Charm and bottom mass determination from sum rules

Vacuum polarization function at $O(\alpha_s^2)$ and $O(\alpha_s^3)$ from Padé approximants

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Taskforce: A. H. Hoang, V.M. and M. S. Zebarjad

References: Taskforce: arXiv:0807.4173 [hep-ph] [1]

+ w.i.p. [2]

Outline

- General remarks on heavy quark masses
  - Different schemes
  - High precision…why?
- How to determine it
  - Perturbative errors
  - Sum rules: need of contour improvement
- Padé approximants
  - Predictions
- Preliminary results

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Remarks on heavy quark masses

Confinement $\longrightarrow$ m_q not physical observable

Parameter in QCD Lagrangian $\longrightarrow$ formal definition (as strong coupling)

Renormalization and scheme dependent

In general running mass $m(\mu)$ (RGE evolution)

$$m_q^{\text{schemeA}} = m_q^{\text{schemeB}}(1 + \alpha_s + \alpha_s^2 + \ldots)$$

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$$m_q^{\text{schemeA}} = m_q^{\text{schemeB}}(1 + \alpha_s + \alpha_s^2 + \ldots)$$

Pole mass M $\rightarrow$ Infrared renormalon $O(\Lambda_{QCD})$

$\overline{\text{MS}}$ scheme:

- short-distance mass
- standard mass for comparison: $\overline{m}_q(\overline{m}_q)$

And free form renormalon ambiguities

Why high precision?

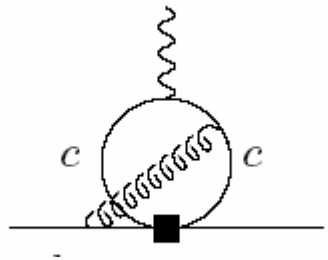
Strong dependence in flavour processes

Constrains new physics

$B \to X_s \gamma$

strong charm mass (scheme) dependence in NLO matrix elements

Misiak, Gambino



$$B(\bar{B} \to X_s\gamma)^{\text{theo,NLO}}_{E_\gamma > 1.6\,\text{GeV}} = \left[3.79 \pm 0.3\,(\overline{m}_c(\mu)) \pm 0.1\,(\delta\overline{m}_c) \pm \ldots \right] \times 10^{-4}$$

100 MeV

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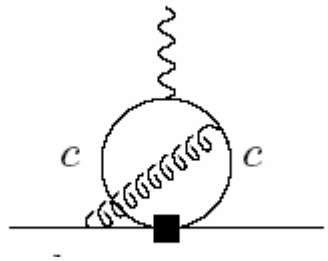

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$\delta\overline{m}_c$ → 100 MeV

$K^+ \to \pi^+ \nu \bar{\nu}$

NNLO QCD computations for charm contributions P_c (See also next slide)

Buras, Gorbahn, Nierste, Haisch (2005)

$$\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu}) = (\,7.96 \pm 0.49_{P_c} \pm 0.84_{\text{other}}\,) \times 10^{-11}$$

$$\pm 0.43\,(\delta m_c) \pm 0.23\,(\delta\alpha_s, \text{pert. error})$$

Why high precision? (Taken from U. Haisch)

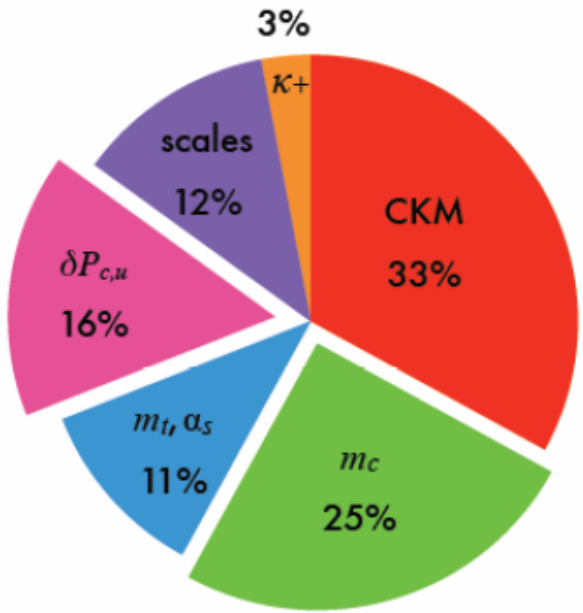
SM prediction(s) of $K^+ \to \pi^+ \nu\bar{\nu}$: error budget

$$\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu}(\gamma)) = \{8.57 \pm 0.93, 8.51 \pm 0.73, 8.04 \pm 0.98\} \times 10^{-11}$$

$m_c(m_c) = (1.30 \pm 0.05)\,\text{GeV}$

$m_c(m_c) = (1.286 \pm 0.013)\,\text{GeV}$ *

$m_c(m_c) = (1.224 \pm 0.017 \pm 0.054)\,\text{GeV}$ †



*Kühn et al. '07 †Hoang & Manohar '05

Why high precision?

Taken from A. H. Hoang, Chamonix (2005)

- Quark masses wanted as accurate as possible
- **bottom mass**

example: $\boxed{B \to X_u l\nu} \quad \Longrightarrow \quad |V_{\rm ub}|$

$$\Gamma \sim G_F^2 |V_{\rm ub}|^2 \, m_{\rm b}^5 \left[1 + \alpha_s + \alpha_s^2 + \ldots + \frac{\Lambda_{\rm QCD}}{m_b} + \frac{\Lambda_{\rm QCD}^2}{m_b^2} + \ldots\right]$$

$$\frac{\delta V_{\rm ub}}{V_{\rm ub}} \sim 2.5 \frac{\delta m_b}{m_b}$$

$B \to X_s\gamma, \ldots$

shape function, ...

Determination of m_c

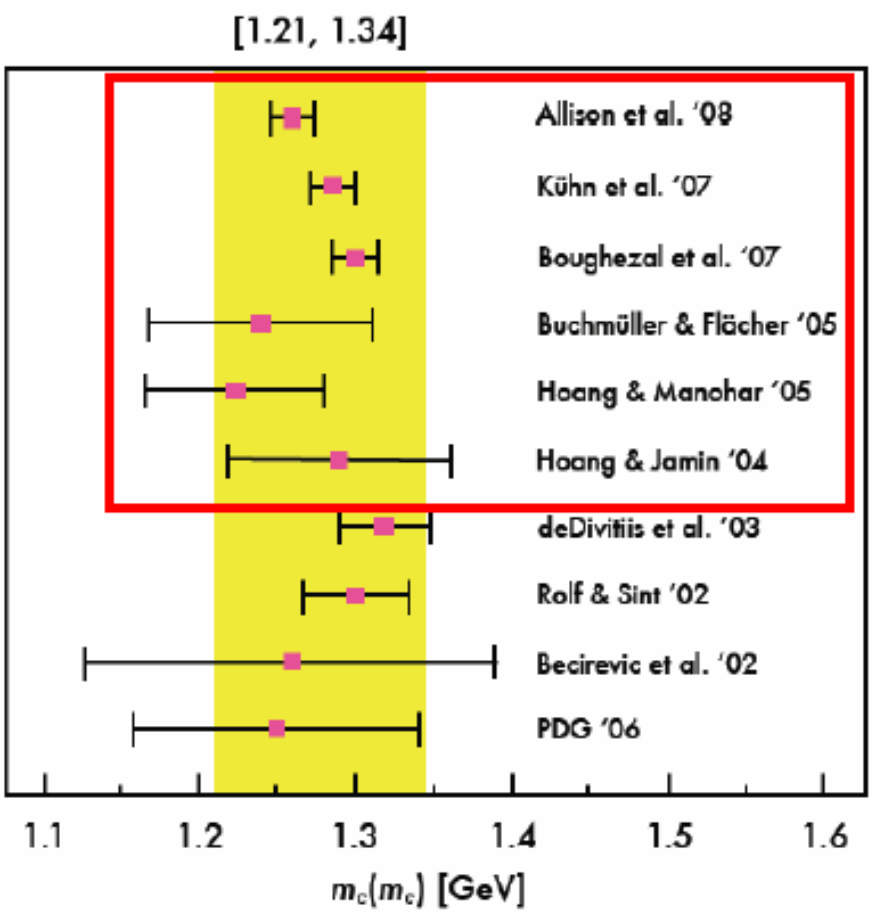
Spectral moments of inclusive B decays (nonrelativistic)

Charmominum sum rules (relativistic)

Lattice

Taken from A. Hoang

Flavor institute CERN 2008



$m_c(m_c)$ [GeV]	method
1.266 ± 0.014	lattice, unquenched, staggered
1.286 ± 0.013	low-momentum sum rules, N^{3}LO
1.295 ± 0.015	low-momentum sum rules, N^{3}LO
1.24 ± 0.07	fit to B-decay distribution, $\alpha_s^2\beta_0$
$1.224 \pm 0.017 \pm 0.054$	fit to B-decay data, $\alpha_s^2\beta_0$
1.29 ± 0.07	NNLO moments
1.319 ± 0.028	lattice, quenched
1.301 ± 0.034	lattice, quenched
$1.26 \pm 0.04 \pm 0.12$	lattice, quenched
1.25 ± 0.09	PDG 2006

Determination of m_c

Inclusive B decays

A.H. Hoang A.V. Manohar 2005

→ perturbative error estimate tricky !

(A): extract $\overline{m}_c(4.2\ \text{GeV}) \xrightarrow{\text{RGE}} \overline{m}_c(\overline{m}_c) = 1521 - 203 - 72 - 15 = 1231 \pm 10$ MeV

(B): extract $\overline{m}_c(\overline{m}_c) = 1279 - 66 - 18 - 6 = 1201 \pm 4$ MeV

$$\overline{m}_c(\overline{m}_c) = 1.224 \pm 0.017\,(\text{exp}) \pm 0.031\,(B) \pm 0.023\,(m_c)\ \text{GeV}$$

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Lattice determinations combined with perturbation theory (most recent analysis)

⇨ Lattice results for $c\bar{c}$ moments of the pseudoscalar & vector current-current correlators ("experimental data") Allison et al

⇨ 3-loop perturbative computations of the moments (fixed order method)

Chetyrkin et al

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Charmonium sum rules

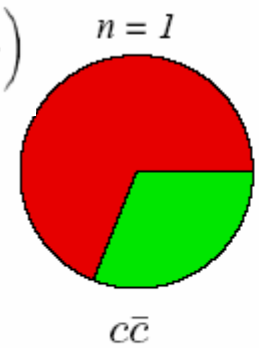
$$M_n = \int_{4m^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} R(s) = \frac{1}{2\pi i} \int_C \frac{ds}{s^{n+1}} \Pi\left(\frac{s}{4\overline{m}_c^2}\right)$$

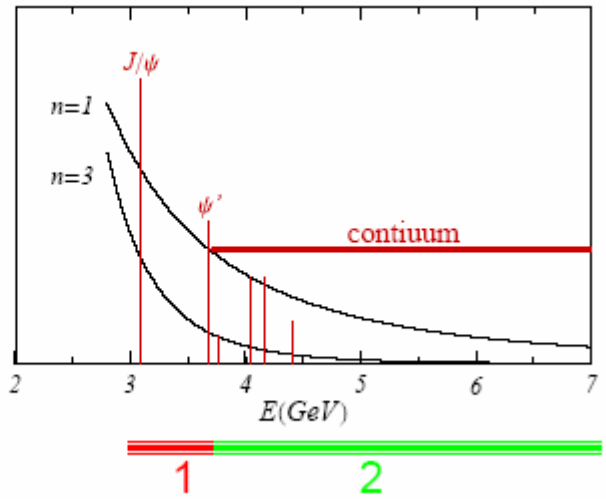
$$\Pi(q^2 \approx 0, m^2) = \frac{1}{12\,\pi^2\,Q_q^2} \sum_{n=1}^{\infty} M_n\, q^{2n}$$

duality bound $n < 3$

relativistic: $n \lesssim 4$

- $\overline{\text{MS}}$ mass
- usual loop expansion in powers of α_s





Charmonium sum rules

$$M_n = \int_{4m^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} R(s) = \frac{1}{2\pi i} \int_{|s|=s_0} \frac{ds}{s^{n+1}} \Pi\left(\frac{s}{4\overline{m}_c^2}\right)$$

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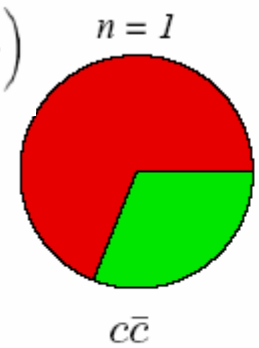

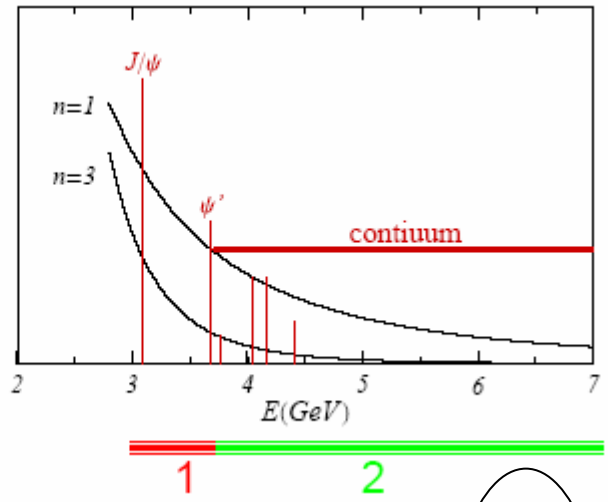

Fixed order analysis (s-independent)

$\mu \approx m_c^2$

Kühn et al ('08) [3] $\quad m(m) = 1.286 \pm 0.009_{\text{exp}} \pm 0.009_{\alpha} \pm 0.002_{\mu}$

Boughez et al ('08) [4] $\quad 1.295 \pm 0.012_{\text{exp}} \pm 0.009_{\alpha} \pm 0.003_{\mu}$

Maier et al (08) [5] $\quad 1.277 \pm 0.006_{\text{exp}} \pm 0.014_{\alpha} \pm 0.005_{\mu}$

Only for n = 1 **[3,4]**, 2 **[5]** 3-loops in pert. theory. Updated experimental data

Tiny errors! (underestimated ?)

Need for contour improved analysis

Contour improved analysis

First applied to hadronic tau decays Liberder Pich ('92) (See talk by A. Pich WG I)

Now μ depends on s $\rightarrow$ rearrangement of higher order contributions

Similar to fixed order $\longrightarrow$ $\mu^2 = \xi^2 M^2 z$

Reweighs threshold versus continuum effects $\longrightarrow$ $\mu^2 = \xi^2 M^2 (1-z)$

$\rightarrow z = \dfrac{q^2}{4m^2}$

2 - loops

Hoang, Jamin (2004) [6]

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Calculations $\longrightarrow$ more convenient through the vacuum polarization function

$$(g_{\mu\nu} q^2 - q_\mu q_\nu)\, \Pi(q^2) = -i \int \mathrm{d}x\, e^{iqx} \langle 0 | T\, j_\mu(x) j_\nu(0) | 0 \rangle$$

$$R(q^2) = 12\pi\, Q_q^2\, \mathrm{Im}\, \Pi(q^2 + i0, m^2)$$

Dispersion relation

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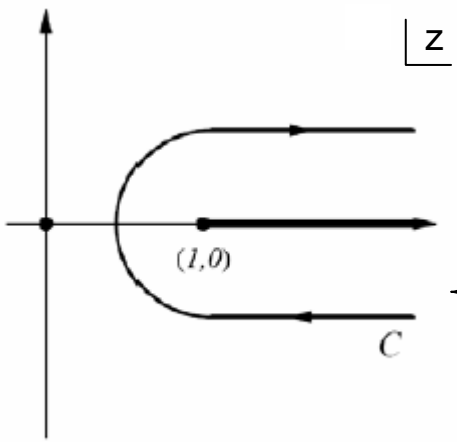


Solve the RGE for strong coupling in the complex plane

Need full dependence of polarization function $\longrightarrow$ Padé

Difference between two methods much higher that individual errors due to μ variation

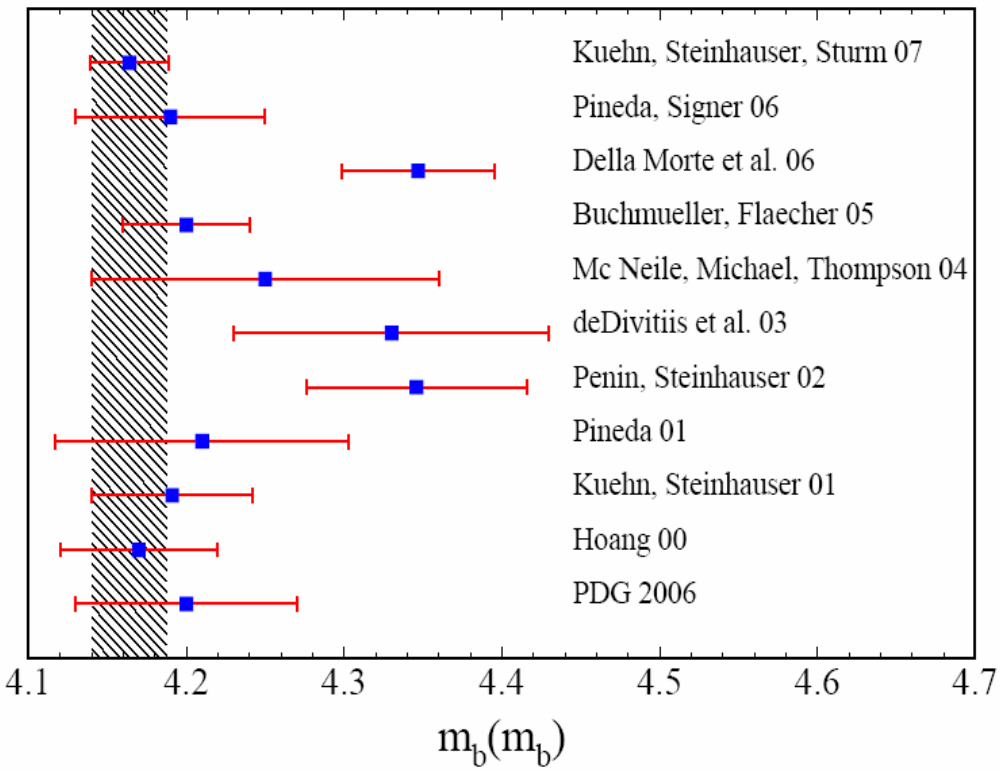
Determination of m_b

Spectral moments of inclusive B decays (nonrelativistic)

Bottomonium sum rules (relativistic)

Lattice

Taken from Kühn et al [3]



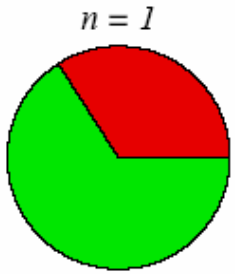
$m_b(m_b)$ (GeV)	Method
4.164 ± 0.025	low-moment sum rules, NNNLO
4.19 ± 0.06	Υ sum rules, NNLL (not complete)
4.347 ± 0.048	lattice (ALPHA), quenched
4.20 ± 0.04	fit to B decay distribution, $\alpha_s^2\beta_0$
$4.25 \pm 0.02 \pm 0.11$	lattice (UKQCD)
4.33 ± 0.10	lattice, quenched
4.346 ± 0.070	$\Upsilon(1S)$, NNNLO
$4.210 \pm 0.090 \pm 0.025$	$\Upsilon(1S)$, NNLO
4.191 ± 0.051	low-moment sum rules, NNLO
4.17 ± 0.05	Υ sum rules, NNLO
4.20 ± 0.07	PDG

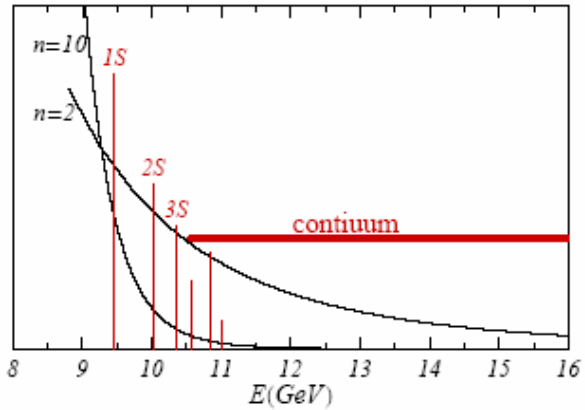
Also low-moment sum rules N^{3}LO Boughezal et al [4]

$$m_b(m_b) = 4.205 \pm 0.058$$

Determination of m_b from sum rules

A) relativistic: $n \lesssim 4$





exp. data:

- $\Upsilon(1S),\ldots,\Upsilon(6S)$
- no data in the R_b continuum, $\rightarrow$ models (!)

Large experimental errors because of the continuum

1 2

Refs [3,5] substitute by theory, that is why they have small errors

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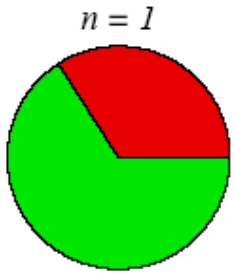

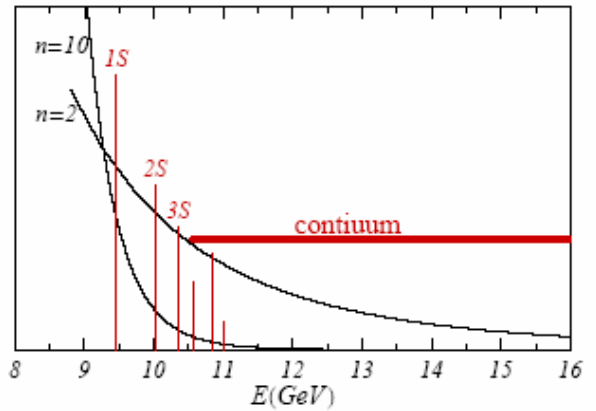

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Boughezal et al ('08) [4] $4.205\pm0.057_{\rm exp}\pm0.010_{\alpha}\pm0.002_{\mu}$

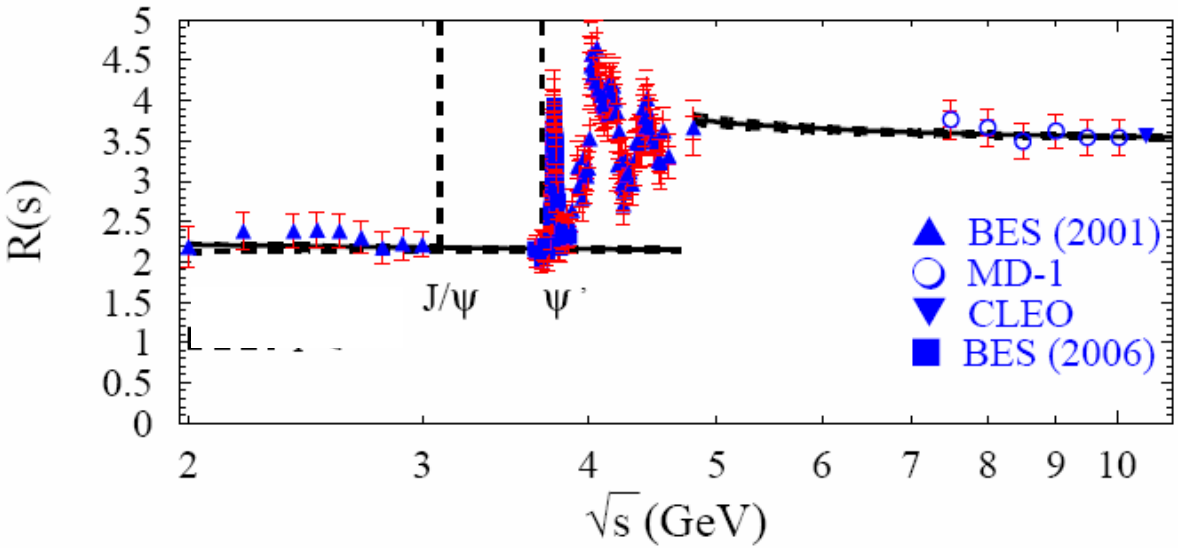
Maier et al ('08) [5] $4.162\pm0.014_{\rm exp}\pm0.012_{\alpha}\pm0.003_{\mu}$

Only for n = 1 **[4]**, 2 **[3,5]** 3-loops in pert. Theory. Fixed order only

Tiny errors! (underestimated ?)

Need for contour improved analysis

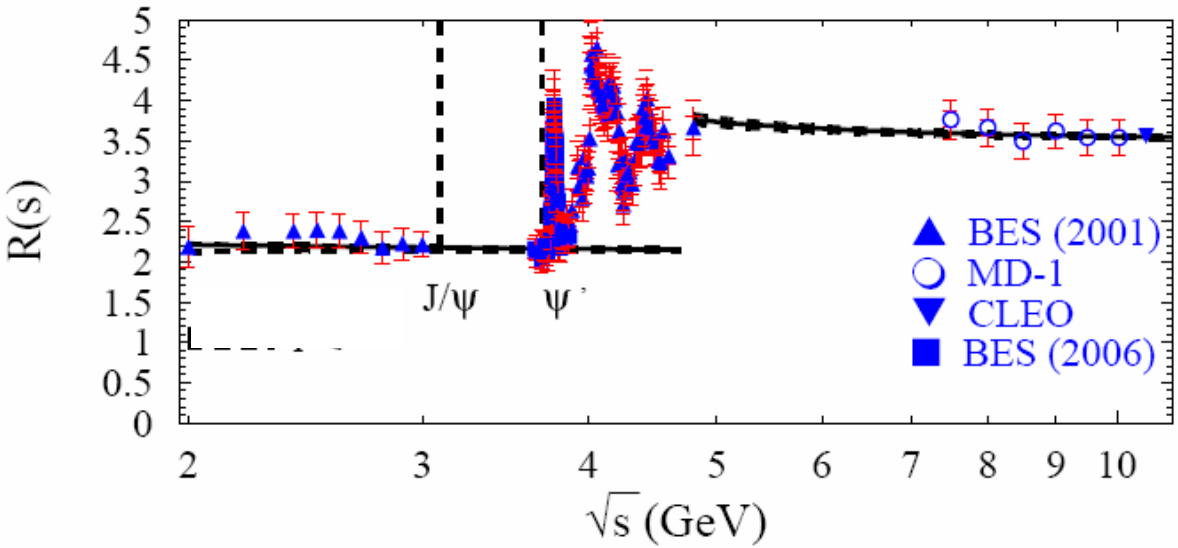
A critical view on sum rules



Experimental input

- Resonances region ☺
- Threshold region 😐
- Continuum region ☹

A critical view on sum rules



Experimental input

- Resonances region ☺
- Threshold region 😐
- Continuum region ☹

n	$1(10^0)$	$2(10^1)$	$3(10^2)$	$4(10^3)$
Kühn;Maier	0.2166(31)	0.1497(27)	0.1312(27)	0.1249(27)
Boughezal	0.2087(42)	–	–	–

charm

n	$1(10^3)$	$2(10^5)$	$3(10^7)$	$4(10^9)$
Kühn;Maier	4.601(43)	2.881(37)	2.370(34)	2.178(32)
Boughezal	4.456(121)	–	–	–

bottom

A critical view on sum rules

Determine m(1,2,3 GeV) and then run down to m(m)

$$[4m(\mu)^2]^n M_n^{pert} = \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c}\right) + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2\right)$$
$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3\right) + \dots .$$

$$l_{m_c} \equiv \ln(m_c^2(\mu)/\mu^2)$$

A critical view on sum rules

Determine m(1,2,3 GeV) and then run down to m(m)

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$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3\right) + \dots .$$

$$l_{m_c} \equiv \ln(m_c^2(\mu)/\mu^2)$$

Let us repeat Boughezal analysis with their input, and fixed order, but determining directly m(m)

$$[4\bar{m}(\bar{m})^2]^n M_n^{pert} = f_n^0 + \frac{\alpha_s(\mu)}{\pi} f_n^1 + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[f_n^2 + \frac{\beta_0}{4} f_n^1 \log\left(\frac{\mu^2}{\overline{m}^2}\right)\right]$$
$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left[f_n^3 + \left(\frac{\beta_0}{2} f_n^2 + \frac{\beta_1}{16} f_n^1\right) \log\left(\frac{\mu^2}{\overline{m}^2}\right) + \frac{\beta_0^2}{16} f_n^1 \log^2\left(\frac{\mu^2}{\overline{m}^2}\right)\right]$$

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A critical view on sum rules

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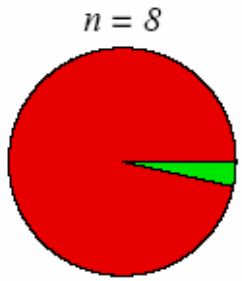
$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left[f_n^3 + \left(\frac{\beta_0}{2} f_n^2 + \frac{\beta_1}{16} f_n^1\right) \log\left(\frac{\mu^2}{\overline{m}^2}\right) + \frac{\beta_0^2}{16} f_n^1 \log^2\left(\frac{\mu^2}{\overline{m}^2}\right)\right]$$

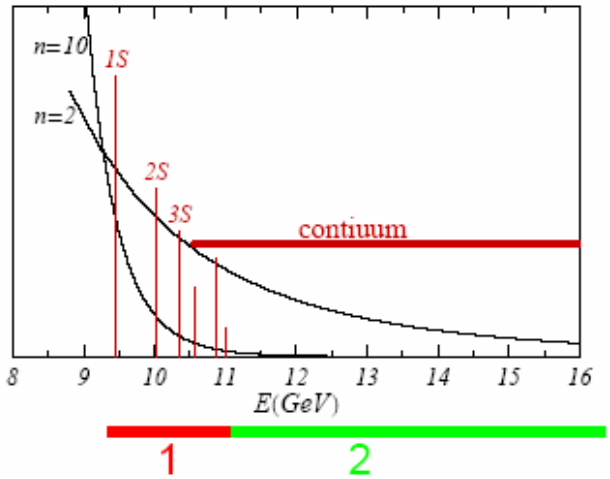
$$1.286 \pm 0.009_{\rm exp} \pm 0.011_{\mu} \pm 0.012_{\rm exp} \longleftrightarrow 1.295 \pm 0.009_{\rm exp} \pm 0.003_{\mu} \pm 0.012_{\rm exp}$$

Different central values, different μ variation. Similarly for bottom.

Determination of m_b from sum rules

B) non-relativistic: $4 \lesssim n \lesssim 10$





exp. data:

- $\Upsilon(1S)$, $\Upsilon(2S)$ dominate
- very rough model for R_b continuum suffi cient

Experimental error much smaller, since continuum region is suppressed

Determination of m_b from sum rules

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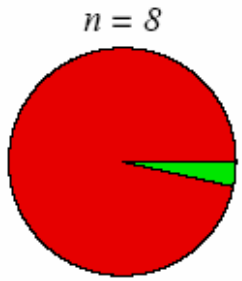

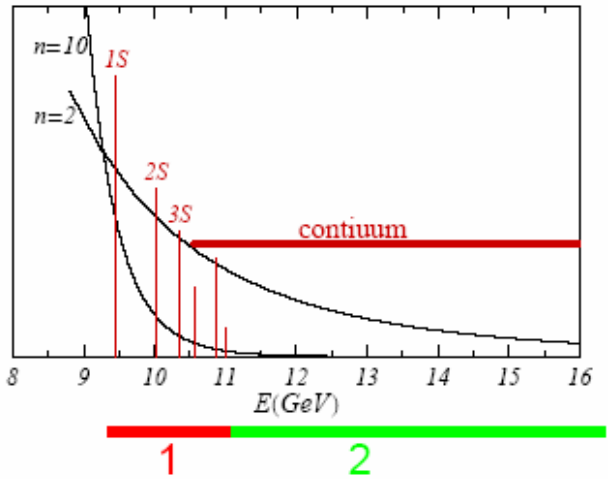

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- very rough model for R_b continuum suffi cient

Experimental error much smaller, since continuum region is suppressed

Melnikov: $m(m) = 4.19 \pm 0.07$ GeV

Beneke: 4.25 ± 0.10 GeV

Hoang: 4.17 ± 0.05 GeV

And also from spectral moments of

Inclusive b decays

Construction of $\Pi^{(2,3)}$

$$\Pi = \Pi^{(0)} + C_F \left(\frac{\alpha_S}{\pi}\right)\Pi^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2 \Pi^{(2)} + \left(\frac{\alpha_S}{\pi}\right)^3 \Pi^{(3)} + \cdots$$

For $\Pi^{(0)}$ and $\Pi^{(1)}$ the full q^2 and mas dependence is exactly known

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30 moments known Maier et al (2008)

For $\Pi^{(2)}$ and $\Pi^{(3)}$ $\rightarrow$

- $q^2 \sim 0$ (fixed order moments) — Regular (no logs)
- $q^2 \sim \infty$ (expansion for small mass) — $\log^n(-4z)$
- $q^2 \sim 4m^2$ (threshold expansion) — $\log^n(1-z)$ + Coulomb singularity

Only two moments known 1st [3,4], 2nd [5]

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Only two moments known 1st [3,4], 2nd [5]

The three regimes can be matched into a single function $\longrightarrow$ Padé

$\Pi^{(2)}$ $\longrightarrow$ Chetyrkin et al ('96)

$\Pi^{(3)}$ $\longrightarrow$ Taskforce ('08), [1]

Prediction of moments and constants

Renders R-ratio at all energies

Reliable estimation of errors

Construction of $\Pi^{(2,3)}$

$$\Pi^{(2,3)}(z) = \Pi^{(2,3)}_{\rm reg}(z) + \Pi^{(2,3)}_{\log}(z)$$

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General form of a Padé

$$P_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{1 + \sum_{j=1}^{m} b_j x^j}$$

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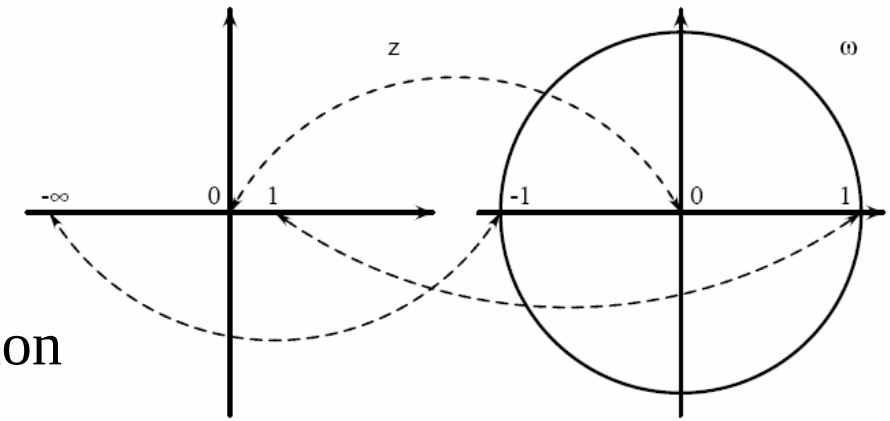
General form of a Padé →

$$P_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{1 + \sum_{j=1}^{m} b_j x^j}$$

It would be unwise using $P_{m,n}(z)$, we better do a conformal mapping

$$\omega = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}$$

and use $P_{m,n}(\omega)$



ω has a cut for z > 1, much as Π

so the Padé contributes to the R-function

Construction of $\Pi^{(2,3)}_{\log}$

$$\Pi^{(2,3)}_{\log}(z) = \Pi^{(2,3)}_{\rm thr}(z) + \Pi^{(2,3)}_{\rm inf}(z) + \Pi^{(2,3)}_{\rm zero}(z)$$

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up to $1/z^2$

$\ln^m(1-z)$	$[\Pi^{(1)}(z)]^m$
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$$G(z) = \frac{2\,u\,\ln u}{u^2-1}, \quad \text{with} \quad u \equiv \frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}$$

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Higher logs tuned by $\dfrac{1+a\,z}{z}$

up to $1/z^2$

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Handle for estimating errors!

The choice of "log-removers"

is not unique

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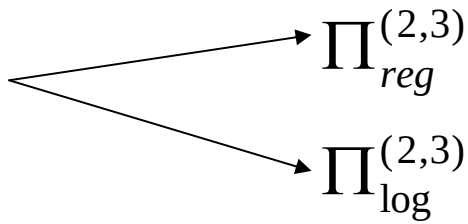
Construction of $\Pi^{(2,3)}_{reg}$

Leading Coulomb singularity has no log(1 - z)



New handle for estimating uncertainties

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FlaviA net

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$\Pi^{(2,3)}_{reg}$

$\Pi^{(2,3)}_{\log}$

New handle for estimating uncertainties

$\Pi^{(2,3)}_{reg}$

Contains all known moments plus

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If unknown, it can be predicted (unknown moments as well)

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For example:

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$$P(\omega) = \frac{1-\omega}{(1+\omega)^2}\left[\Pi^{(2)}_{\rm reg}(z) - \Pi^{(2)}_{\rm reg}(-\infty)\right] \longrightarrow \Pi^{(2)}(z) = \frac{(1+\omega)^2}{1-\omega}P(\omega) - P(0) + \Pi^{(2)}_{\log}(z)$$

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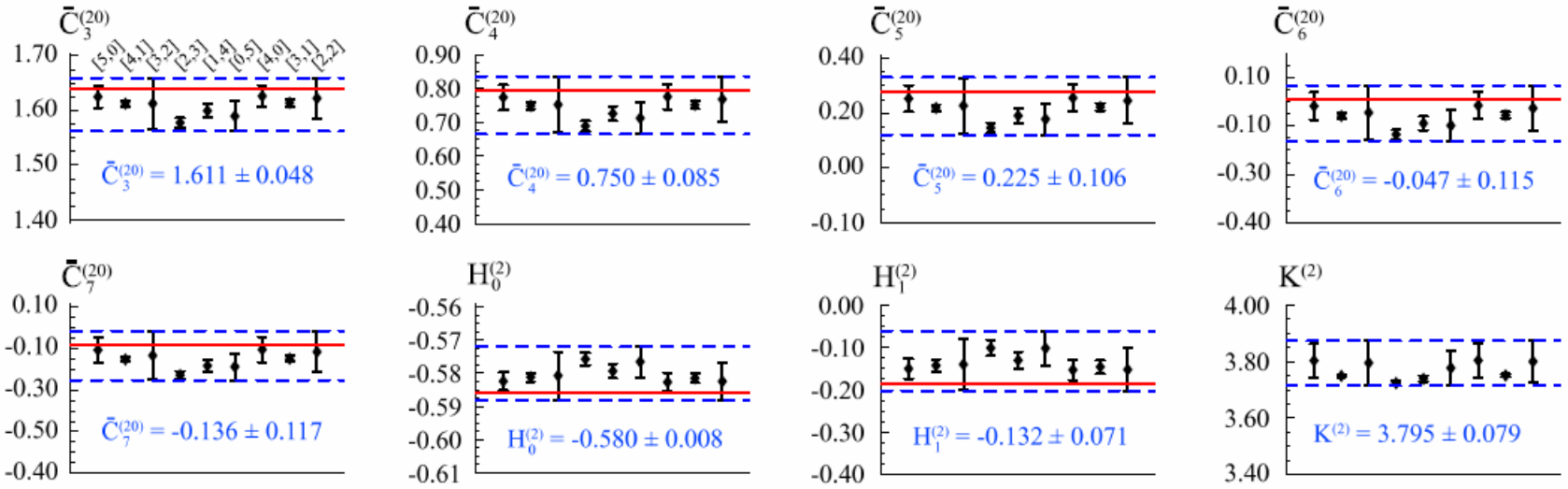
We also demand that P(ω) has no $1/z^{(2n+1)/2}$ terms at infinity up to the order considered

Padé predictions for $\Pi^{(2)}$



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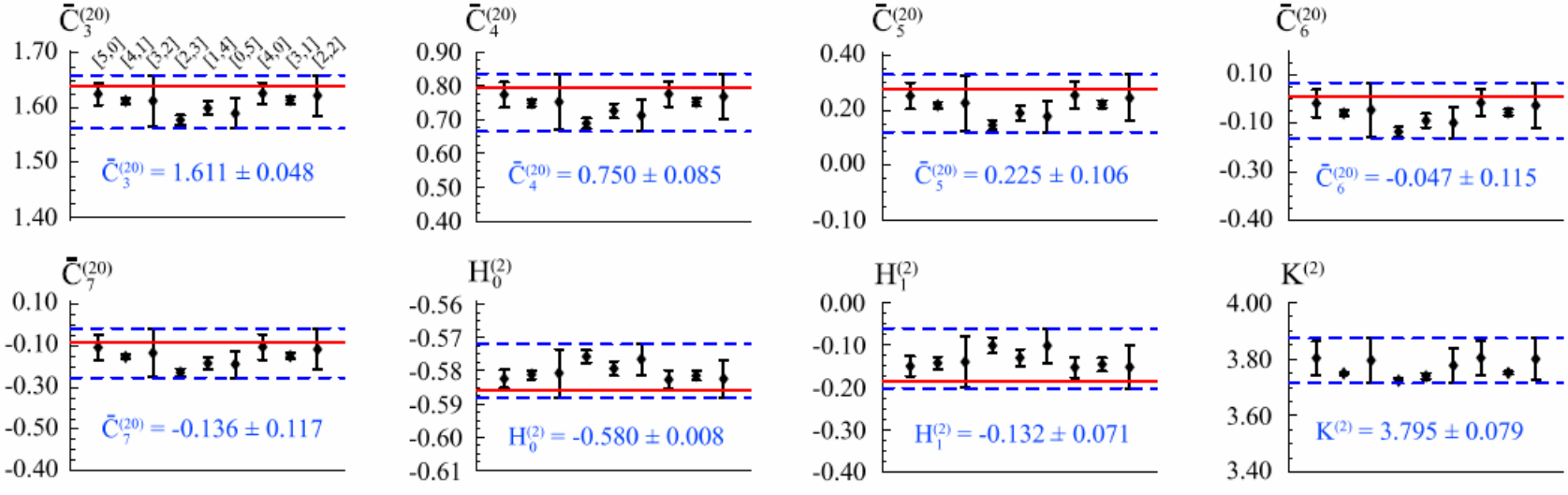
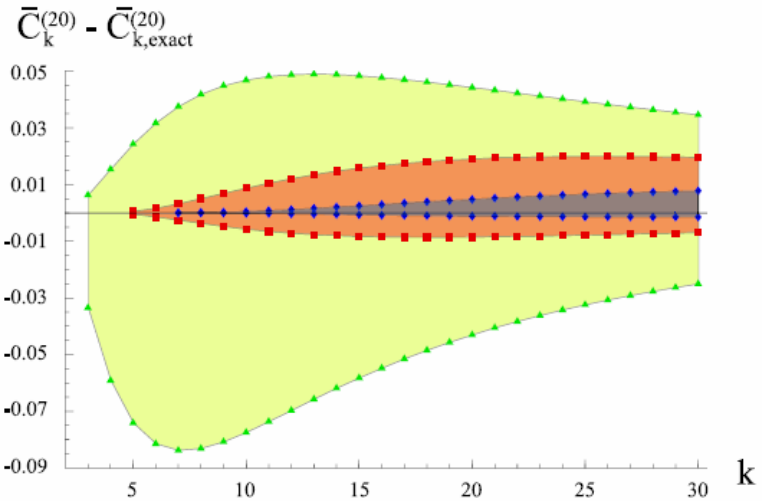
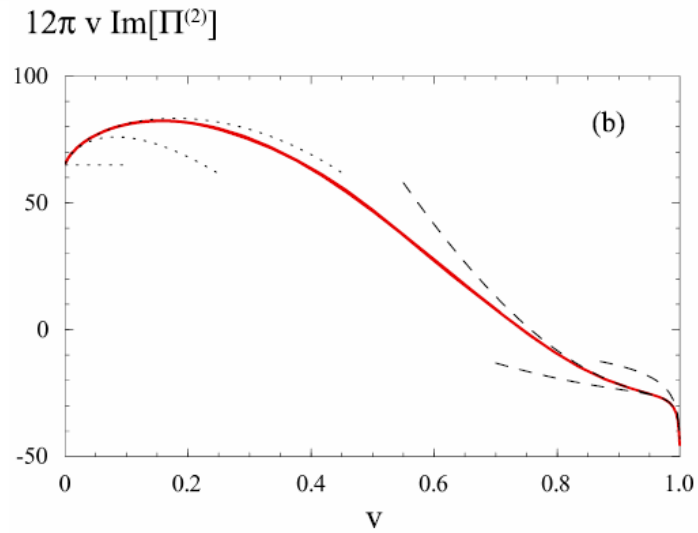
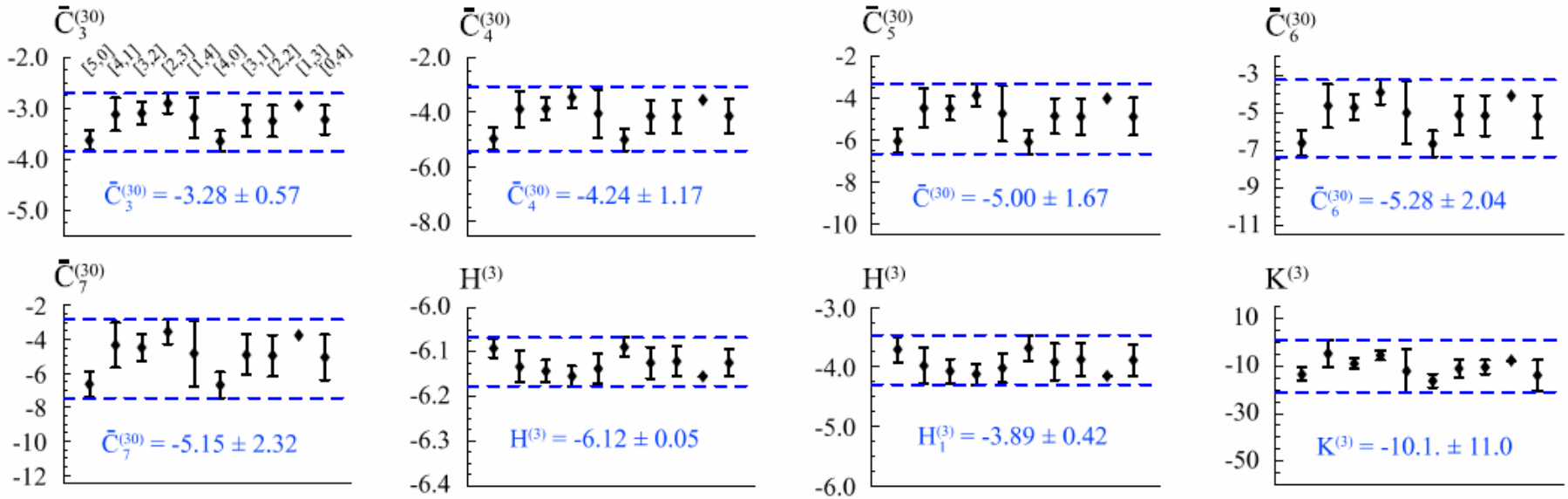






Padé predictions for $\Pi^{(3)}$



	$n_f = 4$	$n_f = 5$
$\bar{C}_1^{(30)}$	-5.6404	-7.7624
$\bar{C}_2^{(30)}$	-3.4937	-2.6438
$\bar{C}_3^{(30)}$	-3.279 ± 0.573	-1.457 ± 0.579
$\bar{C}_4^{(30)}$	-4.238 ± 1.171	-1.935 ± 1.201
$\bar{C}_5^{(30)}$	-4.996 ± 1.666	-2.507 ± 1.732
$\bar{C}_6^{(30)}$	-5.280 ± 2.045	-2.809 ± 2.150
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$H_0^{(3)}$	-6.122 ± 0.054	-4.989 ± 0.053
$H_1^{(3)}$	-3.885 ± 0.417	-3.180 ± 0.405
$K^{(3)}$	-10.09 ± 11.00	-5.97 ± 10.09

Agreement with Chetyrkin !

$$H_1^{(3)} = -4.33306$$

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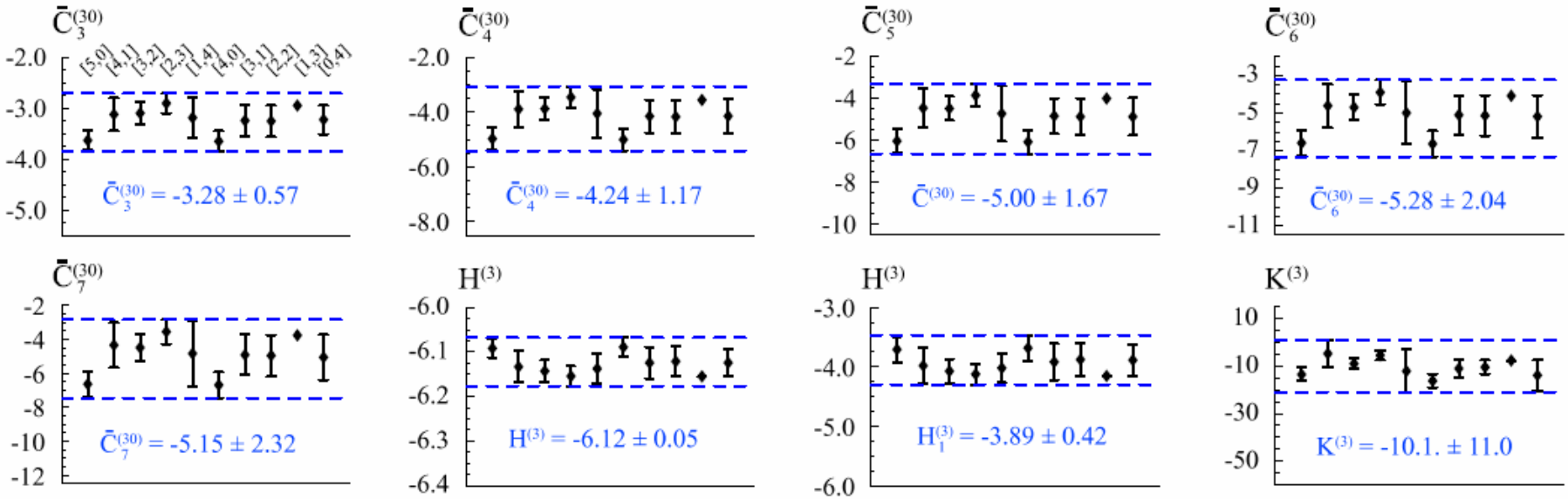

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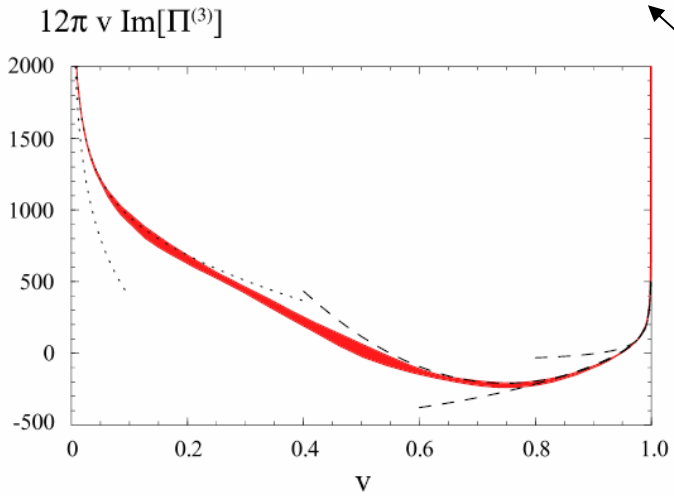

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Preliminary (!) results: charm

It constitutes an update of Joang Jamin [6]. Only the theoretical input is updated

We use more precise two loop results, with the updated Padé's

We use new results for three loop, from Padé approximants

Experimental input still not updated! Do not focus on central values.

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n	1	2	3	4
Method 1	1.277(71+23)	1.272(42+17)	1.267(36+11)	1.264(37+8)
Method 2	1.268(68+24)	1.264(41+15)	1.263(35+10)	1.260(36+7)
Method 3	1.277(74+24)	1.290(45+20)	1.295(37+18)	1.295(38+14)
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$m_c(\delta\exp + \delta\mu)$

Central values for each method approach

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has decreased factor 2

$m_c(\delta\exp+\delta\mu)$

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Method 3	1.277(74+24)	1.290(45+20)	1.295(37+18)	1.295(38+14)
Convined	$1.274(69+24_{\mu}+5_{method})$	1.282(43+18+13)	1.284(36+14+16)	1.282(38+10+18)

three loops [w.i.p.] $1.279\pm0.040_{\exp}\pm0.003_{\mu}\pm0.012_{method}\pm0_{n}$

n	1	2	3	4
Method 1	1.285(71+11)	1.277(42+9)	1.271(36+6)	1.264(37+4)
Method 2	1.272(68+8)	1.267(41+5)	1.264(35+3)	1.259(36+2)
Method 3	1.278(74+6)	1.287(45+4)	1.290(37+2)	1.289(38+1)
Convined	1.282(73+7+6)	1.279(43+4+10)	1.279(36+2+13)	1.274(37+1+15)

Preliminary (!) results: charm

It constitutes an update of Joang Jamin [6]. Only the theoretical input is updated

We use more precise two loop results, with the updated Padé's

We use new results for three loop, from Padé approximants

Experimental input still not updated! Do not focus on central values.

two loops [6] $1.283 \pm 0.040_{\exp} \pm 0.016_{\mu} \pm 0.014_{method} \pm 0.001_{n}$

n	1	2	3	4
Method 1	1.277(71+23)	1.272(42+17)	1.267(36+11)	1.264(37+8)
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has decreased factor 2

$m_c \left(\delta \exp + \delta\mu \right)$

Central values for each method approach

will decreased substantially when including updated experimental data

Even more preliminary: bottom

Experimental input taken from Kühn et al [3], and (for now) ignore errors.

Many things need to be checked and experimental input reconsidered.

Difference of results for various n's, enhances the error

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Experimental input taken from Kühn et al [3], and (for now) ignore errors.

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Difference of results for various n's, enhances the error

two loops $4.152 \pm 0.011_{\mu} \pm 0.012_{method} \pm 0.009_{n}$

n	1	2	3	4
Method 1	4.125(22)	4.146(15)	4.159(10)	4.168(06)
Method 2	4.117(26)	4.134(15)	4.154(10)	4.165(06)
Method 3	4.125(22)	4.162(13)	4.179(07)	4.188(03)
Convined	4.119(24+4)	4.147(14+14)	4.165(8+10)	4.175(4+12)

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Experimental input taken from Kühn et al [3], and (for now) ignore errors.

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3 loops [w.i.p.] $4.153 \pm 0.003_{\mu} \pm 0.009_{method} \pm 0.016_{n}$

n	1	2	3	4
Method 1	4.136(07)	4.154(06)	4.164(04)	4.170(02)
Method 2	4.123(07)	4.147(04)	4.159(03)	4.167(02)
Method 3	4.131(04)	4.163(01)	4.178(02)	4.185(03)
Combined	4.130(7+6)	4.136(3+8)	4.168(2+10)	4.177(3+9)

Even more preliminary: bottom

Experimental input taken from Kühn et al [3], and (for now) ignore errors.

Many things need to be checked and experimental input reconsidered.

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But…

Things go in the right direction!

Perturbative errors go down when more loops are included

Errors due to Padé approximants well under control

Promising results once the analysis is finished

Conclusions and outlook

- It is essential to have a reliable error estimate for charm and bottom masses
- Concerning relativistic sum rules, a contour improved analysis is mandatory.
  - For that we need to know the exact vacuum polarization function.
  - Experimental input must be treated with care (secondary radiation, singlet …)
- The Padé method is the best hope we can have for a semi analytical solution for arbitrary q^2 and masses for three and four loops.
- It can be systematically improved if more individual pieces (moments) are known.
  - It can predict constant pieces, but unfortunately cannot predict logs ☹.
  - It has proven to be useful and predicts known pieces with high accuracy
- The difference between the center values of methods 2 and 3 has gone down, but still is much bigger that the individual errors due to scale variation.
- Errors will go further down when updating exp. input, but still larger than in [3,4,5]
- The analysis can be easily extended to other correlators ⟶ connection to lattice

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