lstituto Nazionale di Fisica Nucleare

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Sources for Flasma Accelerators and Radiation Compton with Lasers And Deam

## A review of plasma based beamline elements for advanced beam manipulation Andrea R. Rossi<sup>\*</sup> INFN - Milan



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# BD issues with plasma beams

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### Plasma beams properties



Courtesy: C. Benedetti

Plasma structures (waves) are small:

typical scales are from few tens to few hundreds of microns, depending on plasma density (and wave regime) through

$$\lambda_{\rm p}[\mu m] \approx \frac{100}{\sqrt{n_0 [10^{17} {\rm cm}^{-3}]}}$$

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### Plasma fields are very intense:

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typical values may vary in the range of few to a thousand GV/m

$$E_z^{(\text{max})}[\text{GV/m}] \approx 100\sqrt{n_0[10^{18}\text{cm}^{-3}]}$$
  
 $\partial_r E_r[\text{GV/m}/\mu\text{m}] \approx 6n_0 [10^{18}\text{cm}^{-3}]$ 



### Plasma beams properties

## High energy spread all:

typical soles are from few tens to few hundreds of micros, depending on plasma density (and wave regime) through through focusing  $\frac{100}{\sqrt{n_0[10^{17} \mathrm{cm}^{-3}]}}$ 

### Courtesy: C. Benede within the plasma channel

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### Plasma beams properties

## High energy spread

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Courtesy: C. Benede within the plasma channel

Plas If the bunch was monochromatic, high divergence and tight focusing would not be thousand (at least in principle) a problem:

 $E_{z}^{(\max)}[\text{GV/m}] \approx 100\sqrt{n_{0}[10^{18}\text{cm}^{-3}]} \qquad \text{os} \qquad Proper matchings of the beam at the solution of$ 

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### Plasma beams properties

Plasma structures (waves) are small:

typical scales are from few tens to few hundreds of microns, depending on plasma density (and wave regime) through

## Since it's not, we must be prepared to face the effects of [1017 cm-3]

Courtesy: C. Benedetti

## Plasma fields are very intense:

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 $\partial_r E_r [\text{GV/m}/\mu\text{m}] \approx 6n_0 [10^{18} \text{cm}^{-3}]$ 



Highly chromatic beam transport: RF acceleration vs plasma acceleration

$$\varepsilon_n^2 = <\gamma>^2 \left(\sigma_E^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon^2\right) \text{ General formula}$$

Emittance dominated beam drifting in vacuum

$$f_n^2 = <\gamma >^2 \left(s^2 \sigma_E^2 \sigma_{x'}^4 + \varepsilon^2\right)$$

### EXAMPLE: RF gun

<٧>	σ <sub>E</sub>	σ <sub>x</sub>	σ <sub>x'</sub>	S	$s\sigma_{E}\sigma_{x'}^{2}$	γε	٤ <sub>n</sub>
12	9 x 10 <sup>-3</sup>	780 um	1.1 x 10 <sup>-4</sup>	1 m	10 nm	1.01 um	1.02 um

### EXAMPLE: GeV class beam from SELF INJECTION

<٧>	$\sigma_{E}^{(0)}$	$\sigma_x^{(0)}$	σ <sub>x'</sub> <sup>(0)</sup>	S	$s\sigma_{E}\sigma_{x'}^{2}$	γε	٤ <sub>n</sub>
1800	6.4 x 10 <sup>-2</sup>	0.49 um	2.9 x 10 <sup>-2</sup>	1 m	0.94 mm	2.5 um	~ 1 mm

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### Highly chromatic beam transport: a closer look



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### Highly chromatic beam transport: a closer look



### Highly chromatic beam transport: a closer look



### Highly chromatic beam transport: a closer look Self-injection beam



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- Transverse PS broadening
  Strong (normalized) emittance dilution
- Broadening (dilution) is driven by energy spread through betatron frequency spread:

$$\left(\frac{\Delta\omega_{\rm b}}{\omega_{\rm b}}\right)^2 \propto \sigma_E^2$$

### Highly chromatic beam transport: some bibliography

- K. Floettmann, Some basic features of beam emittance, Phys. Rev. STAB 6, 034202 (2003).
- T. Mehrling *et al., Transverse growth in staged-wakefield acceleration*, Phys. Rev. STAB 15, 111303 (2012).
- P. Antici et al., Laser-driven electron beamlines generated by coupling laserplasma sources with conventional transport systems, J. App. Phys. 112, 044902 (2012).
- M. Migliorati et al., Intrinsic normalized emittance growth in laser-driven electron accelerators, Phys. Rev. STAB 16, 011302 (2013).

## Plasma beams manipulation

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### Plasma beams manipulations...

- Control initial beam parameters at injection, for internal injection schemes.
- Eventually reduce beam length, for external injection / staging.
- Inject beams into plasma with the correct size, to avoid emittance dilution in subsequent acceleration, for external injection / staging.
- Extract beams from plasma realizing a trade-off between size and divergence at constant emittance (adiabatic matching), to avoid propagating highly divergent beams.
- Control energy spread, keeping it as low as possible to avoid chromaticity induced emittance dilution.

### Injection

- Self-injection.
- Optical injection.
- Density gradient injection.

- Ionization injection.
- Field ionization.
- External injection\*.



\* C.E. Clayton and L. Serafini, IEEE Trans. Plas. Sci. **24**, 400 (1996) N.E. Andreev, S.V. Kuznetsov, Plas. Phys. Contr. Fus. **45**, 39 (2003)

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### Self-injection

Relays on wave breaking due to large amplitude fields, plasma wavefronts distortion, forward Raman scattering<sup>1</sup> and other non linear phenomena: very poor control on injected bunches.





High charge bunches, "easy" to implement. Many experimental results.

[1] Modena, A., Z. Najmudin, A. E. Dangor, C. E. Clayton, K. A. Marsh, C. Joshi, V. Malka, C. B. Darrow, C. Danson, D. Neely, and F. N. Walsh, 1995, Nature London **377**, 606.

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## Optical injection: ponderomotive injection and colliding pulses

Two high intensity laser pulses<sup>1</sup> or three resonant lower energy laser pulses<sup>2</sup>: one pulse is the "pump" and drives the plasma wakefield; the others "push" background electrons into the accelerating bucket. Different blends but usually requires a complex experimental setup and a high degree of precision in space and time.



[1] D. Umstadter, J. K. Kim, and E. Dodd, 1996, Phys. Rev. Lett. **76**, 2073.
 [2] E. Esarey, A. Ting, R. F. Hubbard, W. P. Leemans, J. Krall, and P. Sprangle, Phys. Rev. Lett. **79**, 2682 (1997).

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### Optical injection: cold injection

The driver laser pulse collides head on with a low intensity injection laser. The bubble is temporarily deformed and a high charge bunch of "cold" electrons is injected when the bubble goes back to its original shape<sup>1</sup>.



[1] R. Lehe, A. F. Lifschitz, X. Davoine, C. Thaury, and V. Malka, Phys. Rev. Lett. 111, 085005 (2013)

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### Density gradient injection

A decreasing plasma density<sup>1</sup> causes the plasma wavelength to increase so that background electrons are enclosed in the larger bubble.



V. Petrillo, L. Serafini and P. Tomassini, Phys. Rev. STAB 11, 070703 (2008)



Schmid et al., Phys. Rev. STAB 13, 091301 (2010).

[1] Bulanov, S., N. Naumova, F. Pegoraro, and J. Sakai, 1998, Phys. Rev. E 58, R5257.

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### Injection by ionization/1

The plasma is composed by a mixture of gases, with different ionization energies. One is ionized by the wave driver and forms the plasma wave, the other is ionized either by the plasma field<sup>1</sup> or by an injection laser and the ionized electrons form the injected bunch. In LWFA this last technique takes the name of ionization injection<sup>2</sup>.



[1] A. Martinez de la Ossa, J. Grebenyuk, T. Mehrling, L. Schaper, and J. Osterhoff, Phys. Rev. Lett. **111**, 245003 (2013).
 [2] E. Oz, et al., Phys. Rev. Lett. **98**, 084801 (2007).

### Injection by ionization/2

In PWFA, ionization injection is named Trojan Horse<sup>1</sup>. A rather new mechanism, injection by plasma torch<sup>2</sup>, is between ionization injection and density gradient.





[1] B. Hidding, G. Pretzler, J. B. Rosenzweig, T. Königstein, D. Schiller, and D. L. Bruhwiler, Phys. Rev. Lett. **108**, 035001 (2012).
 [2] G. Wittig, et al., Nuc. Inst. Meth. Phys. Res. A, http://dx.doi.org/10.1016/j.nima.2016.02.027 (in press).

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### Acceleration



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 $\tilde{Q} = Q k_{\rm d}^3 \ll 1$ 

performances can be very interesting.

Beam loading is significant but manageable with bunch charges up to few tens of pC.

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 $\alpha > 1$ 

> 1



manageable with bunch charges up to few tens of pC.

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> 1

### **Beam loading**

Beam loading is the perturbation to the plasma fields due to the witness bunch self-fields. Generally speaking:

- Modifies the total fields acting on the witness.
- May reduce acceleration performances.
- May be used for reducing energy spread.
- Depends on witness current.
- Its effects depend on the intensity of plasma fields, hence also on plasma wave regime.
- Requires very fine tuning if used for reducing energy spread.



### Transverse manipulations and matching



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### Matching into plasma

It's easy to find matching conditions for bubble regime with negligible beam loading:

$$\sigma_{\rm tr,match} = \sqrt[4]{\frac{2}{\gamma}} \sqrt{\frac{\varepsilon_n}{k_{\rm p}}}$$

Typical values are in the order of 0.1 - 1 um.

For a **quasi-linear** plasma wave regime matched spot-sizes have the same order of magnitude. If the plasma driver transverse size is always much larger than the beam size, transverse fields can be considered linear, although they depend on  $\zeta$ .

In **linear regime** the same considerations on the nature of transverse fields hold true, **but beam loading is usually not negligible**, unless charge is very low.

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In **linear regime** the same considerations on the nature of transverse fields hold true, **but beam loading is usually not negligible**, unless charge is very low.

## Tight focusing is needed

### Matching into/out of plasma/1

More conventional solutions: high performance beam optics like permanent magnet quadrupoles...



...reaching many hundreds of T/m gradients, adequate for energies up to few hundreds MeV.

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### Matching into/out of plasma/2

### Plasma lenses: classification



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### Stand alone, active plasma lens: discharge capillary<sup>1</sup>

The beam goes through a capillary filled with gas, while a current is flowing in the capillary. If some (rather restrictive) conditions are met, the bunch is focused by the azimuthal magnetic field generated by the current density.



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### Stand alone, active plasma lens: gas jet<sup>1</sup>

A gas jet, acting as plasma lens, is powered by an *ad hoc* laser pulse.



#### Easily tuned for different bunches Adequate acceptance

#### **Density profile**

$$n(z) = \begin{cases} n_1 & \text{for } z < 0 & (\text{First jet}) \\ 0 & \text{for } 0 < z < L_d & (\text{Drift space}) \\ n_2 & \text{for } L_d < z < L_d + L_2 & (\text{Second jet}) \end{cases}$$

#### Condition for optimal collimation\*

 $\langle k_{\beta} \rangle L_{\rm d} \tan(\langle k_{\beta} \rangle L_2) = 1$  $k_{\beta} = k_p / \sqrt{2\gamma}$ 

The only significant source of aberrations is energy spread (chromaticity)

\* this relation was derived assuming a constant emittance during drift.

Requires two high power lasers to be operated.

[1] R. Lehe, C. Thaury, E. Guillaume, A. Lifschitz, and V. Malka, Phys. Rev. STAB 17, 121301 (2014)

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### Stand alone, passive plasma lens: gas jet<sup>1</sup>

A gas jet, acting as plasma lens, is powered by the same laser extracting and accelerating the bunch



### **Density profile**

$$n(z) = \begin{cases} n_1 & \text{for } z < 0 & (\text{First jet}) \\ 0 & \text{for } 0 < z < L_d & (\text{Drift space}) \\ n_2 & \text{for } L_d < z < L_d + L_2 & (\text{Second jet}) \end{cases}$$

### Condition for optimal collimation\*

$$\frac{\langle k_{\rm foc} \rangle Z_R^2}{L_{\rm d} + L_2} \tan\left(\frac{\langle k_{\rm foc} \rangle Z_R^2}{L_{\rm d}} - \frac{\langle k_{\rm foc} \rangle Z_R^2}{L_{\rm d} + L_2}\right) = 1$$
$$k_\beta = k_p / \sqrt{2\gamma}$$

Requires only one laser

\* this relation was derived assuming a constant emittance during drift.

Beam loading may reduce effectiveness.

[1] R. Lehe, C. Thaury, E. Guillaume, A. Lifschitz, and V. Malka, Phys. Rev. STAB 17, 121301 (2014)

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### Stand alone, passive plasma lens: gas jet<sup>1</sup>

A gas jet, acting as plasma lens, is powered by the same laser extracting and accelerating the bunch



[1] C. Thaury, et al., Nature Comm. 6, 6860 (2015)

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### Integrated, passive plasma lens: plasma ramps

A tapering at the end (beginning) of the plasma channel acts as a plasma lens and defocuses (focuses) the bunch performing matching. Moreover, the focusing (defocusing) of the driver helps in performing the process.



X.L.Xu, et al., arXiv: 1411.4386v2 [physics.acc-ph] 2015

## Integrated, passive plasma lens: plasma ramps & driver focusing/defocusing

A tapering at the end (beginning) of the plasma channel acts as a plasma lens and defocuses (focuses) the bunch performing matching. Moreover, the focusing (defocusing) of the driver helps in performing the process.



## Integrated, passive plasma lens: plasma ramps & tailored driver focusing/defocusing in hollow capillary

A tapering at the end (beginning) of the plasma channel acts as a plasma lens and defocuses (focuses) the bunch performing matching. Moreover, the focusing (defocusing) of the driver is tailored to help in performing the process. Stability vs injected beam parameters jitters



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## Integrated, passive plasma lens: plasma ramps & tailored driver focusing/defocusing

In the raps the regime is linear with dominating beam loading. This working condition allows a beam loading assisted matching. In the acceleration section the regime is quasi-linear and the beam loading negligible.



### Matching into/out of plasma: is it really necessary?



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### Matching into/out of plasma: is it really necessary?



### Longitudinal manipulations



### Longitudinal compression

Longitudinal compression is possible when a bunch has an energy much lower than the resonant one, i.e. when the witness is (initially) much slower than the plasma wake<sup>1</sup>, by velocity bunching<sup>2</sup>.



[1] J.L. Bobin, in Proc. of the ECFA-CAS/CEFN-In-2P3-IRF/CEA-EPS Workshop, p. 58 (1987). C.S. Liu and V.K. Tripathi, Interaction of

M. Ferrario, T. C. Katsouleas, L. Serafini, and Ilan Ben Zvi, IEEE Trans. Plas. Sci. 28, (2000).

electromagnetic waves with electron beams and plasmas, World Scientific, Singapore, 1994.

[2] L. Serafini and M. Ferrario, LNF-00/036, 2000. L. Serafini and M. Ferrario, AIP Conf. Proc. 581, 87 (2001).

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### Energy spread control by beam loading

One way to limit energy spread in plasma is to "flatten out" the longitudinal field along the bunch by properly tailoring the beam loading.



Optimal beam profile for non-linear regime

$$\lambda(\xi) = E_t^2 + \frac{r_b^2}{4} = \frac{R_b^4 + r_t^4}{8r_t^2} - \sqrt{\frac{R_b^4 - r_t^4}{8r_t^2}}(\xi - \xi_t)$$



M. Tzoufras, et al., Phys. Plas. 16, 056705 (2009)

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### Energy spread control by beam loading

One way to limit energy spread in plasma is to "flatten out" the longitudinal field along the bunch by properly tailoring the beam loading.



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### Energy spread control by hollow plasma channel

Another possibility is to use a hollow plasma channel, which also provides further advantages. The properties of a square well shaped hollow channel have been first studied in

T. C. Chiou, T. Katsouleas, C. Decker, W. B. Mori, J. S. Wurtele, G. Shvets, and J. J. Su, Phys. Plasmas 2, (1995)

and exploited, in quasi-linear regime, for acceleration in

C. B. Schroeder, E. Esarey, C. Benedetti, and W. Leemans, Phys. Plasmas 20, 080701 (2013)

We will show the results found in

A. Pukhov, O. Jansen, T. Tueckmantel, J. Thomas, and I. Yu. Kostyukov, Phys. Rev. Lett. **113**, 245003 (2014)

which studies the bubble regime.

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### Energy spread control by plasma dechirper<sup>1</sup>

Following the idea of corrugated pipe dechirper<sup>2</sup>, it is possible to arrange plasma density in order to act as a plasma dechirper.



FACET beam parameters

[2] K.L.F. Bane and G. Stupakov, Nuc. Inst. Meth. Phys. Res. 690, 106 (2012). S. Antipov, et al., Phys. Rev. Lett. 112, 114801 (2014).

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### Energy spread control by alternating focusing<sup>1</sup>

In linear and quasi-linear regimes, the focusing strength is zero on crest. For a finite length bunch, however, the tail would be disrupted by defocusing fields. Modulating the plasma density moves the bunch back and forth the accelerating peak keeping it focused and avoiding chirp build-up.



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### Beam streaking

(for longitudinal diagnostics)

Diagnose the longitudinal properties of a plasma beam (ultra short) is hard with conventional methods. A proposal, shows how it is possible to use plasma fields.



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## A (cherry picked) FEL oriented manipulation (not in plasma): chromatic matching<sup>1</sup>

i.e. a possible answer to the question "how could we run an FEL with the plasma beams we can produce right now?"



[1] A. Loulergue, M. Labat, C. Evain, C. Benabderrahmane, V. Malka and M.E. Couprie, New J. Phys. 17, 023028 (2015).

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### Chromatic matching

Assume a bunch with high energy spread and divergence. After first focusing use a chicane to stretch the bunch and realize an energy sorting (chirp):

#### δ>0 dE/E (%) dE/E (%) δ<0 Energy de-mixing Chicane $\Delta s_{bunch} = r_{56} \delta$ -4--10 10 20 Ō -20 -10 10 20 0 Z (µm) Z (µm)

### Longitudinal phase space

Transverse phase spaces



## If the beam is focused, each quasi-monochromatic slice will have a waist at a different position.

Pictures source: A. Loulergue, LWFA electron beam manipulations for FEL amplification, presented at EAAC 2015.



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### Chromatic matching

The waist slips along the bunch from the tail to the head...







... pretty much like the FEL radiation ...

Source: A. Loulergue, LWFA electron beam manipulations for FEL amplification, presented at EAAC 2015.





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## Cornerstone application: All Optical FEL

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### AOFEL in a nutshell

Simply collide a plasma generated beam with an electromagnetic ondulator (laser):



First proposed in V. Petrillo, L. Serafini and P. Tomassini, in Proc. of the 11<sup>th</sup> European Particle Accelerator Conference, Genova, Italy, 2008. Studied in A. Bacci, C. Maroli, V. Petrillo, A.R. Rossi, L. Serafini and P. Tomassini, Nuc. Inst. Meth. Phys. Res. A **587**, 388 (2008) and in V. Petrillo, L. Serafini and P. Tomassini, Phys. Rev. STAB **11**, 070703 (2008)

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## From FEL to AOFEL: Scale down linac AND ondulator sizes!



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### AOFEL: e.m. undulator

### **RESONANCE CONDITIONS:**



Magnetostatic undulator- FEL

Example : for  $\lambda_R = .1nm$ ,  $\lambda_w = 2cm$  and  $a_w \sim 1$  $\implies E = 10 \text{ GeV}$ 

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### AOFEL: e.m. undulator

### **RESONANCE CONDITIONS:**



Magnetostatic undulator- FEL

Example : for 
$$\lambda_R = .1nm$$
,  $\lambda_w = 2cm$  and  $a_w \sim .1$   
 $\implies E = 10 \text{ GeV}$ 

$$\lambda_{R} = \lambda_{L} \frac{\left(1 + a_{0}^{2}/2\right)}{4\gamma_{0}^{2}} \frac{\text{Electromagnetic undulator}}{\text{Example : for } \lambda_{R} = .1 \text{nm}, \ \lambda_{L} = 0.8 \mu\text{m and } a_{0} \sim 0.2^{*}$$
$$\implies E = 22 \text{ MeV}$$

\* for many practical reasons,  $a_0 \lesssim 1$ 

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### **AOFEL:** plasma electron source

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### **AOFEL:** electron bunch



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### **AOFEL: FEL radiation**



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### **AOFEL: FEL radiation**



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### Conclusion

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### Conclusion



Thanks for your attention!

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## Backup Slides

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### Highly chromatic beam transport: theory

DEF: 
$$\varepsilon_n^2 = \langle x^2 \rangle \langle \beta^2 \gamma^2 x'^2 \rangle - \langle x \beta \gamma x' \rangle^2$$
  
DEF:  $\sigma_E^2 = \frac{\langle \beta^2 \gamma^2 \rangle - \langle \beta \gamma \rangle^2}{\langle \gamma \rangle^2}$   
DEF:  $x' = \frac{p_x^i}{\langle p_z \rangle}$ 

ASSUME: relativistic electrons ( $\beta \sim 1$ ) and no correlation btw *x* and energy

$$\varepsilon_n^2 = <\gamma >^2 \sigma_E^2 < x^2 > < x'^2 > + <\beta\gamma >^2 (< x^2 > < x'^2 > - < xx' >^2)$$

$$\varepsilon_n^2 = <\gamma >^2 \left(\sigma_E^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon^2\right)$$



### Velocity bunching

