta_{x-y,1}$.
- If $a^\pm_{\mathbf{x}} = e^{(H-\mu N)x_0} a^\pm_x e^{-(H-\mu N)x_0}$, $\mathbf{x} = (x, x_0)$, $N = \sum_x a^+_x a^-_x$ and μ the chemical potential, the Grand-Canonical imaginary time 2-point correlation is

  $$< \mathbf{T} a^-_{\mathbf{x}} a^+_{\mathbf{y}} > = \frac{Tr e^{-\beta(H-\mu N)} \mathbf{T}\{a^-_{\mathbf{x}} a^+_{\mathbf{y}}\}}{Tr e^{-\beta(H-\mu N)}}$$

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  where $\mathbf{T}$ is the time-order product and μ is the chemical potential.
- We introduce a counterterm ν so that the renormalized chemical potential is fixed to an interaction independent value $u\cos 2\pi(\omega\hat{x}+\theta)$. Morally this is equivalent to fix the density.

Localized regime

Theorem

For ω Diophantine

$$||\omega x|| \geq C_0 |x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \quad (*)$$

$||.||$ *is the norm on the one dimensional torus of period* 1, *and if* θ *verifies*

$$||\omega x \pm 2\theta|| \geq C_0 |x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \quad (**)$$

$u = 1$, $\mu = \cos 2\pi(\omega \hat{x} + \theta) + \nu$ *there exists an* ε_0 *such that, for* $|\varepsilon|, |U| \leq \varepsilon_0$,*it is possible to choose* ν *so that the limit* $\beta \to \infty$

$$| < \mathbf{T} a_{\mathbf{x}}^- a_{\mathbf{y}}^+ > | \leq C e^{-\xi|x-y|} \log(1+\min(|x||y|))^\tau \frac{1}{1 + (\Delta |x_0 - y_0)|)^N} (***)$$

with $\Delta = (1 + \min(|x|, |y|))^{-\tau}$, $\xi = |\log(\max(|\varepsilon|, |U|))|$.

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- A simple consequence of the theorem proof is a localization result formulated fixing the phase θ and varying the chemical potential; namely if we choose $\theta = 0$, $\mu = \cos 2\pi\omega\bar{x}$, $\bar{x} \in \mathbb{R}$, than (***) if $||\omega x \pm 2\omega\bar{x}|| \geq C|x|^{-\tau}$, $x \neq 0$. If $\bar{x}$ half-integer Δ is replaced by the gap size.

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- The proof can be extended to more general form of quasi-periodic potential; one simply needs that $\phi_x = \bar{\phi}(2\pi(\omega x + \theta))$ with $\bar{\phi} \in C^1$, even $\bar{\phi}(t) = \bar{\phi}(-t)$ and periodic $\bar{\phi}(t) = \bar{\phi}(t+1)$; moreover one needs $\partial\bar{\phi}_{\omega\hat{x}+\theta} \neq 0$.

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- $$S_0(\mathbf{x}, \mathbf{y}) = \frac{1}{\beta L} \sum_{k_0, k} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 + \cos k - \mu}$$

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- Close to the singularity

  $$\cos(k' \pm p_F) - \mu \sim \pm \sin p_F k' + O(k'^2)$$

  linear dispersion relation.

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  $$\hat{g}(x' + \bar{x}_\rho, k_0) \sim \frac{1}{-ik_0 \pm v_0 (\omega x')_{\mathrm{mod}.1}}$$
- The denominator can be arbitrarily large; for $x \neq \rho\hat{x}$ by (*),(**) ,$||\omega x'|| = ||\omega(x - \rho\hat{x}) + 2\delta_{\rho,-1}\theta|| \geq C|x - \rho\hat{x}|^{-\tau}$. $(\omega x')_{\mathrm{mod}.1}$ can be very small for large x (infrared-ultraviolet mixing)

Anti-integrable limit; proof of localization

The 2-point function is given by $\frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}}W|_0$

$$e^{W(\phi)} = \int P(d\psi)e^{-V(\psi)-\mathcal{B}(\psi,\phi)}$$

with $P(d\psi)$ a gaussian Grassmann integral with propagator $\delta_{x,y}\bar{g}(x, x_0 - y_0)$, $\bar{g}(x, x_0)$ is the temporal FT of $\hat{g}(x, k_0)$

$$V(\psi) = U\int d\mathbf{x}\sum_{\alpha=\pm}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{x}+\alpha\mathbf{e}_1}\psi^-_{\mathbf{x}+\alpha\mathbf{e}_1}$$

$$+\varepsilon\int d\mathbf{x}(\psi^+_{\mathbf{x}+\mathbf{e}_1}\psi^-_{\mathbf{x}} + \psi^+_{\mathbf{x}-\mathbf{e}_1}\psi^-_{\mathbf{x}}) + \nu\int d\mathbf{x}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}$$

where $\int d\mathbf{x} = \sum_{x\in\Lambda}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0$, Finally $B = \int d\mathbf{x}(\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} + \psi^+_{\mathbf{x}}\phi^-_{\mathbf{x}})$

Small divisors

- In absence of many body interaction there are only chain graphs, $\alpha_i = \pm$

$$\varepsilon^n \sum_{x_1} \int dx_{0,1} \ldots dx_{0,n} \bar{g}(x_1, x_0 - x_{0,1}) \bar{g}(x_1 + \sum_{i \leq n} \alpha_i, (x_{0,n} - y_0))$$

$$\prod_{i=1}^{n} \bar{g}(x_1 + \sum_{k \leq i} \alpha_k, x_{0,i+1} - x_{0,i})$$

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- Propagators $g(k_0, x)$ can be arbitrarily large (small divisors)

$$|\hat{g}(x' \pm \bar{x}, k_0)| \leq C_0 |x'|^\tau$$

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Small divisors

- In absence of many body interaction there are only chain graphs, $\alpha_i = \pm$

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- When $U \neq 0$ there also loops producing additional divergences, absent in KAM or in the non interacting case.
- To establish localization in presence of interaction one has to prove that such small divisors are harmless, even with loops.

Some idea of the proof

- We perform an *RG* analysis decomposing the propagator as sum of propagators living at $\gamma^{2h-1} \leq k_0^2 + |\phi_x - \phi_{\hat{x}}|^2 \leq \gamma^{2h+1}$, $h = 0, -1, -2...$, $\gamma > 1$, $\phi_x = \cos 2\pi(\omega x + \theta)$; this correspond to two regions, around $\bar{x}_+$ and $\bar{x}_-$.

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- This implies that the single scale propagator has the form $\sum_{\rho=\pm} g_\rho^{(h)}$ with $|g_\rho^{(h)}(\mathbf{x})| \leq \frac{C_N}{1+(\gamma^h(x_0-y_0))^N}$; the corresponding Grasmann variable is $\psi^{(h)}_{\mathbf{x},\rho}$.

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- We integrate the fields with decreasing scale; for instance $W(0)$ (the partition function) can be written as

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- The effective potential V^h sum of monomials of any order in $\sum_{x_1'} \int dx_{0,1} ... dx_{0,n} W^h \prod_i \psi^{\varepsilon_i}_{x_i', x_{0,i}, \rho_i}$ (we have integrated the deltas in the propagators).

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- One has to distinguish among the monomials $\prod_i \psi^{\varepsilon_i}_{x'_i, x_{0,i}, \rho_i}$ in the effective potential between resonant and non resonant terms. Resonant terms; $x'_i = x'$. Non Resonant terms $x'_i \neq x'_j$ for some i, j. (In the non interacting case only two external lines are present).

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- It turns out that the non resonant terms are irrelevant (even if they are relevant according to power counting).
- Roughly speaking, the idea is that if two propagators have similar (not equal) small size (*non resonant subgraphs*) , then the difference of their coordinates is large and this produces a "gain" as passing from x to $x + n$ one needs n vertices. That is if $(\omega x'_1)_{\mathrm{mod}1} \sim (\omega x'_2)_{\mathrm{mod}1} \sim \Lambda^{-1}$ then by the Diophantine condition

$$2\Lambda^{-1} \geq ||\omega(x'_1 - x'_2)|| \geq C_0 |x'_1 - x'_2|^{-\tau}$$

that is $|x'_1 - x'_2| \geq \bar{C} \Lambda^{\tau^{-1}}$

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- Consider two vertices w_1, w_2 such that x'_{w_1} and x'_{w_2} are coordinates of the external fields, and let be c_{w_1,w_2} the path (vertices and lines) in $\bar{T}_v$ connecting w_1 with w_2; we call $|c_{w_1,w_2}|$ the number of vertices in c_{w_1,w_2}. The following relation holds, if $\delta^i_w = \pm 1$ it corresponds to an ε end-point and $\delta^i_w = (0, \pm 1)$ is a U end-point

$$x'_{w_1} - x'_{w_2} = \bar{x}_{\rho_{w_2}} - \bar{x}_{\rho\, w_1} + \sum_{w \in c_{w_1,w_2}} \delta^{i_w}_w$$

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- As $x_i - x_j = M \in \mathbb{Z}$ and $x'_i = x'_j$ then $(\bar{x}_{\rho_i} - \bar{x}_{\rho_j}) + M = 0$, so that $\rho_i = \rho_j$ as $\bar{x}_+ = \hat{x}$ and $\bar{x}_- = -\hat{x} - 2\theta/\omega$ and $\hat{x} \in \mathbb{Z}$.

Some idea of the proof

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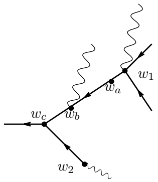

FIG. 1: A tree $\bar{T}_v$ with attached wiggly lines representing the external lines P_v; the lines represent propagators with scale $\geq h_v$ connecting w_1, w_a, w_b, w_c, w_2, representing the end-points following v in τ.

.

Some idea of the proof

- By the Diophantine condition a) $\rho_{w_1} = \rho_{w_2}$ the (*); b)if $\rho_{w_1} = -\rho_{w_2}$ by (**)

$$2cv_0^{-1}\gamma^{h_{\bar{v}'}} \geq$$
$$||(\omega x'_{w_1})||_1 + ||(\omega x'_{w_2})||_1 \geq ||\omega(x'_{w_1} - x'_{w_2})||_1 \geq C_0(|c_{w_2,w_1}|)^{-\tau}$$

so that $|c_{w_1,w_2}| \geq A\gamma^{\frac{-h_{\bar{v}'}}{\tau}}$. If two external propagators are small but not exactly equal, you need a lot of hopping or interactions to produce them

- If $\bar{\varepsilon} = \max(|\varepsilon|, |U|))$ from the $\bar{\varepsilon}^n$ factor we can then extract (we write $\bar{\varepsilon} = \prod_{h=-\infty}^{0} \bar{\varepsilon}^{2^{h-1}}$)

$$\bar{\varepsilon}^{\frac{n}{4}} \leq \prod_{v \in L} \varepsilon^{N_v 2^{h_{v'}}}$$

where N_v is the number of points in v ; as $N_v \geq |c_{w_1,w_2}| \geq A\gamma^{\frac{-h_{v'}}{\tau}}$ then

$$\bar{\varepsilon}^{\frac{n}{4}} \leq \prod_{v \in L} \bar{\varepsilon}^{A\gamma^{\frac{-h_{v'}}{\tau}} 2^{h_{v'}}}$$

where L are the non resonant vertices If $\gamma^{\frac{1}{\tau}}/2 > 1$ then $\leq C^n \prod_{v \in L} \gamma^{3h_v S_v^L}$ where S_v^L is the number of non resonant clusters in v.

Ideas of proof

- We localize the resonant terms $\mathbf{x} = x_{0,i}, x$ with all x_i' equal

$$\mathcal{L}\psi^{\varepsilon_1}_{\mathbf{x}_1,\rho}...\psi^{\varepsilon_n}_{\mathbf{x}_n,\rho} = \psi^{\varepsilon_1}_{\mathbf{x}_1,\rho}...\psi^{\varepsilon_n}_{\mathbf{x}_1,\rho}$$

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- In order to sum over the number of external fieds one uses both the cancellations due to anticommutativity and the diophantine condition.

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- This concludes the discussion of the localized regime; we discuss briefly the extended regime.

Extended regime

- Different behavior is found close to the integrable limit. Fix $\varepsilon = 1, \theta = 0$, U, u small, $\mu = \cos p_F$, $||2\pi\omega n||_{2\pi} \geq C|n|^{-\tau}$, $n \neq 0$, then (Mastropietro, CMP99, PRB2016) :

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- 2) If $p_F = n\omega\pi$ a faster than any power decay with rate

$$\Delta_{n,U} \sim [u^{2n}(a_n + F)]^{X_n}$$

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- In the case of Aubry-Andre' potential all gaps persists instead; no quantum phase transition at small coupling.

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- This is true for quasi-periodic functions with fast decaying Fourier transform; With other quasi-random noise, is believed instead that there are infinitely many rcc.

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- Spin? Coupled chains? other eigenstates? 2 or 3 dimension?