Dispersive approaches for K_{l3} form factors.

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  + In preparation

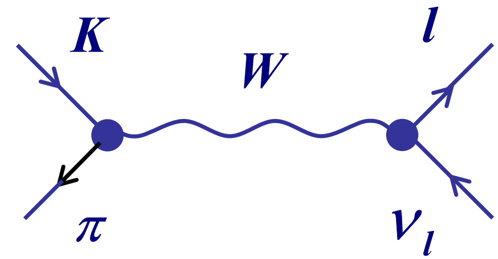
In the memory of Jan Stern

Outline

1. Introduction and Motivation
2. Dispersive representation of the scalar and vector form factors
3. Results of the dispersive analysis
4. Matching of the 2 loop ChPT with the dispersive representation of the $K\pi$ scalar form factor

1. Introduction and Motivation

1.1 Definition



- K_{l3} decays $K \rightarrow \pi l \nu_l$
- The hadronic matrix element :

$$\boxed{\langle \pi(p_\pi) | \; \bar{s}\gamma_\mu u \; | K(p_K) \rangle = f_+(t)(p_K + p_\pi)_\mu + f_-(t)(p_K - p_\pi)_\mu}$$

→ $f_+(t)$, $f_-(t)$: form factors

→ $t = q^2 = (p_\mu + p_{\nu_\mu})^2 = (p_K - p_\pi)^2$

- Analysis in terms of $f_+(t)$ and $\boxed{f_s(t) = f_+(t) + \frac{t}{m_K^2 - m_\pi^2} f_-(t)}$

→ Normalization $\bar{f}_+(t) = \frac{f_+(t)}{f_+(0)}$ and $\bar{f}_0(t) = \frac{f_s(t)}{f_+(0)}$, $\bar{f}_+(t) = \bar{f}_0(0) = 1$

- $\bar{f}_+(t)$ accessible in K_{e3} and $K_{\mu 3}$ decays
- $\bar{f}_0(t)$ only accessible in $K_{\mu 3}$ (suppressed by m_l^2/M_K^2) + correlations

⇨ difficult to measure.

1.2 Extraction of Vus

- Decay rate formula for K_{l3} $\quad l=(e,\mu)$

$$\Gamma_{K^{+/0}l3} = \frac{Br_{K^{+/0}l3}}{\tau_{K^{+/0}}} = \frac{C_K^2 G_F^2 m_{K^{+/0}}^5}{192\pi^3} S_{EW} \left(1 + 2\Delta_{K^{+/0}l}^{EM}\right) \left| f_+^{K^{+/0}}(0) V_{us} \right|^2 I_{K^{+/0}}^l$$

Kaon life time

½ for K^+, 1 for K^0

$$I_{K^{+/0}}^l = \int dt \frac{1}{m_{K^{+/0}}^8} \lambda^{3/2} F\left(t, \overline{f}_+(t), \overline{f}_0(t)\right)$$

- Experimental inputs: → $\boxed{Br_{K^{+/0}l3}, \tau_{K^{+/0}}}$ Kl3 branching ratios, Kaon life time, with good treatment of radiative corrections

  → $\boxed{I_{K^{+/0}}^l}$ Phase space integrals, need form factor shapes extracted from Dalitz plot, from **NA48, KTeV, KLOE and ISTRA+.**

- Theoretical inputs: → $\boxed{S_{EW}}$ Short distance EW corrections **[Marciano&Sirlin]**

  → $\boxed{\Delta_{K^{+/0}l}^{EM}}$ Long distance EM corrections (use of ChPT)

[Bijnens&Prades'97],[Knecht et al'00], [Cirigliano et al '02'04] [Andre'04], [Gatti'05],[Moussallam et al'06], [Cirigliano, Giannotti, Neufeld '08]

1.2 Extraction of Vus

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Kaon life time

½ for K^+, 1 for K^0

$$I^l_{K^{+/0}} = \int dt \frac{1}{m^8_{K^{+/0}}}\lambda^{3/2} F\left(t, \overline{f}_+(t), \overline{f}_0(t)\right)$$

- Extraction of $\boxed{f_+(0)|\mathcal{V}_{us}|} \xrightarrow{f_+(0)} \boxed{|\mathcal{V}_{us}|}$: test of the CKM unitarity

$$\boxed{|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 \overset{?}{=} 1}$$

$0^+ \to 0^+$ β decays $\qquad$ K_{l3} decays $\qquad$ Negligible (B decays)

- Extraction of $\Delta_{SU(2)}$: $\boxed{\dfrac{f_+^{K^+}(0)}{f_+^{K^0}(0)} = 1 + \Delta_{SU(2)}}$, access to the ratio of light quark mass

$\Rightarrow$ $\boxed{R = \dfrac{m_s - \hat{m}}{m_d - m_u}}$ Important role in ChPT, Prediction : $\boxed{\begin{array}{l}\Delta_{SU(2)} = 2.36(22)\% \\ R = 40.8(3.2)\end{array}}$

[Leutwyler'96], [Cirigliano et al'01]

1.3 Theoretical knowledge for the scalar FF: CT relation

- Callan-Treiman Theorem: SU(2) x SU(2) theorem

$$\boxed{C = \overline{f_0}(\Delta_{K\pi}) = \frac{F_K}{F_\pi f_+^{K^0}(0)} + \Delta_{CT}}$$

$$\Delta_{K\pi} = m_K^2 - m_\pi^2$$

- Corrections of order m_u, m_d
  - → No chiral logarithms : $\Delta_{CT} \sim \mathcal{O}\left(m_{u,d} / 4\pi F_\pi\right)$
  - → Isospin limit m_d=m_u : $\boxed{\Delta_{CT}^{NLO} \sim -3.5.10^{-3}}$ **[Gasser & Leutwyler]**
  - → K^0 decay : no small denominators due to π^0 - η mixing $\left(\mathcal{O}\left((m_d - m_u)/m_s\right)\right)$.
  - → K^+ decay case : enhancement by π^0 - η mixing in the final state

⇒ $\boxed{\Delta_{CT}^{K^+} \sim few\ 10^{-2}}$ (K^0 ideal decay)

- Estimations of the higher order terms: corrections in $\mathcal{O}\left(m_{u,d}.m_s\right)$

⇒ $\boxed{\Delta_{CT} = (-3.5 \pm 8).10^{-3}}$

1.3 Theoretical knowledge for the scalar FF: CT relation

- Callan-Treiman Theorem: SU(2) x SU(2) theorem

$$\boxed{C = \overline{f_0}(\Delta_{K\pi}) = \frac{F_K}{F_\pi f_+^{K^0}(0)} + \Delta_{CT}}$$

$$\Delta_{K\pi} = m_K^2 - m_\pi^2$$

- Estimations of the higher order terms: corrections in $\mathcal{O}\left(m_{u,d}.m_s\right)$

⇨ $\Delta_{CT} = (-3.5 \pm 8).10^{-3}$

- In agreement with estimate
  - at $\mathcal{O}\left(p^4, e^2p^2, (m_d - m_u)p^2\right)$

$$\Delta_{CT}^{K^0} = (1.7 \pm 7).10^{-3}$$

$$\Delta_{CT}^{K^+} = (-10.4 \pm 7).10^{-3}$$

[Kastner & Neufeld'08]

1.4 Test of the Standard Model.

$$C_{SM} = \overline{f}_0(\Delta_{K\pi}) = \underbrace{\frac{F_K |V^{us}|}{F_\pi |V^{ud}|} \frac{1}{f_+(0)|V^{us}|} |V^{ud}|}_{B_{exp}} + \Delta_{CT}$$

- C is predicted in the Standard Model using the measured Br: $Br(K_{l2}/\pi_{l2})$, $\Gamma(K_{e3})$ and $|V_{ud}|$. ($|V_{us}|$ not needed in this prediction.)

$\Rightarrow$ $\boxed{B_{exp} = 1.2446 \pm 0.0041}$ $\ln C_{SM} = 0.2188(35) + \Delta_{CT}$

- Relation which tests the Standard Model very accurately for K^0. If physics beyond the SM: ~1% difference between C and B_{exp}. Uncertainties from Δ_{CT} and B_{exp} on the permile level $\Rightarrow$ opportunity to see a possible effect.

- Possible test of the lattice results for F_K/F_π, $f_+(0)$.

1.5 Matching with the ChPT 2 loop calculations

- Measurement of slope and curvature of the scalar form factor from experiments ⇨ determination of
  - F_K/F_π
  - $f_+(0)$
  - C_{12} and C_{34}, 2 $\mathcal{O}(p^6)$ LECs which enters K_{l3} decays
  - Callan-Treiman correction Δ_{CT}

- Why ?
  - LECs play an important role in ChPT calculations, enter different processes. Ex: C_{12} into $\eta \to 3\pi$ **[Bijnens&Ghorbani'07].**
  - Possibility to test the lattice calculation and resonance model estimates
  - Test of the Standard Model, knowledge of $f_+(0)$ ⇨ extraction of V_{us}

1.6 How to measure the form factor shapes ?

- Data available from **KTeV**, **NA48** and **KLOE** for K^0 and from **ISTRA+**, **NA48** and **KLOE** for K^+.
- Necessity to parametrize the 2 form factors $\overline{f}_+(t)$ and $\overline{f}_0(t)$ to fit the measured distributions.
- Different parametrizations available, 2 classes of parametrizations :

→ 1rst class: parametrizations based on mathematical rigourous expansion, the slope and the curvature are free parameters :

– Taylor expansion

$$\boxed{\overline{f}_{+,0}(t) = 1 + \lambda'_{+,0}\frac{t}{m_\pi^2} + \frac{1}{2}\lambda''_{+,0}\frac{t}{m_\pi^2} + \ldots}$$

– Z-parametrization, conformal mapping from t to z variable with |z|<1 improve the convergence of the series.

$$\boxed{f_{+,s}(t) = f_{+,s}(t_0)\frac{\phi(t_0,t_0,Q^2)}{\phi(t,t_0,Q^2)}\sum_{k=0}^{n} a_k\left(t_0,Q^2\right)z\left(t,t_0\right)^k}$$ **[Hill'06]**

Theoretical error can be estimated : for a specific choice of ϕ, $\sum_{k=0}^{n} a_k^2$ bounded ⇨ use of some high-energy inputs (τ data …).

→ 2nd class: parametrizations which by using physical inputs impose specific relations between the slope and the curvature

⇒ reduce the correlations, only one parameter fit.

– Pole parametrization, the dominance of a resonance is assumed

$$\boxed{\bar{f}_{+,0}(t) = \frac{m_{V,S}^2}{m_{V,S}^2 - t}}$$

$m_{V,S}$ is the parameter of the fit

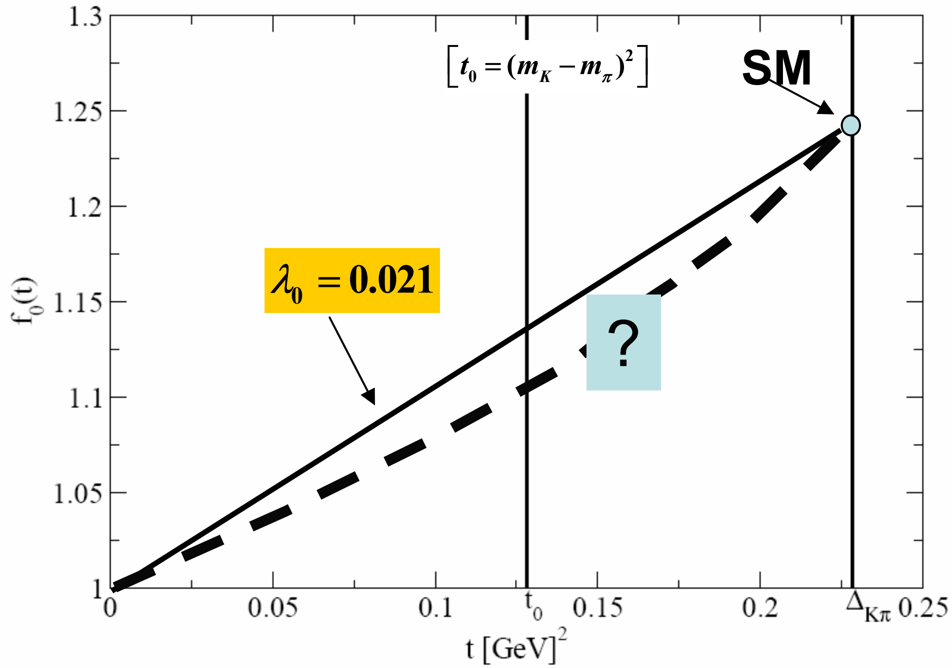
– Dispersive parametrization: use of the low energy Kπ scattering data and presence of resonances to contrain by dispersion relations the higher order terms of the expansion. Analysis from **[Jamin, Oller, Pich'04], [Bernard, Oertel, E.P, Stern'06],** for the scalar form factor and from **[Moussallam'07], [Jamin, Pich &Portoles'08]** for the vector form factor using τ data.

- Two requirements in the measurements of the form factor shapes from the K_{l3} data
  - Try to measure the form factor shapes from the data with the best accuracy and the slope and curvature of the scalar form factor are of special interest to allow for a matching with the 2 loop ChPT calculations.
  - Measurement of $\bar{f}_0(\Delta_{K\pi}) \equiv C$ to test the Standard Model via the CT correction
- Experimental constraints: if one uses a parametrization from 1rst class, for example a Taylor expansion,
  - Only two parameters measurable for the vector form factor, λ'_+ and λ_+''
  - Only one parameter accessible for the scalar form factor λ'_0
  - The correlations are strong,

λ'_0	1	-0.9996	-0.97	0.91
λ''_0		1	0.98	-0.92
λ'_+			1	-0.98
λ''_+				1

[Franzini]

- Using a linear parametrization, it is impossible to extrapolate with a good precision up to the CT point !



- Necessity to use a second class parametrization which reduces the correlation, only one parameter is fitted.
  - For the vector form factor ⇨ pole parametrization with dominance of the K*(892) in good agreement with the data.
  - For the scalar form factor, not a such obvious dominance ⇨ necessity of a dispersive parametrization to improve the extraction of the ff parameters.

2. Dispersive Representation of the $K\pi$ form factors

2.1 Dispersive parametrization for the scalar FF

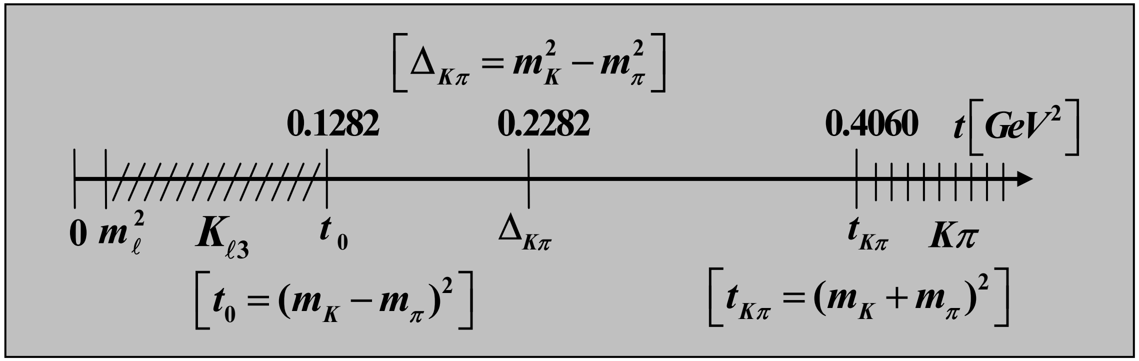
- Problem : How to construct a very precise representation of $\bar{f}_0(t)$ between 0 and $\Delta_{K\pi}$?
- Knowledge :
  - → $\bar{f}_0(0) = 1$
  - → $\bar{f}_0(\Delta_{K\pi}) = C$, Callan-Treiman point
  - → $K\pi$ scattering phase
  - → Asymptotic behaviour of the form factor : $\bar{f}_0(s) \overset{s\to\infty}{=} \mathcal{O}(1/s)$
- A dispersion relation with two substractions at 0 and $\Delta_{K\pi}$ for $\ln(\bar{f}_0(t))$, assuming that $\bar{f}_0(t)$ has no zero

$$\bar{f}_0(t) = \exp\left[\frac{t}{\Delta_{K\pi}}\left(\ln C - G(t)\right)\right]$$

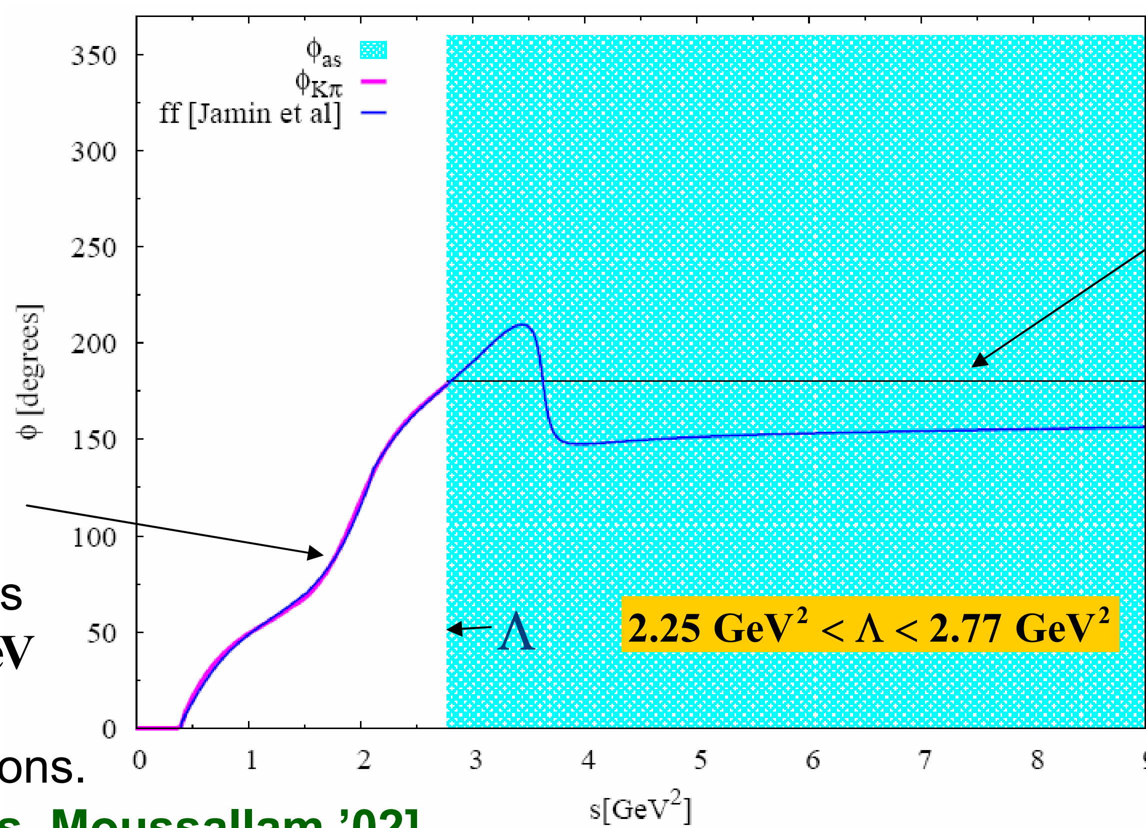
with

$$G(t) = \frac{\Delta_{K\pi}(\Delta_{K\pi} - t)}{\pi}\int_{t_{K\pi}}^{\infty}\frac{ds}{s}\frac{\phi(s)}{(s-\Delta_{K\pi})(s-t)}$$

  - → $\phi(t)$ phase of the form factor : $\bar{f}_0(t) = \left|\bar{f}_0(t)\right| e^{i\phi(t)}$



- Phase used



Asymptotic value

$\phi(s) \xrightarrow[s\to\infty]{} \pi$

$\overline{f}_0(t) = \mathcal{O}(1/t)$

$K\pi$ scattering data

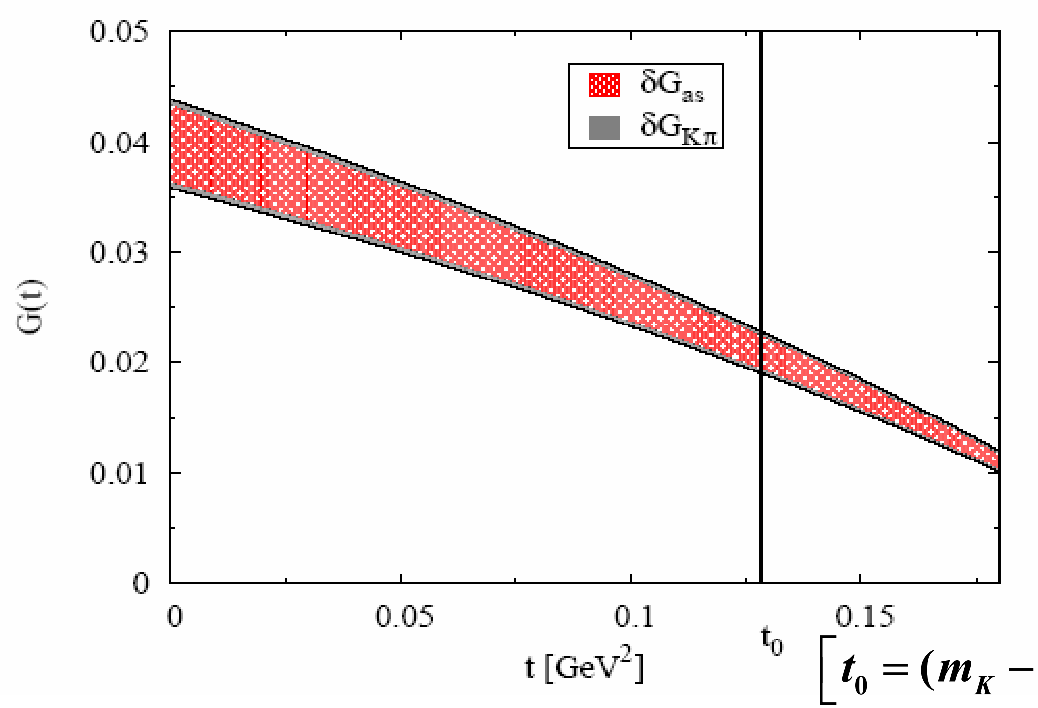
Experimental inputs for $1\,\mathrm{GeV} < E < 2.5\,\mathrm{GeV}$ **[Aston et al]**+ Roy Steiner equations. **[Buettiker, Descotes, Moussallam '02]**

- Elastic up to ~1.5 GeV ⇨ $t < \Lambda:\ \phi(t) = \phi_{K\pi}(t) = \delta^{s,\frac{1}{2}}_{\pi,K}(t) \pm \Delta\delta^{s,\frac{1}{2}}_{\pi,K}(t)$

[Watson theorem]

$t > \Lambda:\ \phi(t) = \phi_{as}(t) = \pi \pm \pi$

- 2 subtractions ⇨ Rapid convergence of G(t)



$\left[t_0 = (m_K - m_\pi)^2 \right]$

G(t) with the uncertainties added in quadrature

$$G(0) = 0.0398 \pm 0.0040$$

G(t) does not exceed 20% of the expected value of lnC

lnC~0.20

- Study of the robustness :
  - Result not sensitive to Λ (G(0) remains stable)
  - Isospin breaking corrections
  - Influence of a zero

- Apart from the parameter (lnC) to be determined by the fit, very precise parametrization of the form factor in the physical region.

2.2 Dispersive parametrization of the Kπ vector form factor

- We can also write a dispersion relation for the vector form factor, improving the pole parametrization. In this case the presence of K*(892) is assumed.
- In the same way as for the scalar form factor, a dispersion relation with two subtractions for ln($\bar{f}_+(t)$) : 2 subtraction points at low energy :

  → $\bar{f}_+(0) = 1$

  → $\bar{f}'_+(0) = \Lambda_+ / m_\pi^2$

  Assuming $\bar{f}_+(t)$ has no zero

$$\bar{f}_+(t) = \exp\left[\frac{t}{m_\pi^2}\left(\Lambda_+ + H(t)\right)\right]$$

with

$$H(t) = \frac{m_\pi^2 t}{\pi}\int_{t_{\pi K}}^{\infty}\frac{ds}{s^2}\frac{\varphi(s)}{(s-t)}$$

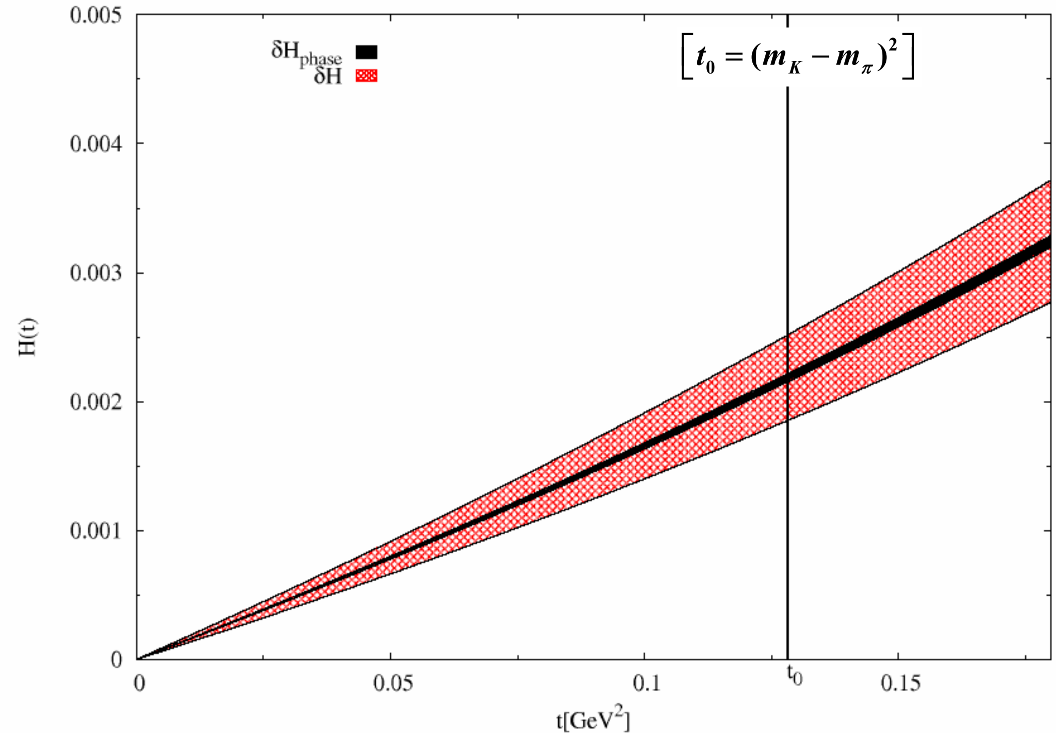
  → φ(t) phase of form factor : $\bar{f}_+(t) = \left|\bar{f}_+(t)\right| e^{i\varphi(t)}$

- φ(s) a priori unknown but

  → $\bar{f}_+(s) \overset{s \to -\infty}{=} \mathcal{O}(1/s)$ ⟹ For large s, φ(s) → π. Rapid convergence of H(t)

  → At « low energy » $\varphi(s) = \delta_{K\pi}^{1/2}(s)$, P wave I=1/2 Kπ scattering phase [**Watson theorem**].

- $K\pi$ scattering phase
  - Experimental input for $0.825\ \mathrm{GeV} < E < 2.5\ \mathrm{GeV}$ **[Aston et al].**
  - Extrapolation of the phase down to threshold complicated ⇨ lack of relevant experimental inputs.
  - Construction of the partial wave amplitude : Breit-Wigner (K*(892)) a la Gounaris-Sakourai (Analyticity, Unitarity and Correct threshold behavior) Inputs: mass and width of K*(892).



- Study of the robustness: H(t) precisely known.

3. Results of the dispersive analysis

NA48	$K_{\mu 3}$ seuls
$\Lambda_+ \times 10^3$	23.3 ± 0.9
$\ln C$	0.1438 ± 0.0138
$\rho(\Lambda_+, \ln C)$	-0.44
χ^2/dof	595/582
$\lambda'_+ \times 10^3$	23.33 ± 0.9
$\lambda''_+ \times 10^3$	1.3 ± 0.1
$\lambda'_0 \times 10^3$	8.9 ± 1.2
$\lambda''_0 \times 10^3$	0.50 ± 0.05

KLOE	$K_{\mu 3}$ et K_{e3} combinés
$\Lambda_+ \times 10^3$	25.7 ± 0.6
$\ln C$	0.204 ± 0.025
$\rho(\Lambda_+, \ln C)$	-0.27
χ^2/dof	2.6/3
$\lambda'_+ \times 10^3$	25.7 ± 0.6
$\lambda''_+ \times 10^3$	1.1 ± 0.1
$\lambda'_0 \times 10^3$	14.0 ± 2.1
$\lambda''_0 \times 10^3$	0.50 ± 0.06

KTeV	K_{e3} seuls	$K_{\mu 3}$ seuls	K_{e3} et $K_{\mu 3}$ combinés
$\Lambda_+ \times 10^3$	25.17 ± 0.58	24.57 ± 1.10	25.09 ± 0.55
$\ln C$	-	0.1947 ± 0.0140	0.1915 ± 0.0122
$\rho(\Lambda_+, \ln C)$	-	-0.557	-0.269
χ^2/dof	66.6/65	193/236	0.48/2
$\lambda'_+ \times 10^3$	25.17 ± 0.58	24.57 ± 1.10	25.09 ± 0.55
$\lambda''_+ \times 10^3$	1.22 ± 0.03	1.19 ± 0.05	1.21 ± 0.03
$\lambda'_0 \times 10^3$	-	13.22 ± 1.20	12.95 ± 1.04
$\lambda''_0 \times 10^3$	-	0.59 ± 0.03	0.58 ± 0.03

Experiment Ke3+Kμ3	ln C
KTeV+BOPS Prel.	0.192(12)
KLOE'08	0.204(25)
NA48'07 ($K_{\mu 3}$ only)	0.144(14)

- To be compared with

$$\ln C_{SM} = 0.2160(35)(64)$$

KLOE and KTeV in agreement and in agreement with the SM.
NA48 4.5σ away !

- A deviation from the SM prediction can be explained :
  - Test of RHCs appearing at NLO of an EW low energy effective theory as a signature of exchange of new particules (W_R,…) at high energy. [Bernard, Oertel, E.P., Stern'06]
  - Presence scalar couplings (charged Higgs) : [Hou]
    MFV + large tanβ : hard to explain a 4.5σ effect (~several% level) [Isidori, Paradisi'06]
  - Existence of a complex zero and its complex conjugate for the form factor [Bernard, Oertel, E.P., Stern, work in progress]

3. Matching of the 2 loop ChPT with the dispersive representation of the $K\pi$ scalar form factor

3.1 Computation of K_{l3} form factors at 2 loops in the isospin limit [Bijnens&Talavera'03]

- The scalar form factor at two loops in the isospin limit

$$f_S(t) = f_+(0) + \overline{\Delta}(t) + \frac{F_K/F_\pi - 1}{\Delta_{K\pi}} t + \frac{8}{F_\pi^4}\left(2C_{12}^r + C_{34}^r\right)\left(m_K^2 + m_\pi^2\right) t - \frac{8}{F_\pi^4} t^2 C_{12}^r$$

- The vector form factor $f_+(0)$ at 2 loops in the isospin limit is expressed as $$f_+(0) = 1 + \Delta(0) - \frac{8}{F_\pi^4}\left(C_{12}^r + C_{34}^r\right)\Delta_{K\pi}^2$$

- In these expressions, no dependence on the L_i at p^4, only via p^6 contribution. Only 2 LECs C_{12} and C_{34} which can be determined by the measurement of the slope and the curvature of the scalar form factor.

- $\overline{\Delta}(t)$ and $\Delta(0)$: contributions from loops: → F_π, the LECs L_i ($L_5 \Longleftrightarrow F_K/F_\pi$) can be calculated at $\mathcal{O}(p^6)$ with the knowledge of the L_i at $\mathcal{O}(p^4)$ in the physical region.

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$$f_+(0) = 1 + \Delta(0) - \frac{8}{F_\pi^4}\left(C_{12}^r + C_{34}^r\right)\Delta_{K\pi}^2$$

- $\overline{\Delta}(t) = -0.25763t + 0.833045t^2 + 1.25252t^3 \quad \left[K_{l3}^0\right]$

$$\Delta(0) = -0.0080 \pm 0.0057\left[loops\right] \pm 0.0028\left[L_i^r\right]$$

⇨ To be updated with the new experimental inputs (K_{l4})

3.3 Matching

$$f_S(t) = f_+(0) + \bar{\Delta}(t) + \frac{F_K/F_\pi - 1}{\Delta_{K\pi}} t + \frac{8}{F_\pi^4}\left(2C_{12}^r + C_{34}^r\right)\left(m_K^2 + m_\pi^2\right) t - \frac{8}{F_\pi^4} t^2 C_{12}^r$$

- Taking the derivative:

$$\Longrightarrow \lambda_0' f_+(0) = \frac{m_\pi^2}{\Delta_{K\pi}}\left(\frac{F_K}{F_\pi} - 1\right) + \frac{8 m_\pi^2 \Sigma_{K\pi}}{F_\pi^4}\left(2C_{12}^r + C_{34}^r\right) + m_\pi^2 \bar{\Delta}'(0)$$

- And derivate 2 times:

$$\Longrightarrow \lambda_0'' f_+(0) = -\frac{16 m_\pi^4}{F_\pi^4} C_{12}^r + m_\pi^4 \bar{\Delta}''(0) \qquad (1)$$

- Combine with the two loop result for $f_+(0)$

$$\Longrightarrow f_+(0) = 1 + \Delta(0) - \frac{8}{F_\pi^4}\left(C_{12}^r + C_{34}^r\right)\Delta_{K\pi}^2 \qquad (2)$$

- From (1)+(2) $\Longrightarrow$ $\boxed{2C_{12}^r + C_{34}^r}$ $\Longrightarrow$ $\boxed{f_+(0) = f\left(\frac{F_K}{F_\pi}, \lambda_0'\right)}$

or

$\boxed{\lambda_0' = f\left(\frac{F_K}{F_\pi}, f_+(0)\right)}$

- From DR $\lambda_0'' = \lambda_0'^2 - 2\frac{m_\pi^4}{\Delta_{K\pi}} G'(0)$

$= \lambda_0'^2 + \left(4.16 \pm 0.50\right) \times 10^{-4}$

Results

- We will present trends and not exact results: use of $\Delta(0)$ and $\Delta(t)$ from **[Bijnens&Talavera]** determined with F_K/F_π =1.22 and F_π =92.4 MeV

  ⇨ Redo the fit varying F_K/F_π and F_π.

- We vary $\Delta(0)$ in its error bars, give the largest uncertainty.

- For instance, take the most recent and precise value for F_K/F_π from lattice

$$\boxed{\frac{F_K}{F_\pi} = 1.189 \pm 0.007}$$

[HPQCD-UKQCD'07]

	λ_0 10^{-3}	$f_+(0)$	C_{12} 10^{-6}	C_{34} 10^{-6}	Δ_{CT} 10^{-2}
KLOE	14.0 ± 2.1	0.9700(218)	0.463(537)	3.387(4.226)	0.028(1.011)
KTeV	12.95 ± 1.04	0.9803(127)	0.720(251)	1.323(2.233)	-0.180(933)
NA48	8.88 ± 1.24	1.0212(149)	1.523(200)	-6.634(2.586)	-0.963(905)

- Uncertainties from $\Delta(0)$, F_K/F_π and λ_0'
- Uncertainties on $f_+(0)$ between 1.5% and 2%, not competitive with the most recent lattice result (uncertainties of ~ 0.5%)
- Limiting uncertainty from λ_0', average ?
- Uncertainties on $\Delta(0)$ and $\bar{\Delta}(t)$ should decrease with new fits.

3.4 Standard Model framework

- Unitarity of V_{CKM} : $|V_{ud}|^2 + |V_{us}|^2 + \cancel{|V_{ub}|^2} = 1$
- V_{ud} from $0^+ \rightarrow 0^+$ superallowed β decays: $V_{ud} = 0.97418(26)$

  [Towner&Hardy'07]
- V_{us} from unitarity $|V_{us}|^2 = 1 - |V_{ud}|^2$

$\Rightarrow$ F_K/F_π, $f_+(0)$ are known from experiments.

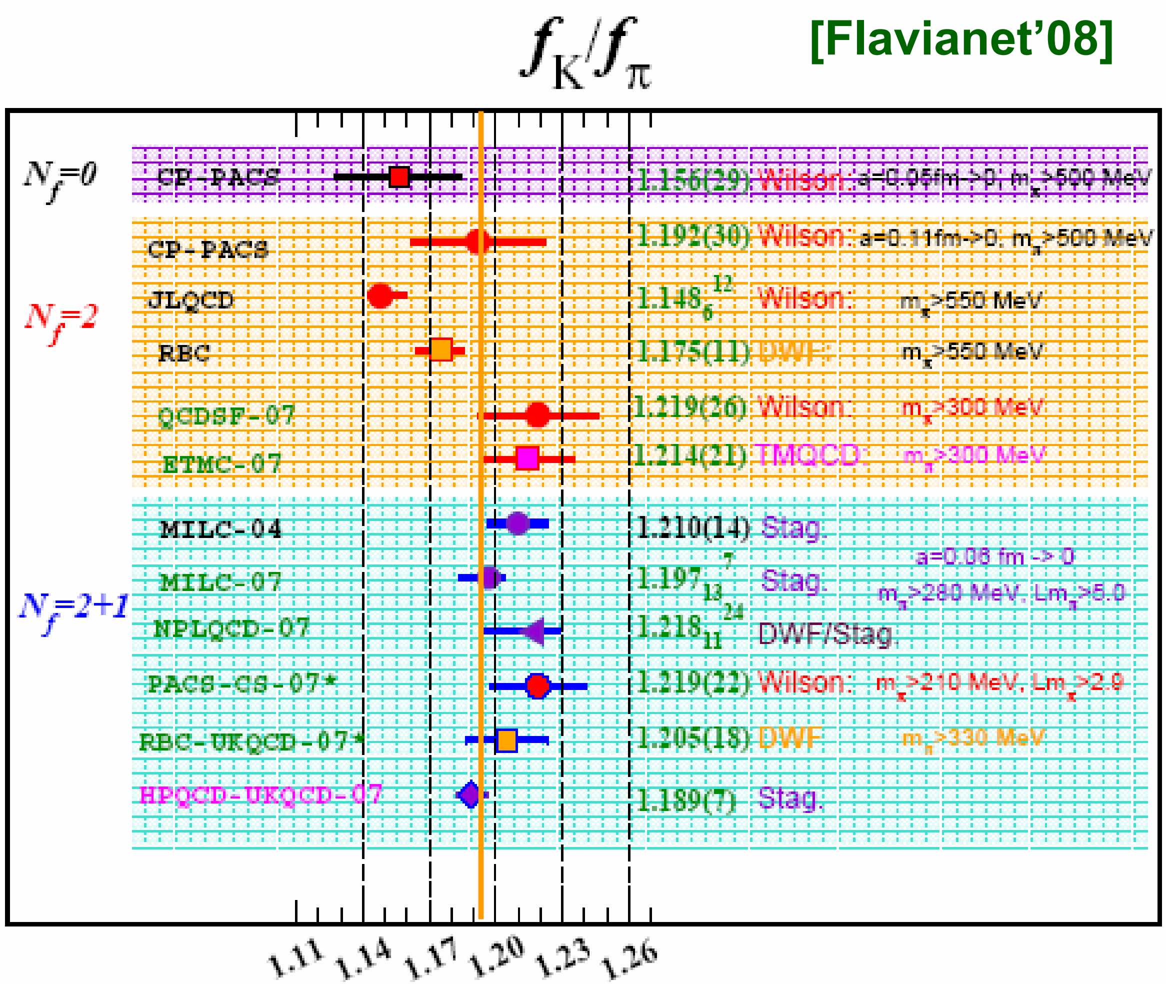
- From $\Gamma(\pi \to \mu\nu(\gamma)) \sim |F_\pi V^{ud}|$ $\Rightarrow$ $\hat{F}_\pi = (92.1 \pm 0.2)$ MeV
- From $\dfrac{\Gamma(K \to \mu\nu(\gamma))}{\Gamma(\pi \to \mu\nu(\gamma))} \sim \left|\dfrac{F_K V^{us}}{F_\pi V^{ud}}\right|$ $\Rightarrow$ $\dfrac{\widehat{F}_K}{\widehat{F}_\pi} = 1.192 \pm 0.007$
- From $\Gamma(K^0 \to \pi^+ e^- \nu) \sim |f_+(0)\, V^{us}|$ $\Rightarrow$ $\widehat{f}_+(0) = 0.9574 \pm 0.0052$

Results

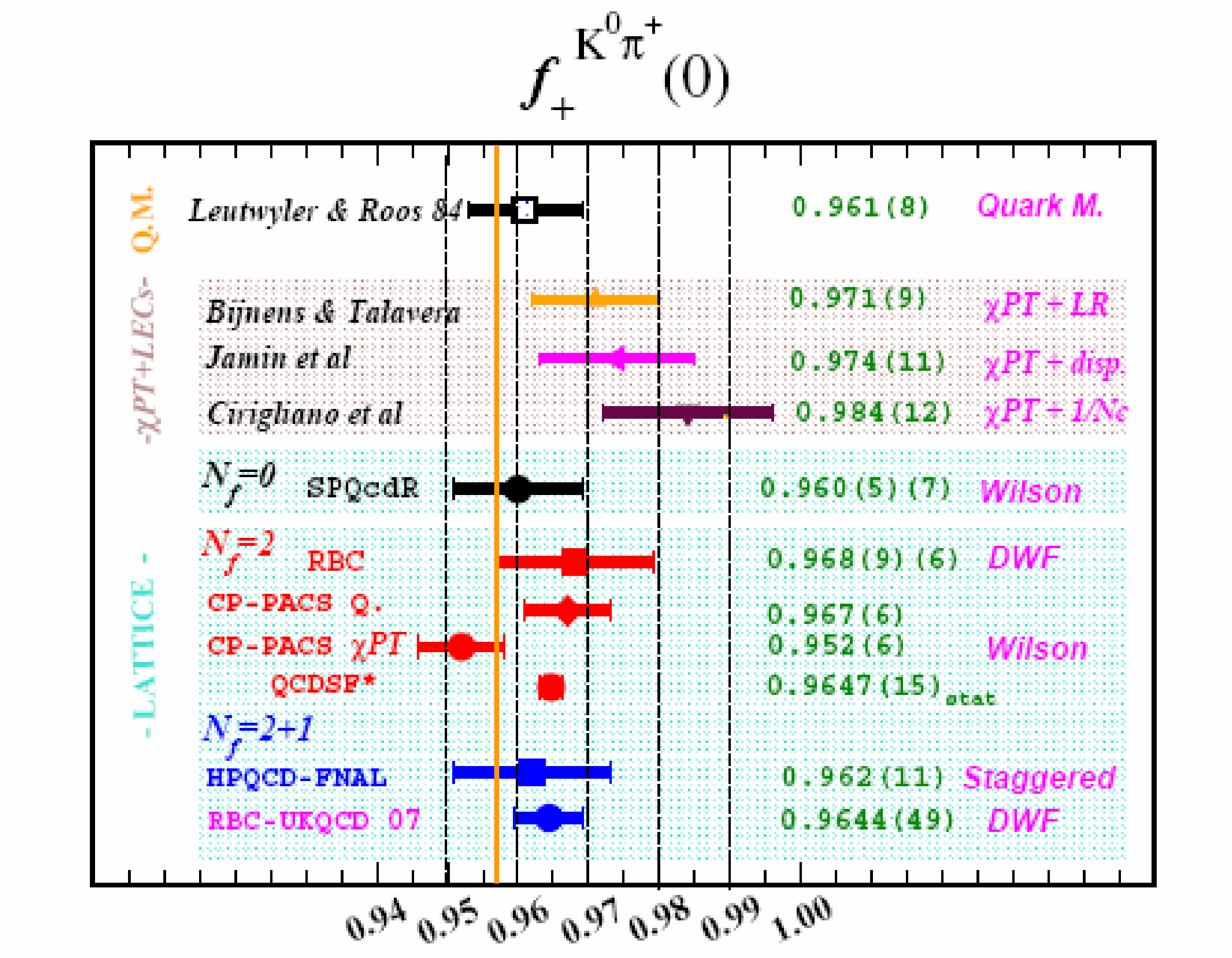
$\Delta(0)$	F_K/F_π	$f_+(0)$	λ_0' (10^{-3})	C_{12} (10^{-6})	C_{34} (10^{-6})	Δ_{CT} (10^{-2})
-0.0165	1.192	0.957	14.46	-0.170	4.741	-1.193
-0.008	1.192	0.957	15.20	-0.421	6.480	-0.118
0.0005	1.192	0.957	15.93	-0.683	8.229	0.948

- F_K/F_π and $f_+(0)$ in agreement with the lattice results: on the lower side of the lattice results for F_K/F_π.
- $f_+(0) = 0.961(8)$ **[Leutwyler&Roos]**
- $f_+(0) = 0.984(12)$ **[Cirigliano et al]**

Lattice Results



Lattice Results



Results

$\Delta(0)$	F_K/F_π	$f_+(0)$	λ'_0 (10^{-3})	C_{12} (10^{-6})	C_{34} (10^{-6})	Δ_{CT} (10^{-2})
-0.0165	1.192	0.957	14.46	-0.170	4.741	-1.193
-0.008	1.192	0.957	15.20	-0.421	6.480	-0.118
0.0005	1.192	0.957	15.93	-0.683	8.229	0.948

- $f_+(0)$ in agreement with $f_+(0) = 0.961(8)$ **[Leutwyler&Roos'84]**
- In desagreement with $f_+(0) = 0.984(12)$ **[Cirigliano et al],** recently updated by **[Kastner&Neufeld'08]**

$$\Rightarrow \boxed{f_+(0) = 0.986 \pm 0.007_{1/N_C} \pm 0.002_{M_S, M_P}}$$

due to a difference in the value C_{34}

Results

$\Delta(0)$	F_K/F_π	$f_+(0)$	λ'_0 (10^{-3})	C_{12} (10^{-6})	C_{34} (10^{-6})	Δ_{CT} (10^{-2})
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0.0005	1.192	0.957	15.93	-0.683	8.229	0.948

$$\left(C_{12}^{SP} = -\frac{F_\pi^4}{8M_S^2} \right)$$

- RChPT models: $C_{12} \sim -1\text{x}10^{-5}$ for M_s~980MeV (a_0)
  For $1GeV \leq M_s \leq 1.5GeV$ ⇨ $-9\ 10^{-6} \leq C_{12} \leq -1.8\ 10^{-6}$
  Evolving to the ρ scale ⇨ $-7.8\ 10^{-6} \leq C_{12} \leq 4\ 10^{-6}$

- C_{12}=(0.3±5.4)x10^{-7} , λ'_0=0.0157(1) with c.v. $f_+(0)$=0.976 **[Jamin, Oller & Pich]**
  λ'_0=0.0147(4) with $f_+(0)$=0.972(12) and F_K/F_π =1.203(16)

- RChPT models: $C_{34} \sim 6\text{x}10^{-5}$ ⇨ $C_{34}(M_\rho) \sim \left(2.9^{+1.3}_{-5.0}\right)\text{x}10^{-6}$
  in desagreement at 2σ **[Kastner&Neufeld'08]**

- λ'_0 on the large side of experimental results

Results

$\Delta(0)$	F_K/F_π	$f_+(0)$	λ'_0 (10^{-3})	C_{12} (10^{-6})	C_{34} (10^{-6})	Δ_{CT} (10^{-2})
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0.0005	1.192	0.957	15.93	−0.683	8.229	0.948

- Small value of Δ_{CT} in agreement with the NLO result $\Delta_{CT}^{NLO} = (-3.5 \pm 8)\text{x}10^{-3}$
  One recovers $\Delta_{CT}^{loops} = -6.2\text{x}10^{-3}$ [Bijnens&Ghorbani'07]

- Rather large variations of C_i Δ_{CT} and λ_0 with $\Delta(0)$
  ⇨ large uncertainties

3.5 Matching in presence of RHCs

- Change in the values of F_K/F_π and $f_+(0)$ compared to the SM, apparition of V_L and V_R $\Rightarrow$ $\mathcal{V}_{\text{eff}}$ and $\mathcal{A}_{\text{eff}}$

$$\boxed{\left(\frac{F_K}{F_\pi}\right)^2 = \left(\frac{\widehat{F}_K}{\widehat{F}_\pi}\right)^2 \frac{1+2\left(\varepsilon_S-\varepsilon_{NS}\right)}{1+\frac{2}{\sin^2\hat{\theta}}\left(\delta+\varepsilon_{NS}\right)}}$$

and

$$\boxed{\left[f_+^{K^0\pi^-}(0)\right]^2 = \left[\widehat{f}_+^{K^0\pi^-}(0)\right]^2 \frac{1-2\left(\varepsilon_S-\varepsilon_{NS}\right)}{1+\frac{2}{\sin^2\hat{\theta}}\left(\delta+\varepsilon_{NS}\right)}}$$

with $(\delta+\varepsilon_{NS})$ and $(\varepsilon_S - \varepsilon_{NS})$, combination of new physics parameters.

- Use experimental knowledge of λ_0 and $\Delta\varepsilon$ obtained from dispersive fits to determine F_K/F_π, $f_+(0)$, C_{12}, C_{34}, Δ_{CT}

$$\ln C = 0.2188(35) + \underbrace{2\left(\varepsilon_S-\varepsilon_{NS}\right) + \Delta_{CT}/B_{\exp}}_{\Delta\varepsilon}$$

$\Rightarrow$ KLOE compatible with lattice results + no RHCs

NA48, RHCs + small F_K/F_π ($F_K/F_\pi \sim 1.15$)

4. Conclusion and outlook

- Dispersive parametrization very useful to analyse $K^{L}_{\mu 3}$ decays: parametrization physically motivated which allows with one parameter to determine the shape of the form factor, quite robust
  - Allows for a test of the SM electroweak couplings via the CT theorem
  - Allows for a matching with the 2 loop ChPT calculation

- Experimental results from dispersive analysis: KLOE and KTeV agree with the SM and NA48 at 4.5σ $\Rightarrow$ results for K^{+}

- Matching the K_{l3} two loop computation + experimental results using dispersive representation offer the opportunity to determine $f_{+}(0)$, C_{12}, C_{34}, Δ_{CT} as a function of F_K/F_π

- Uncertainties too large at the moment to extract these quantities, need of
  - more precise and consistent fits
  - more precise lattice determinations
  - more precise scalar form factor measurements

4. Conclusion and outlook

- Dispersive parametrization very useful to analyse $K^L_{\mu 3}$ decays: parametrization physically motivated which allows with one parameter to determine the shape of the form factor, quite robust
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- Matching the K_{l3} two loop computation + experimental results using dispersive representation offer the opportunity to determine $f_+(0)$, C_{12}, C_{34}, Δ_{CT} as a function of F_K/F_π

- Uncertainties too large at the moment to extract these quantities, need of
  - more precise and consistent fits
  - more precise lattice determinations
  - more precise scalar form factor measurements

- Outlook: -systematic study of uncertainties
  -consistent matching.
  - include IB

Aditional Slides

- NA48 result $\Rightarrow$ new physics

- The other way round, take the most precise and recent value from lattice for F_K/F_π and f+(0)

$$\boxed{\frac{F_K}{F_\pi} = 1.189 \pm 0.007}$$ **[HPQCD-UKQCD'07]**

$$\boxed{f_+(0) = 0.964 \pm 0.005}$$ **[RBC-UKQCD'07]**

$$\Rightarrow \quad \lambda_0' = \left(14.62 \pm 1.07\right) x 10^{-3}$$

- Compatible with the result from the dispersive analysis with $\ln C_{SM}$

$$\Rightarrow \quad \lambda_0' = \left(15.04 \pm 0.73\right) x 10^{-3}$$

and with the prediction λ'_0=0.0147(4) **[Jamin, Oller&Pich]**

3.5 Matching in presence of RHCs

- Change in the values of F_K/F_π and $f_+(0)$ compared to the SM, apparition of V_L and V_R $\Longrightarrow$ $\mathcal{V}_{eff}$ and $\mathcal{A}_{eff}$

$$\boxed{\left(\frac{F_K}{F_\pi}\right)^2 = \left(\frac{\widehat{F}_K}{\widehat{F}_\pi}\right)^2 \frac{1+2\left(\varepsilon_S-\varepsilon_{NS}\right)}{1+\frac{2}{\sin^2\hat{\theta}}\left(\delta+\varepsilon_{NS}\right)}}$$

and

$$\boxed{\left[f_+^{K^0\pi^-}(0)\right]^2 = \left[\widehat{f}_+^{K^0\pi^-}(0)\right]^2 \frac{1-2\left(\varepsilon_S-\varepsilon_{NS}\right)}{1+\frac{2}{\sin^2\hat{\theta}}\left(\delta+\varepsilon_{NS}\right)}}$$

with $(\delta+\varepsilon_{NS})$ and $(\varepsilon_S - \varepsilon_{NS})$, combination of new physics parameters.

$$\left|\mathcal{V}_{eff}^{ud}\right|^2 + \left|\mathcal{V}_{eff}^{us}\right|^2 = 1 + \Delta_{\mathbf{unitarity}} = 1 + 2(\delta+\varepsilon_{NS}) + 2(\varepsilon_S-\varepsilon_{NS})\sin^2\hat{\theta}$$

$$\Delta_{\mathbf{unitarity}} = \sin^2\hat{\theta}\left(\left|\frac{\widehat{f}_+^{K^0\pi^-}(0)}{f_+^{K^0\pi^-}(0)}\right|^2 - 1\right) \Longrightarrow$$ $\Delta_{unitarity}$ small

- Use experimental knowledge of λ_0 and $\Delta\varepsilon$ obtained with dispersive fits to determine F_K/F_π, $f_+(0)$, C_{12}, C_{34}, Δ_{CT} $\Longrightarrow$ KLOE compatible with lattice results + no RHCs and NA48, RHCs + small F_K/F_π ($F_K/F_\pi \sim 1.15$)

$$\ln C = 0.2188(35) + \underbrace{2\left(\varepsilon_S-\varepsilon_{NS}\right) + \Delta_{CT}/B_{\exp}}_{\Delta\varepsilon}$$

Breaking of $\Delta_{unitarity}$ and ε_S-ε_{NS}=0

$\Delta(0)$	$\epsilon_s - \epsilon_{ns}$	$\Delta_{\rm unitarity}$ (10^{-3})	λ'_0 (10^{-3})	Δ_{CT} (10^{-2})	$f_+(0)$	F_K/F_π	C_{12} (10^{-6})	C_{34} (10^{-6})
-0.0165	SM	SM	14.46	-1.193	0.957 *	1.192 *	-0.170	4.741
	0	-1.5	14.30	-1.428	0.972	1.210	-0.235	2.235
-0.008	SM	SM	15.20	-0.118	0.957 *	1.192 *	-0.421	6.480
	0	-1.5	15.03	-0.368	0.972	1.210	-0.484	3.971
	0	-3.1	14.85	-0.622	0.987	1.229	-0.550	1.344
	0	1.5	15.37	0.127	0.943	1.174	-0.362	8.879
	0	3.1	15.53	0.369	0.930	1.157	-0.306	11.176
0.0005	SM	SM	15.93	0.948	0.957 *	1.192 *	-0.683	8.229
	0	-1.5	15.75	0.684	0.972	1.210	-0.743	5.718

- When $\Delta_{unitarity}$ increases, λ_0 , Δ_{CT} , C_{12} and F_K/F_π decrease whereas C_{34} , and $f_+(0)$ increase.

$$\Delta_{\mathbf{unitarity}} = \sin^2\hat{\theta}\left(\left|\frac{\hat{f}_+^{K^0\pi^-}(0)}{f_+^{K^0\pi^-}(0)}\right|^2 - 1\right)$$

Lattice variation for $\Delta_{unitarity}$

$\Rightarrow$ $\boxed{0.0148 \leq \lambda_0 \leq 0.0154}$

3) Allow for physics beyond the SM: use experimental knowledge of λ_0 and $\Delta\varepsilon$ obtained with dispersive fit.

$$\ln C = 0.2188(35) + \underbrace{2\left(\varepsilon_S - \varepsilon_{NS}\right) + \Delta_{CT} / B_{\exp}}_{\Delta\varepsilon}$$

- NA48 $\lambda_0 = (8.88 \pm 1.24)\text{x}10^{-3}$ and $\Delta\varepsilon = -0.0075(14)$ **[Phys.Letter. B.647]**
- KLOE $\lambda_0 = (14.0 \pm 2.10)\text{x}10^{-3}$ and $\Delta\varepsilon = -0.0015(25)$ **[JHEP'08]**

$\Delta(0)$	$\epsilon_s - \epsilon_{ns}$	λ_0 (10^{-3})	$\Delta_{\text{unitarity}}$ (10^{-3})	Δ_{CT} (10^{-2})	$f_+(0)$	F_K/F_π	C_{12} (10^{-6})	C_{34} (10^{-6})
-0.008	-0.005	14.0	-2.80	-0.623	0.984	1.213	-0.234	1.534
	-0.032	9.01	-3.148	-1.178	0.987	1.152	1.107	-2.158
-0.0165	-0.0012	13.99	-2.41	-1.579	0.980	1.218	-0.202	0.666
	-0.028	9.00	-2.760	-2.130	0.983	1.157	1.132	-1.092
0.0005	-0.0088	14.0	-3.19	0.325	0.988	1.209	-0.264	2.400
	-0.0358	9.01	-3.535	-0.234	0.991	1.148	1.084	0.659

3) Allow for physics beyond the SM: use experimental knowledge of λ_0 and $\Delta\varepsilon$

- NA48 $\lambda_0 = (8.88 \pm 1.24)\mathrm{x}10^{-3}$ and $\Delta\varepsilon = -0.0075(14)$ [Phys.Letter. B.647]

$\Delta(0)$	$\epsilon_s - \epsilon_{ns}$	λ_0 (10^{-3})	$\Delta_{\text{unitarity}}$ (10^{-3})	Δ_{CT} (10^{-2})	$f_+(0)$	F_K/F_π	C_{12} (10^{-6})	C_{34} (10^{-6})
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	-0.0358	9.01	-3.535	-0.234	0.991	1.148	1.084	0.659

- F_K/F_π rather small
- Large contribution from RHCs
- Δ_{CT} on the large side of the NLO result,
- C_{34} becomes negative, C_{12} is positive.

3) Allow for physics beyond the SM: use experimental knowledge of λ_0 and $\Delta\varepsilon$

- KLOE $\lambda_0 = (14.0 \pm 2.10)\mathrm{x}10^{-3}$ and $\Delta\varepsilon = -0.0015$ **[JHEP'08]**

$\Delta(0)$	$\epsilon_s - \epsilon_{ns}$	λ_0 (10^{-3})	$\Delta_{\text{unitarity}}$ (10^{-3})	Δ_{CT} (10^{-2})	$f_+(0)$	F_K/F_π	C_{12} (10^{-6})	C_{34} (10^{-6})
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	-0.0358	9.01	-3.535	-0.234	0.991	1.148	1.084	0.659

- Results for F_K/F_π and $f_+(0)$ compatible with lattice results, small RHCs for the central value.

1.3 Callan-Treiman relation: test of EW couplings of the SM

- Callan-Treiman Theorem: SU(2) x SU(2) theorem

$$C = \overline{f}_0(\Delta_{K\pi}) = \underbrace{\frac{F_K}{F_\pi f_+^{K^0}(0)}}_{B} + \Delta_{CT}$$

$\Delta_{K\pi} = m_K^2 - m_\pi^2$

Δ_{CT} estimated in ChPT $\sim \mathcal{O}\left(m_{u,d} / 4\pi F_\pi\right)$

$\boxed{\Delta_{CT}^{NLO} = (-3.5 \pm 8).10^{-3}}$ **[Gasser&Leutwyler]**

- Test of the SM EW couplings

$$B = \frac{F_K \left|\mathcal{V}^{us}\right|}{F_\pi \left|\mathcal{V}^{ud}\right|} \frac{1}{f_+(0)\left|\mathcal{V}^{us}\right|} \left|\mathcal{V}^{ud}\right|$$

is predicted in the Standard Model using the measured Br: $\mathrm{Br}(K_{l2}/\pi_{l2})$, $\Gamma(K_{e3})$ and $\left|\mathcal{V}_{ud}\right|$. ($\left|\mathcal{V}_{us}\right|$ not needed in this prediction.)

⇨ $\boxed{B = 1.2439 \pm 0.0042}$ $\ln C_{SM} = 0.2183(34) + \Delta_{CT}$

2.1 Why a dispersive parametrization ?

- Use of the physical knowledge :
  - ➢ $K\pi$ scattering phase.
  - ➢ Presence of resonances....
- 2 subtractions ⇨ insensitive to the unknown high energy phase. This allows for a good precision in the region of interest (at low energy).
- Only one free parameter to be fitted.

3.2 Dispersive representation of the scalar Kπ form factor

- From the dispersive parametrization of the scalar form factor

$$\bar{f}_0(t) = \exp\left[\frac{t}{\Delta_{K\pi}}\left(\ln C - G(t)\right)\right]$$

⇨ Slope and curvature

$$\lambda_0' = \frac{m_\pi^2}{\Delta_{K\pi}}\left(\ln C - G(0)\right)$$

and

$$\lambda_0'' = \lambda_0'^2 - 2\frac{m_\pi^4}{\Delta_{K\pi}} G'(0)$$
$$= \lambda_0'^2 + \left(4.16 \pm 0.50\right) \text{x } 10^{-4}$$

$$G(0) = 0.0398(40)$$

2.3 Description

$$\bar{f}_0(t) = \exp\left[\frac{t}{\Delta_{K\pi}}(\ln C - G(t))\right]$$

with

$$G(t) = \frac{\Delta_{K\pi}(\Delta_{K\pi} - t)}{\pi}\int_{t_{\pi K}}^{\infty}\frac{ds}{s}\frac{\phi(s)}{(s-\Delta_{K\pi})(s-t)}$$

→ ϕ(t) phase of form factor : $\bar{f}(t) = \left|\bar{f}(t)\right| e^{i\phi(t)}$

- Two unknowns: ϕ(s) and lnC=ln($\bar{f}_0$(t=$\Delta_{K\pi}$))

- ϕ(s) a priori unknown but
  → $\bar{f}_0(s) \overset{s\to-\infty}{=} \mathcal{O}(1/s)$ ⟹ For large s, $\phi(s) \to \pi$. Rapid convergence of G(t).
  → At « low energy » $\phi(s) = \delta^{1/2}_{K\pi}(s)$, S wave I=1/2 $K\pi$ scattering phase, [**Watson theorem**] well known : **[Buettiker, Descotes, Moussallam '02]**
    - Experimental inputs for $1\ \mathrm{GeV} < E < 2.5\ \mathrm{GeV}$ **[Aston et al].**
    - Extrapolation of the phase down to threshold solving the Roy Steiner equations.

General Case

- Knowing $\lambda_0^{'}$, $\lambda_0^{''}$ + F_K/F_π allows to determine $f_+(0)$

$$\boxed{f_+(0) = \frac{\frac{m_\pi^2}{\Delta_{K\pi}}\left(F_K/F_\pi - 1\right) + \frac{m_\pi^2 \Sigma_{K\pi}}{\Delta_{K\pi}}\left(1 + \Delta(0)\right) + m_\pi^2 \bar{\Delta}'(0) + \frac{m_\pi^2 \Sigma_{K\pi}}{2}\bar{\Delta}''(0)}{\frac{m_\pi^2 \Sigma_{K\pi}}{\Delta_{K\pi}^2} + \lambda_0^{'} + \frac{\Sigma_{K\pi}}{2m_\pi^2}\lambda_0^{''}}}$$

- Once $f_+(0)$ is known $\Rightarrow$ C_{12}, C_{34}, Δ_{CT}

$$\boxed{\Delta_{CT} = \exp\left[\frac{\Delta_{K\pi}}{m_\pi^2}\lambda_0^{'} + G(0)\right] - \frac{F_K}{F_\pi}\frac{1}{f_+(0)}}$$

General Case

- Or equivalently, knowing F_K/F_π and $f_+(0)$ allows te determine λ_0'
  - Use of the dispersive parametrization

$$\lambda_0'' = \lambda_0'^2 - 2\frac{m_\pi^4}{\Delta_{K\pi}}G'(0)$$
$$= \lambda_0'^2 + \left(4.16 \pm 0.50\right) \text{x } 10^{-4}$$

$$\Rightarrow \boxed{\lambda_0' = -\frac{m_\pi^2}{\Sigma_{K\pi}}\left(1 - \sqrt{1 - 2\frac{\Sigma_{K\pi}^2}{\Delta_{K\pi}}\left(\frac{Y}{\Delta_{K\pi}} - G'(0)\right)}\right)}$$ with

$$\boxed{Y = 1 - \frac{\Delta_{K\pi}}{\Sigma_{K\pi}}\frac{F_K}{F_\pi}\frac{1}{f_+(0)} - \frac{1}{f_+(0)}\left(1 + \Delta(0) + \frac{\Delta_{K\pi}^2}{2}\bar{\Delta}''(0) - \frac{\Delta_{K\pi}}{\Sigma_{K\pi}}\left(1 - \Delta_{K\pi}\bar{\Delta}'(0)\right)\right)}$$