Pfaffian Correlation Functions of Planar Dimer Covers

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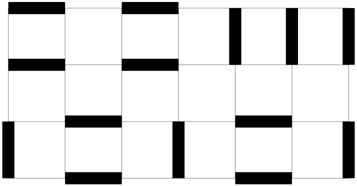
Given a finite graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, a **dimer cover** is a subset of the edge set, $\omega \subset \mathcal{E}$, such that every vertex is covered by exactly one edge.



Given an edge weight $K : \mathcal{E} \mapsto \mathbb{C}$ the **dimer-cover partition function** is

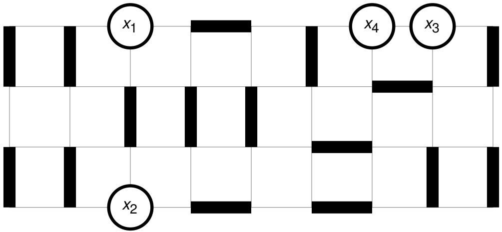
$$\boxed{Z_{\mathbb{G},K} := \sum_{\omega \in \Omega_{\mathbb{G}}} \chi_K(\omega)}$$

with $\Omega_{\mathbb{G}}$ the set of dimer covers and $\chi_K(\omega) := \prod_{b \in \omega} K_b$.

(In the following, wlog: $Z_{\mathbb{G},K} \neq 0$)

Monomer correlation functions

For a collection of monomer sites, $M \subset \mathcal{V}$, the partition function of the monomer-depleted graph is: $\quad Z_{\mathbb{G},K}(M) := \sum_{\omega \in \Omega_{\mathbb{G}}(M)} \chi_K(\omega).$



Monomer correlation function for an even collection of disjoint sites $\{x_1, \ldots, x_{2n}\} \subset \mathcal{V}$:

$$\boxed{S_{2n}(x_1, \ldots, x_{2n}) := \frac{Z_{\mathbb{G},K}(\{x_1, \ldots, x_{2n}\})}{Z_{\mathbb{G},K}} = \Big\langle \prod_{j=1}^{2n} \eta_j \Big\rangle_{\mathbb{G},K}}$$

The case of planar graphs

Pfaffian structure of the dimer partition function: P. W. Kasteleyn '63

$$\boxed{Z_{\mathbb{G},K} = |\operatorname{Pf} D_K|}$$

Kasteleyn adjacency matrix D_K

Selected further properties:

1. Asymptotic bulk monomer correlation function for $\mathbb{G} = \mathbb{Z}^2$ and $K \equiv 1$

$$S_2(x_1, x_2) \sim |x_1 - x_2|^{-1/2} \qquad (|x_1 - x_2| \to \infty)$$

Fisher / Stephenson '63, ...

2. Close relation between the partition functions of the dimer cover and of the Ising model Kasteleyn '63, Fisher '66, Yang / Park '80
3. (Non-)existence of phase transitions Heilmann / Lieb '72
4. Arctic circle phenomenon Cohn / Elkies / Propp '96
5. Continuum limits and their description in terms of (conformal) field theory Kenyon '14
6. and ...

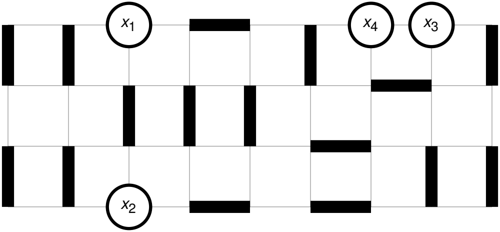
Pfaffian structure of boundary monomer correlations

Theorem

For any finite planar graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ the boundary values of the monomer correlation functions satisfy

$$S_{2n}(x_1, \ldots, x_{2n}) = \sum_{\pi \in \Pi_{2n}} \mathrm{sgn}(\pi) \prod_{j=1}^{n} S_2(x_{\pi(2j-1)}, x_{\pi(2j)}) \equiv \mathrm{Pf}_n\left(S_2(x_i, x_j)\right)$$

where $M := \{x_1, \ldots, x_{2n}\}$ ranges over sequences of disjoint vertices positioned in a cyclic order along any boundary of $\mathbb{G}$.



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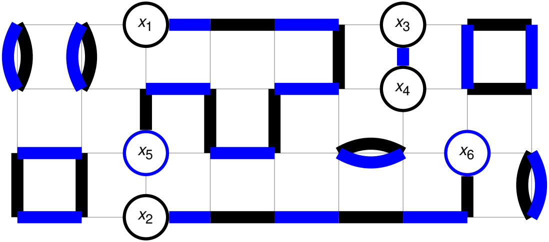
- Case $\mathbb{G} = \mathbb{Z} \times \mathbb{Z}_+$ and $K \equiv 1$: V. B. Priezzhev / P. Ruelle '08

$$S_2((\xi, 0), (\eta, 0)) = \begin{cases} -\frac{2}{\pi\, |\xi - \eta|} & \text{if } |\xi - \eta| \text{ is odd} \\ 0 & \text{else.} \end{cases}$$

- General planar graphs: A. Guilliani / I. Jauslin / E. H. Lieb '15
- Main new point: elementary proof highlighting the topological origin through path integral techniques

Doubled dimer covers with disjoint monomers $M_1, M_2 \subset \mathcal{V}$

$$\omega^{(2)} = (\omega_1, \omega_2) \in \Omega_{\mathbb{G}}(M_1) \times \Omega_{\mathbb{G}}(M_2) =: \Omega_{\mathbb{G}}^{(2)}(M_1, M_2).$$



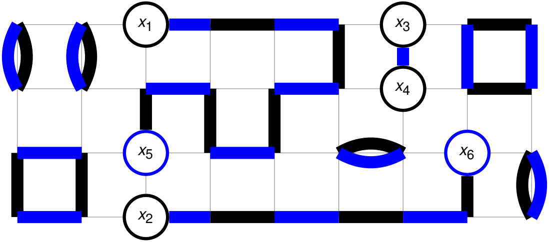
The edge multiplicity of $\omega^{(2)}$ coincides with that of a collection $\Gamma = \Gamma(\omega^{(2)})$ of **2-colored edge-disjoint loops and paths** on a 2-multigraph where each $\gamma \in \Gamma$ is either

i. a double loop covering a single edge,

ii. a simple loop of an even number of non-repeated edges,

iii. a simple path with boundary set $\partial\gamma \subset M_1 \sqcup M_2$.

Doubled dimer covers with disjoint monomers $M_1, M_2 \subset \mathcal{V}$



Double-dimer partition function as path / loop integral

$$Z^{(2)}_{\mathbb{G},K}(M_1, M_2) := Z_{\mathbb{G},K}(M_1)\, Z_{\mathbb{G},K}(M_2) = \sum_{\omega^{(2)} \in \Omega^{(2)}(M_1, M_2)} \chi_K(\omega_1)\, \chi_K(\omega_2)$$

Connection amplitudes for $\{x_j, y_j\}_{j=1,\ldots,N}$ pairs of sites in $M_1 \sqcup M_2$:

$$Z^{(2)}_{\mathbb{G},K}(M_1, M_2; x_j \leftrightarrow y_j \; j = 1, \ldots, N) := \sum_{\omega^{(2)} \in \Omega^{(2)}(M_1, M_2)} \chi_K(\omega_1)\, \chi_K(\omega_2) \prod_{j=1}^{N} \mathbb{1}\left[x_j \overset{\omega^{(2)}}{\longleftrightarrow} y_j \right].$$

Example: monomer correlation function as path integral

Let Ω_n^{A} stand for the set of non-intersecting simple paths Γ_P on the graph $\mathbb{G}$.

Theorem (Path integral for correlations)

For any finite graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ and disjoint sites $\{x_1, \ldots, x_{2n}\} \subset \mathcal{V}$ the monomer correlation function admits the representation

$$S_{2n}(x_1, \ldots, x_{2n}) = \sum_{\substack{\Gamma_P = \{\gamma_1, \ldots, \gamma_n\} \subset \Omega_n^{\mathrm{A}} \\ \partial \Gamma_P = \{x_1, \ldots, x_{2n}\}}} w_K(\Gamma_P) \prod_{\gamma \in \Gamma_P} \mathbb{1}\left[\gamma \text{ is odd}\right] ,$$

with the weight function

$$w_K(\Gamma_P) := \left(\frac{Z_{\mathbb{G},K}(\mathcal{V}(\Gamma_P))}{Z_{\mathbb{G},K}} \right)^2 \prod_{\gamma \in \Gamma_P} \chi_K(\gamma) .$$

Proof idea: Summation over the loops in the complement of Γ_P.

Switching symmetries of connection amplitudes

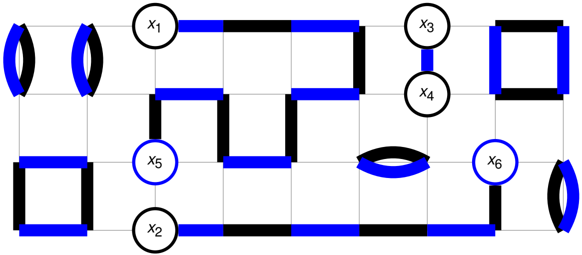
Lemma (Switching principle)

For any finite graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$*, pair of disjoint monomer sets* M_1, M_2 *and* $\{x, y\} \subset \mathcal{V} \backslash (M_1 \sqcup M_2)$*:*

$$Z^{(2)}_{\mathbb{G},K}(M_1 \sqcup \{x,y\}, M_2; x \leftrightarrow y, C) = Z^{(2)}_{\mathbb{G},K}(M_1, M_2 \sqcup \{x,y\}; x \leftrightarrow y, C) \ ,$$

$$Z^{(2)}_{\mathbb{G},K}(M_1 \sqcup \{x\}, M_2 \sqcup \{y\}; x \leftrightarrow y, C) = Z^{(2)}_{\mathbb{G},K}(M_1 \sqcup \{y\}, M_2 \sqcup \{x\}; x \leftrightarrow y, C)$$

where C *stands for any collection of other connection conditions among monomers in* $M_1 \sqcup M_2$*.*



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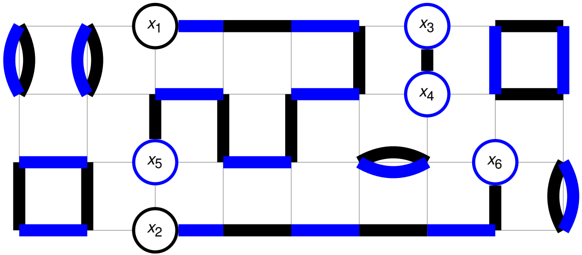

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$$Z^{(2)}_{\mathbb{G},K}(M_1 \sqcup \{x\}, {\color{blue}M_2 \sqcup \{y\}}; x \leftrightarrow y, C) = Z^{(2)}_{\mathbb{G},K}(M_1 \sqcup \{y\}, {\color{blue}M_2 \sqcup \{x\}}; x \leftrightarrow y, C)$$

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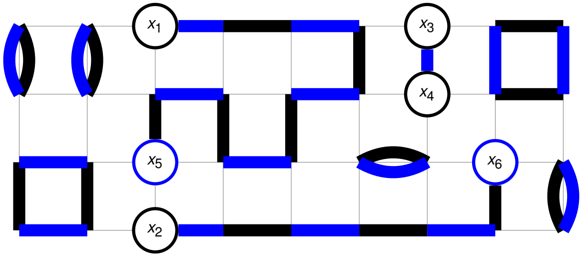

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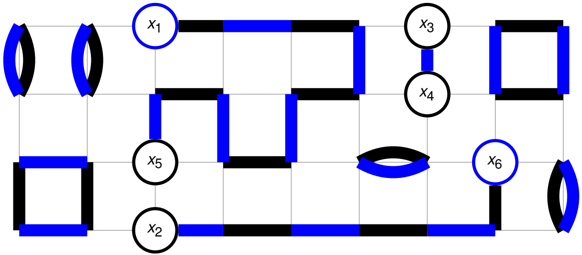


Complete proof of the Pfaffian structure of boundary monomer correlations

Sufficient: $$S_{2n}(x_1,\dots,x_{2n}) = \sum_{k=2}^{2n}(-1)^k\, S_2(x_1,x_k)\, S_{2(n-1)}\,.(\cancel{x_1},x_2,\dots,\cancel{x_k},\dots,x_{2n}).$$

Proof. $$\text{r.s.}\times(Z_{\mathbb{G},K})^2 = \sum_{k=2}^{2n}(-1)^k\, Z^{(2)}_{\mathbb{G},K}(\{x_1,x_k\},\{\cancel{x_1},x_2,\dots,\cancel{x_k},\dots,x_{2n}\};x_1\leftrightarrow x_k)$$

$$+\sum_{k=2}^{2n}(-1)^k \sum_{\substack{l,m=2\\ k\neq l\neq m\neq k}}^{2n} Z^{(2)}_{\mathbb{G},K}\left(\{x_1,x_k\},\{\cancel{x_1},x_2,\dots,\cancel{x_k},\dots,x_{2n}\};\begin{array}{c} x_1\leftrightarrow x_m\\ x_k\leftrightarrow x_l\end{array}\right)$$

Planarity implies: $x_i\leftrightarrow x_j \implies (-1)^{i-j}=-1$

1. term: $(-1)^k=1$ and by switching lemma equal to l.s. $\times(Z_{\mathbb{G},K})^2$.
2. term: $(-1)^{k-l}=-1$ and hence vanishes by switching symmetry $(x_k\leftrightarrow x_l)$.

$\square$

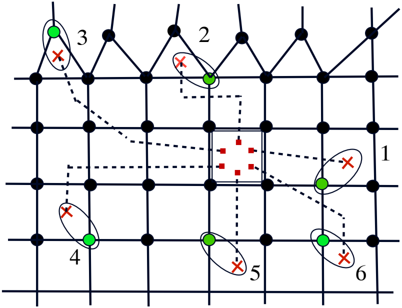
Generalization to order-disorder variables

For a planar graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with edge weights $K : \mathcal{E} \mapsto \mathbb{C}$:

i. the **disorder operators** τ_{ℓ_j} are associate with open-ended, site-avoiding lines $\ell_1, \dots, \ell_n$. These give rise to a partial gauge transformations

$$K \mapsto T_{\ell_j} K$$

corresponding to sign flips of K over edges which are crossed by ℓ_j an odd number of times.



ii. the **expectation values** of products of such operators are defined as:

$$\langle \prod_{j=1}^{n} \tau_{\ell_j} \rangle_{\mathbb{G},K} := \frac{Z_{\mathbb{G}, T_{\ell_1} \circ \cdots \circ T_{\ell_n} K}}{Z_{\mathbb{G},K}} .$$

We will consider **canonical pairs of order-disorder variables** $p_j = (x_j, \ell_j)$ and their operators $\mu_j := \eta_{x_j} \tau_{\ell_j}$.

Pfaffian structure of order-disorder correlations

Theorem

For a finite planar graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with edge weights $K : \mathcal{E} \mapsto \mathbb{C}$, and any collection of canonical pairs of order-disorder variables $p_j = (x_j, \ell_j)$, $j \in \{1, \ldots, 2n\}$, which are cyclicly ordered:

$$\langle \prod_{j=1}^{2n} \mu_j \rangle_{\mathbb{G},K} \; = \; \mathrm{Pf}_n \left(\langle \mu_j \mu_k \rangle_{\mathbb{G},K} \right) \, .$$

1. Generalizes the result for boundary
2. In case monomers $\{x_{2j-1}, x_{2j}\}$ are pairwise adjacent, the disorder lines may be chosen so that their actions are pairwise equivalent, and thus cancel each other, i.e. $\tau_{2j-1}\tau_{2j} = \eta_{x_{2j-1}}\eta_{x_{2j}}$ and
$$\left\langle \prod_{j=1}^{2n} \tau_j \right\rangle_{\mathbb{G},K} = \left\langle \prod_{j=1}^{n} \mu_{x_{2j-1}} \mu_{x_{2j}} \right\rangle_{\mathbb{G},K} .$$
3. Proof idea is similar to boundary case - main modification: count intersection parities of loops.

Key elements in the proof for the order-disorder variables

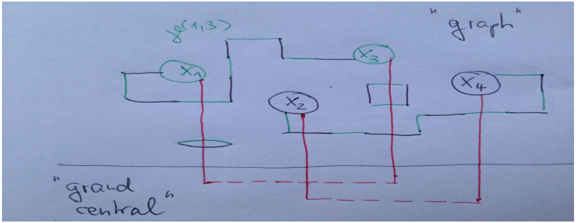
Connection amplitudes for order disorder pairs $(M_1, \mathcal{L}_1)$, $(M_2, \mathcal{L}_2)$:

$$W^{(2)}_{\mathbb{G},K}(\{M_1, \mathcal{L}_1\}, \{M_2, \mathcal{L}_2\}; C) :=$$

$$\sum_{\omega^{(2)} \in \Omega^{(2)}(M_1, M_2)} \mathbb{1}\left[\omega^{(2)} \text{ satisfies } C\right] \chi_K(\omega_1)\,(-1)^{(\omega_1 \,|\, \mathcal{L}_1)}\, \chi_K(\omega_2)\,(-1)^{(\omega_2 \,|\, \mathcal{L}_2)}$$

where $(\omega_j \,|\, \mathcal{L}_j)$ is the number of intersections of ω_j with $\mathcal{L}_j$.
Illustration of **switching principle**, e.g.

$$W^{(2)}_{\mathbb{G},K}(\{p_1, p_k\}, \{\cancel{p_1}, p_2, \ldots, \cancel{p_k}, \ldots, p_{2n}\}; x_1 \leftrightarrow x_k)$$
$$= (-1)^k\, W^{(2)}_{\mathbb{G},K}(\emptyset, \{p_1, \ldots, p_{2n}\}; x_1 \leftrightarrow x_k)$$



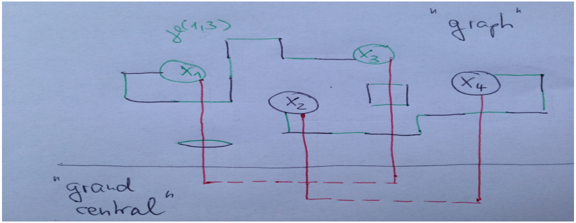
Key elements in the proof for the order-disorder variables

Switching transform bijection: $(\omega_1, \omega_2) \mapsto (\omega_1 \Delta \gamma^{(1,k)}, \omega_2 \Delta \gamma^{(1,k)})$

Effect on the intersection parity:

$$\frac{(-1)^{(\omega_2 \Delta \gamma^{(1,k)} | \mathcal{L})}}{(-1)^{(\omega_1 | \ell_{1,k})} (-1)^{(\omega_2 | \mathcal{L} \setminus \ell_{1,k})}} = (-1)^{(\gamma^{(1,k)} | \mathcal{L})} (-1)^{(\omega^{(2)} | \ell_{1,k})} = (*)$$

Main idea: Consider the loop ensemble composed of $\gamma^{(1,k)}$ concatenated with $\ell_{1,k}$ and the remaining loops arising from concatenating the rest:



$$(*) = (-1)^{(\gamma^{(1,k)} | \mathcal{L} \setminus \ell_{1,k})} (-1)^{(\ell_{1,k} | \omega^{(2)} \setminus \gamma^{(1,k)})}$$

$$= (-1)^k \quad \text{intersection parity of lines } \mathcal{L} \text{ within grand central} .$$

Conclusion

1. As in the Ising model and its random current representation, **doubling the dimer model** reveals its underlying symmetries.
   - **path /loop integral representation**
   - **switching lemmata**
2. Similar reasoning leads to analogous results / proofs for Pfaffian structure of boundary correlation functions of Ising spins and order-disorder correlation functions.

   M. Aizenman / H. Duminil-Copin / V. Tassion / S.W. '16