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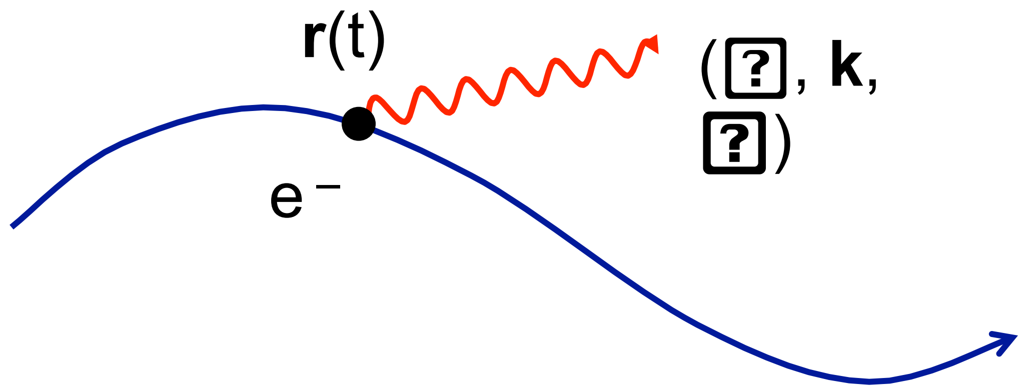
Derivation of the classical radiation amplitude from stimulated emission

X. Artru
Institut de Physique Nucléaire de Lyon

Classical radiation formula in vacuum



applies to :

- Synchrotron radiation in **weak** uniform or *non-uniform* field, e.g. undulator radiation
- **Soft** Compton effect (Thompson regime)
- **Soft** Coherent Bremsstrahlung (? << E)
- Channeling Radiation (classical regime)

• it ignores **quantum recoil effects**

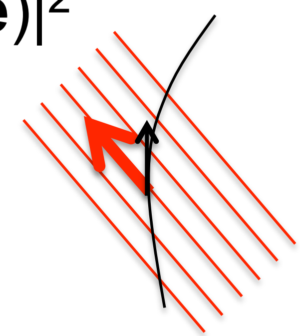
The formula

Jackson book (14.67) :

$$dW/(d\boxed{?}d\boxed{?}) = \boxed{?}\boxed{?}^2/(4\boxed{?}^2)\ \Big| \boxed{?}\ dt\ \mathbf{n}\times\mathbf{n}\times\mathbf{v}(t)\ \ \exp\{i\boxed{?}(t-nr(t)\}\ \Big|^2$$

hbar = c = 1 ; $\boxed{?} = e^2/(4\boxed{?}) = 1/137$ (rationalized system with $\varepsilon_0 = \mu_0 = 1$)

For a polarization $\hat{\mathbf{e}}$: $dW(\boxed{?})\ /\ d^3\mathbf{k} = 1/(16\boxed{?}^3)\ |a_{rad}(\mathbf{k},\hat{\mathbf{e}})|^2$



Radiation amplitude : $a_{rad}(\mathbf{k},\hat{\mathbf{e}}) = -\ e\ \boxed{?}\ \ \mathbf{E}^*_{\mathbf{k},\hat{\mathbf{e}}}(\mathbf{r},t)\cdot d\mathbf{r}(t)$

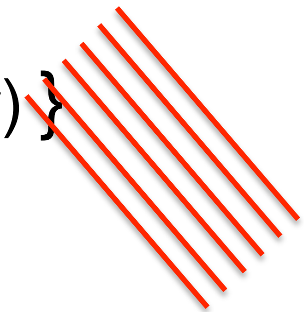
= work done by the electron in the complex plane wave

$$\mathbf{E}_{\mathbf{k},\mathbf{e}}(t,r) = \hat{\mathbf{e}}\ \exp\{i\ \mathbf{k}\cdot\mathbf{r} - i\boxed{?}t\}$$

Precise definition of a_{rad}

- a_{rad} is a Fourier coefficient of the *radiated* field $\mathbf{E}_{rad} = \mathbf{E}_{retarded} - \mathbf{E}_{advanced}$:

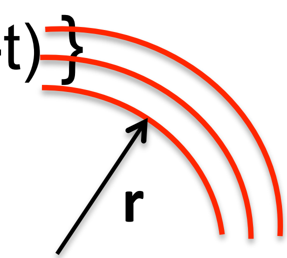
$$\mathbf{E}_{rad}(r,t) = \boxed{?}\ d^3\mathbf{k}\ /(2\boxed{?})^3\ \mathrm{Re}\{\ \Sigma_{\hat{\mathbf{e}}}\ a_{rad}(\mathbf{k},\hat{\mathbf{e}})\ \mathbf{E}_{\mathbf{k},\hat{\mathbf{e}}}\ (t,r)\ \}$$



- a_{rad} is related to the assymptotic radiated field (or *far field*) by

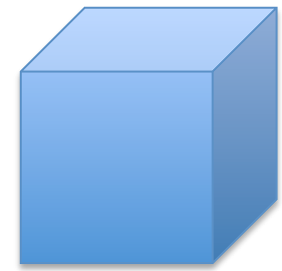
$$\mathbf{E}_{far}(r,t) = \boxed{?}\ \boxed{?}d\boxed{?}\ \Sigma_{\hat{\mathbf{e}}}\ \hat{\mathbf{e}}/r\ \ \mathrm{Re}\{\ -i\ a_{rad}(\mathbf{k},\hat{\mathbf{e}})\ \exp\{i\boxed{?}(r-t)\}\ \}$$

with $\mathbf{k} = \boxed{?}\ \mathbf{r}/r$



Discretisation of the free field modes

Fourier decomposition in discrete modes, in a **box** of volume $V=L^3$ with the **periodic condition** $f(x+L) = f(x)$ (idem in y and z)



$$\boxed{?}\ d^3\mathbf{k}/(2\boxed{?})^3\ \boxed{?}\ V^{-1}\ \Sigma_{\mathbf{k}}$$

In particular, $E_{rad,far}(r,t) = V^{-1}\ \Sigma_{\mathbf{k}}\ \Sigma_{\hat{\mathbf{e}}}\ \mathrm{Re}\{a_{rad}(\mathbf{k},\hat{\mathbf{e}})\ \mathbf{E}_{\mathbf{K},\hat{\mathbf{e}}}(t,r)\ \}$

For any free field : $\mathbf{E}(r,t) = V^{-1}\ \Sigma_{\mathbf{k}}\Sigma_{\hat{\mathbf{e}}}\ \mathrm{Re}\{a(\mathbf{k},\hat{\mathbf{e}})\ \mathbf{E}_{\mathbf{k},\hat{\mathbf{e}}}(t,r)\ \}$

The partial energy in mode $(\mathbf{k},\hat{\mathbf{e}})$ is

$$w(\mathbf{k},\hat{\mathbf{e}}) = |a(\mathbf{k},\hat{\mathbf{e}})|^2 \times 0.5\ \boxed{?}\ d^3\mathbf{r}\ \{[\mathrm{Re}\ \mathbf{E}_{\mathbf{k},\hat{\mathbf{e}}}(r,t)]^2 + [\mathrm{Re}\ \mathbf{B}_{\mathbf{k},\hat{\mathbf{e}}}(r,t)]^2\ \}$$
$$= |a(\mathbf{k},\hat{\mathbf{e}})|^2 \times (2V)^{-1}$$

The total field energy is $\Sigma_{\mathbf{k}}\Sigma_{\hat{\mathbf{e}}}\ w(\mathbf{k},\hat{\mathbf{e}})$

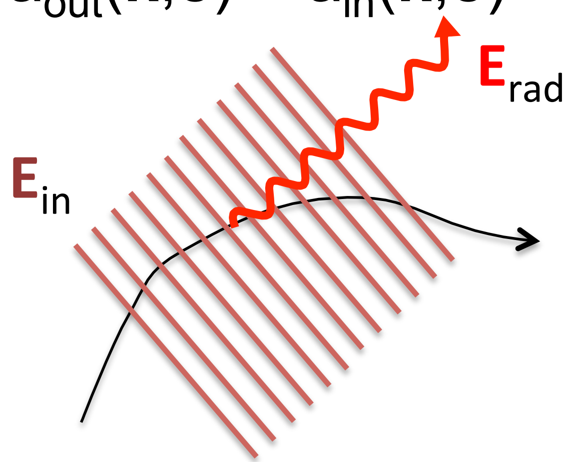
Incoming and outgoing fields

Let us assume that there is a **pre-existing** *incoming* field in the mode (**K**,**ê**).
At t = − [?] ,

$$E_{in}(r,t) = V^{-1} \sum_{\mathbf{k}} \sum_{\hat{\mathbf{e}}} \mathrm{Re}\{a_{in}(\mathbf{k},\hat{\mathbf{e}})\, \mathbf{E}_{\mathbf{k},\hat{\mathbf{e}}}(t,r)\}$$

At t [?] + [?] the *outgoing* field is $\mathbf{E}_{out}(r,t) = \mathbf{E}_{in}(r,t) + \mathbf{E}_{rad}(r,t)$.

In terms of modes : $a_{out}(\mathbf{k},\hat{\mathbf{e}}) = a_{in}(\mathbf{k},\hat{\mathbf{e}}) + a_{rad}(\mathbf{k},\hat{\mathbf{e}})$



Energy gain ΔW of the field

We have two expressions of ΔW :

$$\Delta W = W_{out} - W_{in} = (2V)^{-1} \Sigma_{\mathbf{k}} \Sigma_{\hat{\mathbf{e}}} \; \{ |a_{out}(\mathbf{k},\hat{\mathbf{e}})|^2 - |a_{in}(\mathbf{k},\hat{\mathbf{e}})|^2 \}$$

$$= (2V)^{-1} \; \Sigma_{\mathbf{k}} \, \Sigma_{\hat{\mathbf{e}}} \; \mathrm{Re}\{ (a_{in}+a_{out}) \times a^{*}_{rad} \} \qquad (1)$$

ΔW = work done by the particle in the field $(\mathbf{E}_{in}+\mathbf{E}_{out})/2$

$$= - (e/2) \; \boxed{?} \; (\mathbf{E}_{in}+\mathbf{E}_{out}) \cdot d\mathbf{r}(t)$$

$$= - e \, (2V)^{-1} \Sigma_{\mathbf{k}} \Sigma_{\hat{\mathbf{e}}} \; \mathrm{Re}\{(a_{in}+a_{out}) \; \boxed{?} \; \mathbf{E}_{\mathbf{k},\hat{\mathbf{e}}} (\mathbf{r},t) \cdot d\mathbf{r}(t) \qquad (2)$$

Comparing (1) and (2), we get the radiation formula :

$$a_{rad}(\mathbf{k},\hat{\mathbf{e}}) = - e \; \boxed{?} \; \mathbf{E}^{*}_{\mathbf{k},\hat{\mathbf{e}}} (\mathbf{r},t) \cdot d\mathbf{r}(t)$$

Spontaneous and stimulated radiation

recall : $\Delta W = (2V)^{-1} \sum_{\mathbf{k}} \sum_{\hat{\mathbf{e}}} \; \{ \, |a_{out}(\mathbf{k},\hat{\mathbf{e}})|^2 - \; |a_{in}(\mathbf{k},\hat{\mathbf{e}})|^2 \, \}$

For a single incoming mode ($\mathbf{k}$,$\hat{\mathbf{e}}$),

$$\Delta w(\mathbf{k},\hat{\mathbf{e}}) \;=\; (2V)^{-1} \left[\, 2\,\mathrm{Re}\{\, a_{in} \times a^{*}_{rad} \,\} + |a_{rad}|^2 \,\right]$$

$$= \Delta w(\mathbf{k},\hat{\mathbf{e}})_{stimulated} + \Delta w(\mathbf{k},\hat{\mathbf{e}})_{spontaneous}$$

- The radiation formula $dW(\boxed{?}) / d^3\mathbf{k} = 1/(16\boxed{?}^3)\, |a_{rad}(\mathbf{k},\hat{\mathbf{e}})|^2$ applies to the spontaneous radiation only.

- $\Delta W_{stimulated}$ comes from the interference between incoming and emitted waves. It can be positive or negative. That depends on the relative phase of a_{in} and a_{rad}.

About classical stimulated radiation

Classical stimulated radiation is the basis of free electron lasers (FEL).

The electron **energy loss** in **spontanous** radiation is generated by the *radiation reaction* force. In the electron rest frame,

$$\mathbf{f}_{reac} = \tfrac{2}{3}\, e^2/(4\pi)\ \ d^3\mathbf{r}/d\tau^3$$

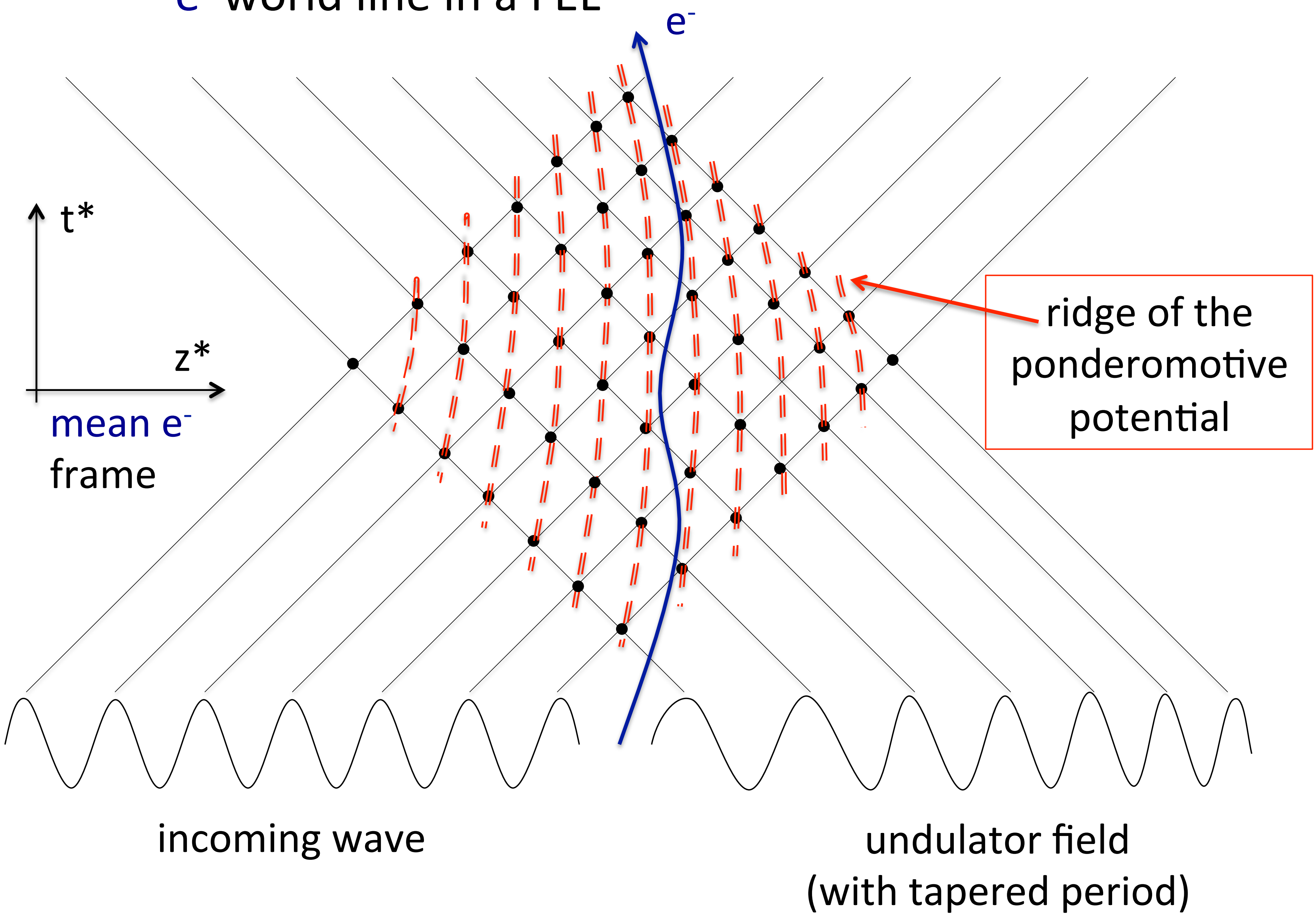
The loss or gain due to **stimulated** radiation can be attributed to the *ponderomotive* force,

$$\mathbf{f}_{pond} = -\, e^2/(2m\omega^2)\ \mathbf{grad}\, \langle \mathbf{E}^2 \rangle \quad \text{(in electron frame), with} \quad \mathbf{E} = \mathbf{E}_{in} + \mathbf{E}_{undul}$$

$\mathbf{f}_{pond}$ is a semi-local average of the Lorentz force. In a FEL this force guides the electrons in *space-time channels*, in such a way that $\langle \Delta W_{stimulated} \rangle > 0$.

For space-time channels, see [X. Artru, *Analogy between free electron laser and channeling by crystal planes,* Channeling 2004, Frascati ; arXiv:hep-ph/0503206]

e⁻ world line in a FEL

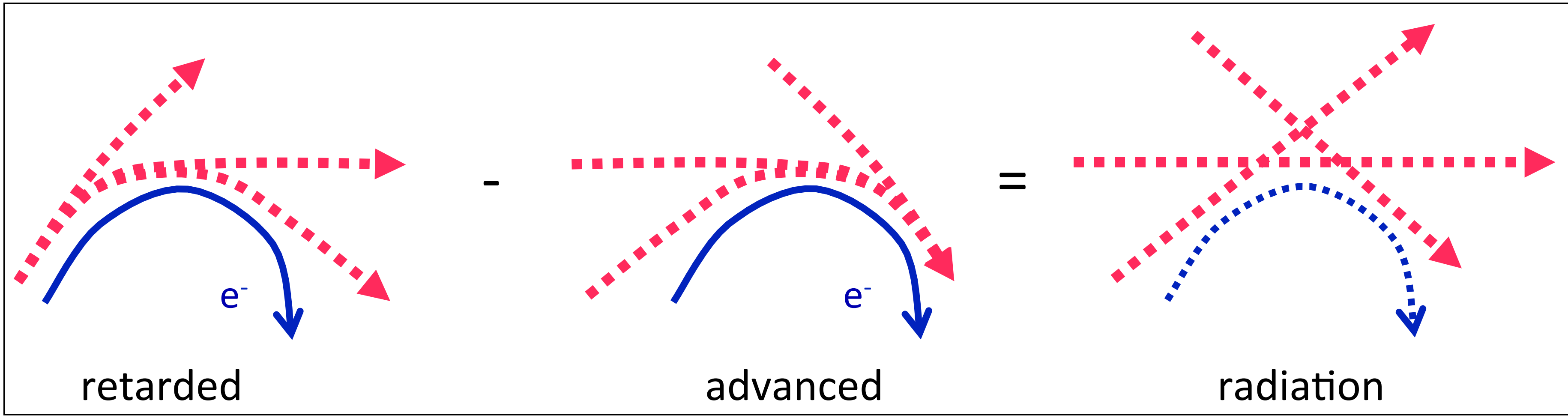


Conclusion

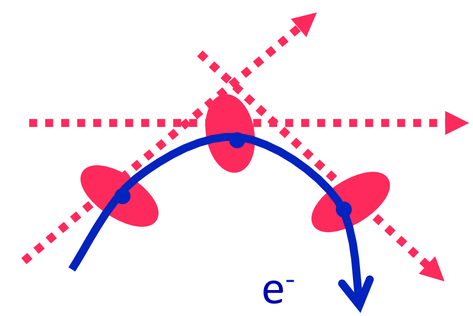
- The classical amplitude of the radiation emitted by an accelerated particle in vacuum can be derived without solving the Maxwell equations or using the Liénard-Wiechert potential.
- It suffices to use the conservation of energy in the classical stimulated radiation. The latter is the basis of FEL.
- The method directly expresses the amplitude as the work of a complex plane wave field.

Thank you for attention !

Radiation field, *self*-field, *near* zone, *far* zone



The *sum*
= self-field.
~ Coulomb
field



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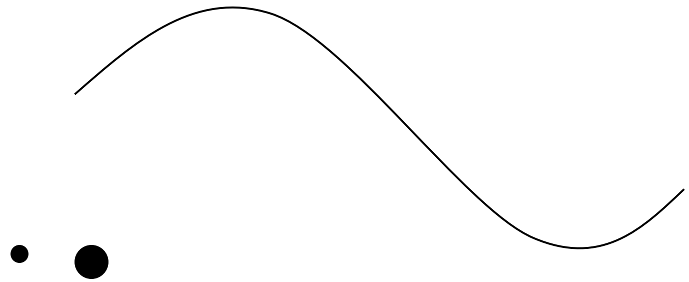

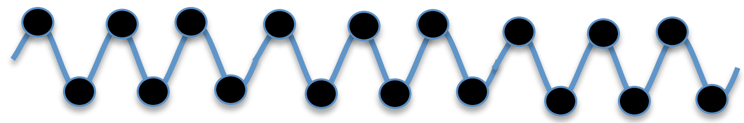

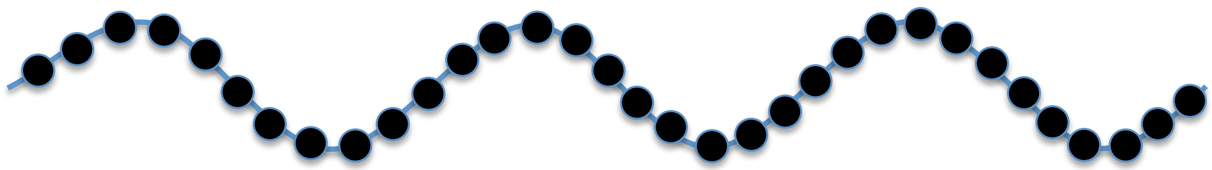

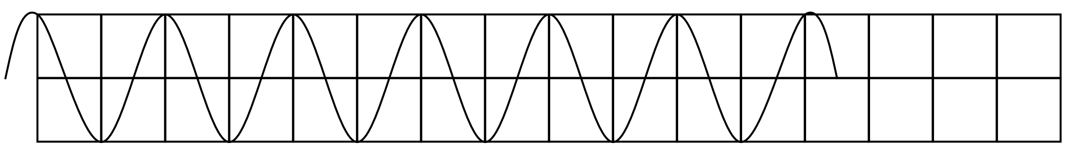

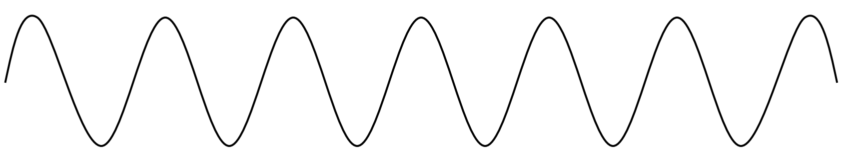

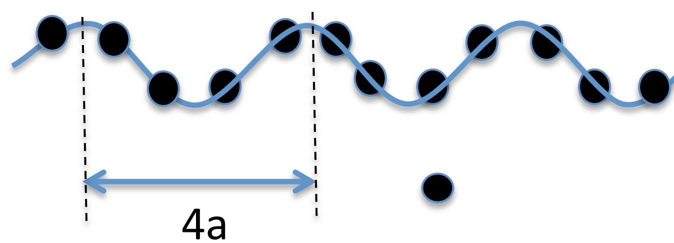

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?
= ?’ = (E/E’) ??????