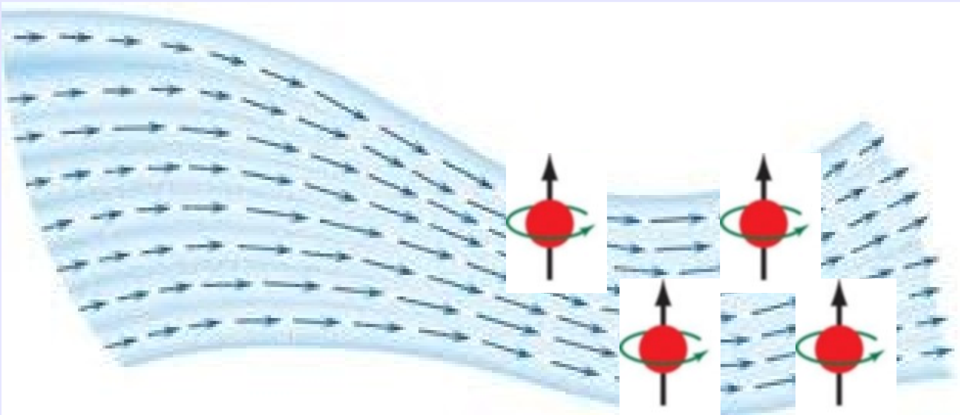

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Polarization and vorticity in relativistic heavy ion collisions

OUTLINE

- Rotation and polarization: an overview
- Quantum-relativistic theory
- Λ polarization in relativistic heavy ion collisions
- Conclusions and outlook

INFN 2016, Frascati, 15 Nov 2016

Introduction: rotation and polarization

Take an ideal gas in a rigidly rotating vessel. At thermodynamical equilibrium (Landau) the gas will also have a velocity field

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$$

WARNING The potential term has a + sign as it stems from both centrifugal and Coriolis potentials

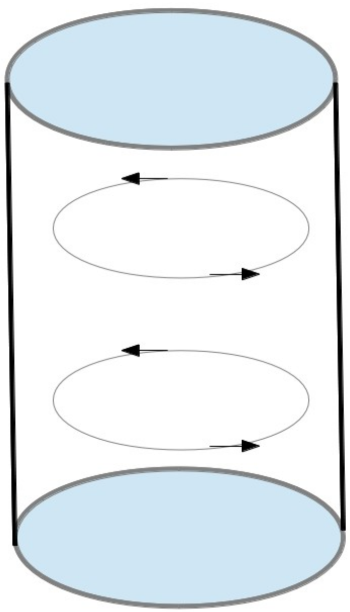
For a *comoving* observer the equilibrium particle distribution function will be given by:

$$f(\mathbf{x}, \mathbf{p}) \propto \exp[-\mathbf{p}'^2/2mT + m(\boldsymbol{\omega} \times \mathbf{x}')^2/2T]$$

If we transform this back to the (primed) inertial observer

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{x} \qquad \mathbf{x} = \mathbf{x}'$$

$$\begin{aligned} f(\mathbf{x}, \mathbf{p}) &\propto \exp[-\mathbf{p}^2/2mT + \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{x})/T] \\ &= \exp[-\mathbf{p}^2/2mT + \boldsymbol{\omega} \cdot \mathbf{L}/T] \end{aligned}$$



It seems quite *natural* to extend this to particle with spin

$$f(\mathbf{x},\mathbf{p},\mathbf{S}) \propto \exp[-\mathbf{p}^2/2mT + \boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S})/T]$$

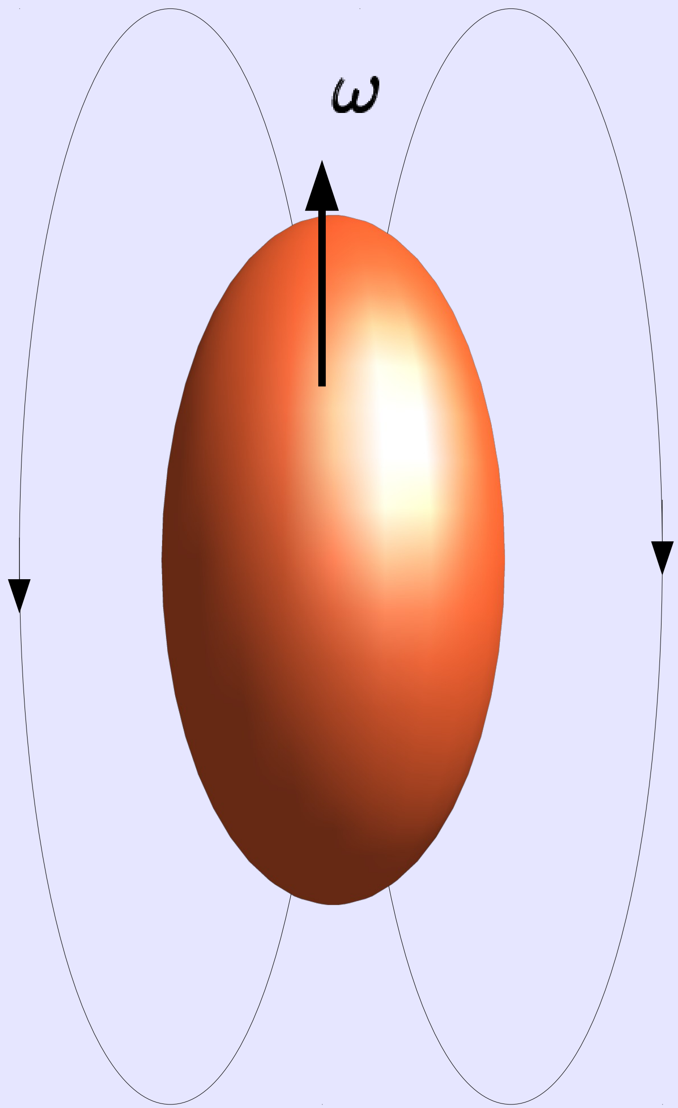
Which implies that particles (and antiparticles) are *POLARIZED*, in a rotating ideal gas, along the direction of the angular velocity vector by an amount

$$\frac{\hbar\omega}{KT}$$

WARNING: The above formula states that the polarization states are unevenly populated in a rotating gas at thermodynamical equilibrium. It does not imply that there is a direct dynamical coupling between spin and rotation (see Mashhoon, Phys. Rev. Lett. 1988).
In other words, the above formula does not imply that – for whatever reason - the hamiltonian of the inertial observer contains a term $\boldsymbol{\omega}\cdot\mathbf{S}$ which would make the spin precessing if particle is accelerated *(not to be confused with Thomas precession, which is a purely relativistic effect).*

Polarization by rotation: have we observed it? Yes: Barnett effect

S. J. Barnett, Magnetization by Rotation, Phys. Rev.. 6, 239–270 (1915).



Spontaneous magnetization of an uncharged body when spun around its axis, in quantitative agreement with the previous polarization formula

$$M = \frac{\chi}{g}\omega$$

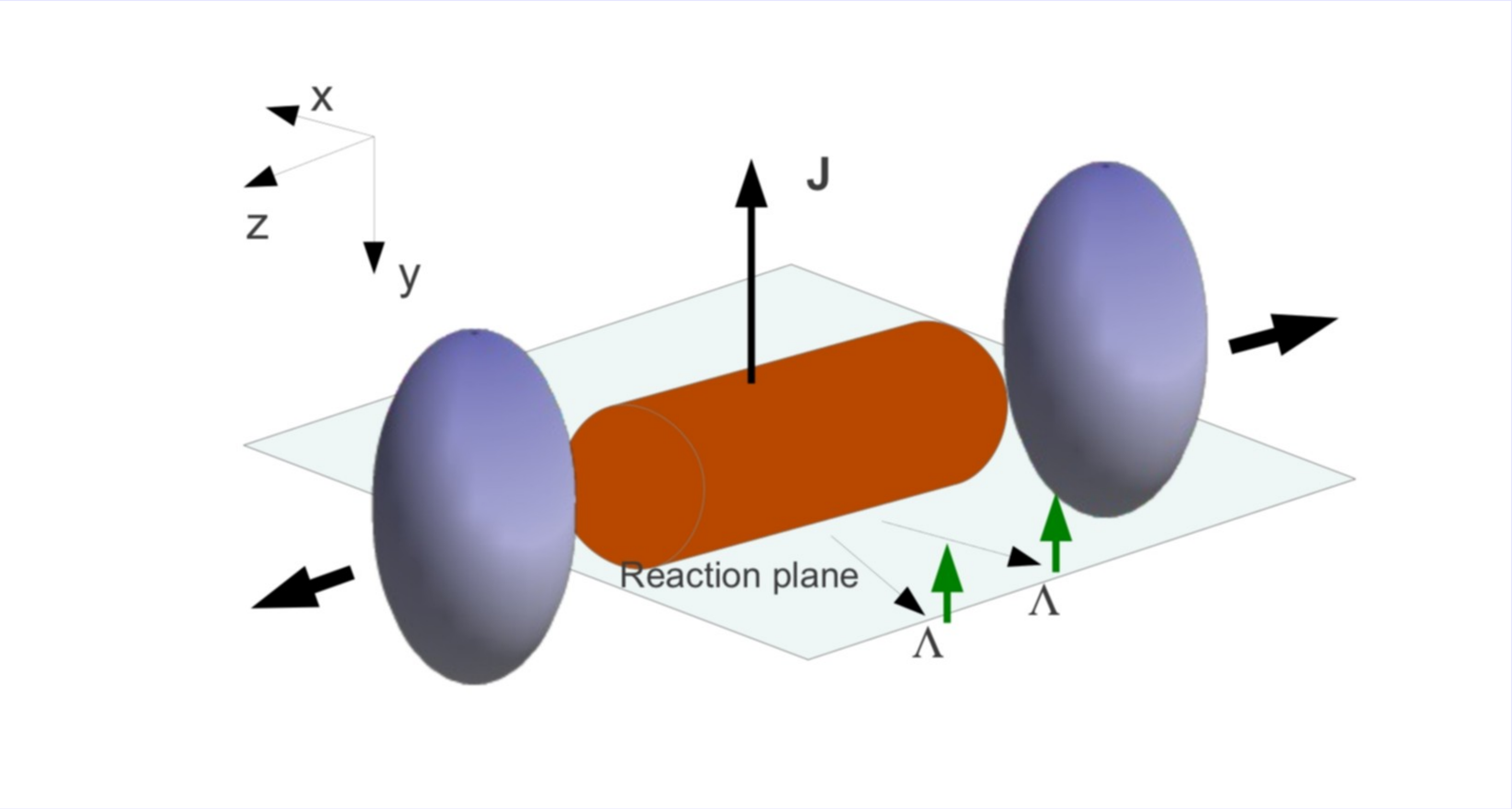
It is a dissipative transformation of the orbital angular momentum into spin of the constituents. The angular velocity decreases and a small magnetic field appears; this phenomenon is accompanied by a heating of the sample. Requires a spin-orbit coupling.

Heavy ion collisions

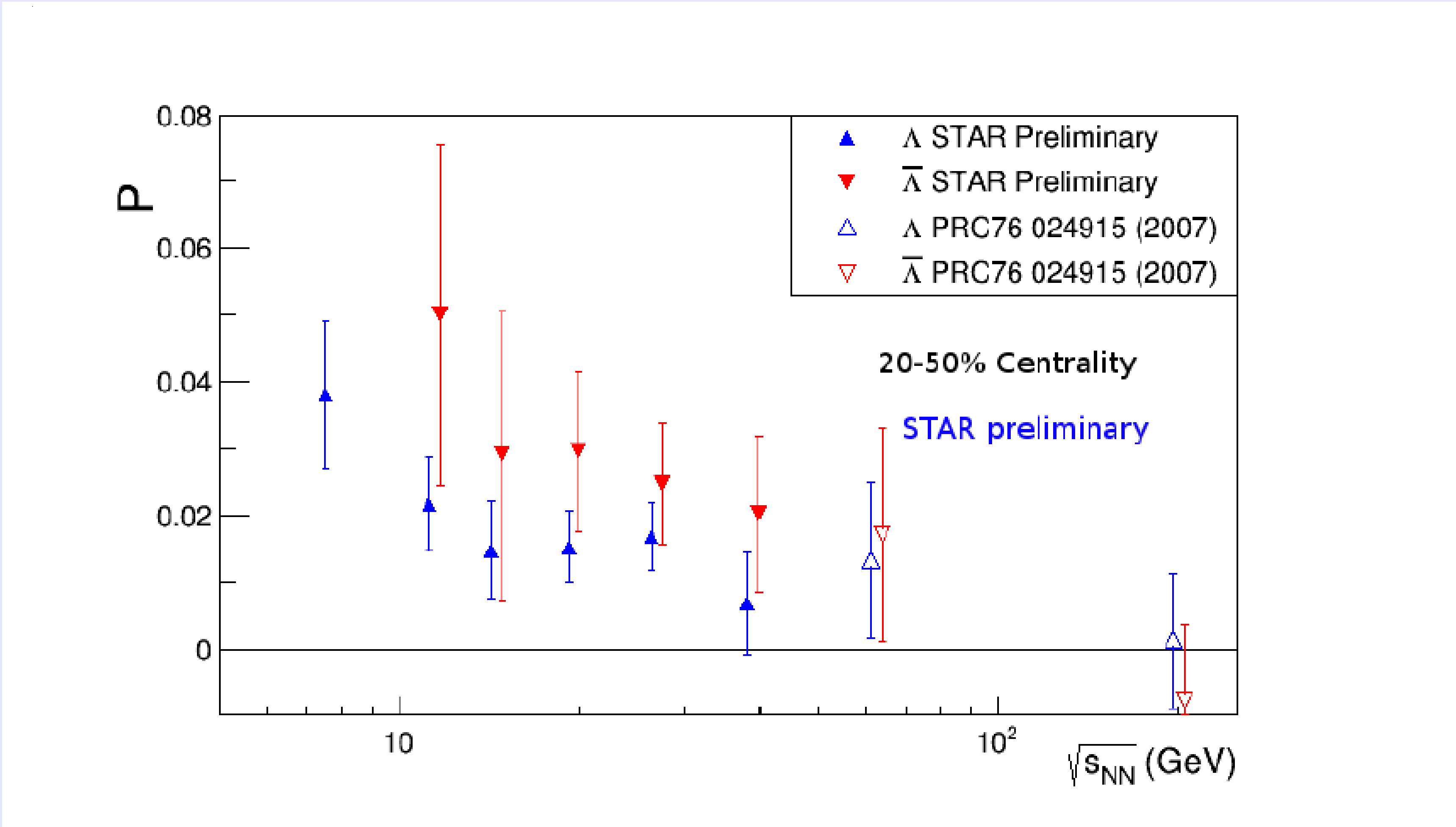
Peripheral collisions ⇨ Angular momentum ⇨ Global polarization w.r.t reaction plane

By parton spin-orbit coupling: Z. T. Liang, X. N. Wang, Phys. Rev. Lett. 94 (2005) 102301

By local equilibration: F. B., F. Piccinini, Ann. Phys. 323 (2008) 2452; F. B., F. Piccinini, J. Rizzo, Phys. Rev. C 77 (2008) 024906



First evidence of global polarization of Λ hyperons



The measured polarization matches the expectations from the thermo-hydro model and is mostly determined by rotation (C-even effect)

First evidence of a quantum effect in hydrodynamics

Polarization in a relativistic fluid: Theory

F. B., V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338 (2013) 32

What happens if particles are relativistic and we have only local equilibrium (hydro)?

Spin, local equilibrium and relativity

We need the single particle distribution function, given the scalar form (for bosons):

$$f(x,p) = \frac{1}{\mathrm{e}^{\beta(x)\cdot p - \xi(x)} - 1}$$

$$\beta^\mu = \frac{1}{T} u^\mu \qquad \xi = \mu/T$$

And of the so-called Cooper-Frye formula

$$\varepsilon \frac{\mathrm{d}N}{\mathrm{d}^3\mathrm{p}} = \int_\Sigma \mathrm{d}\Sigma_\mu p^\mu f(x,p)$$

Polarization as a mean value

Definition of a *relativistic spin* four-vector

For a single particle

$$S^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\lambda\rho}\langle \widehat{J}_{\nu\lambda}\widehat{P}_\rho\rangle \qquad \langle \widehat{X}\rangle = \mathrm{tr}(\widehat{\rho}\widehat{X})$$

Relativistic Spin vs Pauli-Lubanski vs Polarization

$$S^\mu = \frac{1}{m}W^\mu = SP^\mu$$

For a particle with momentum *p*

$$S^\mu(p)N(p) = -\frac{1}{2m}\epsilon^{\mu\nu\lambda\rho}\int_\Sigma \mathrm{d}\Sigma_\tau \; \langle \widehat{\mathcal{S}}^\tau_{\nu\lambda}\rangle_p p_\rho$$

The rank 3 operator is the SPIN TENSOR and we need *its momentum-resolved mean value*

[Warning: it depends on the choice of *T*, the stress-energy tensor]

For the Dirac field

$$\widehat{\mathcal{S}}^{\lambda\mu\nu} = \frac{i}{8}\bar{\Psi}\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}\Psi$$

Wigner function of the free Dirac field

$$W(x,k)_{AB} = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\langle : \Psi_A(x-y/2)\overline{\Psi}_B(x+y/2) : \rangle$$
$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\langle : \overline{\Psi}_B(x+y/2)\Psi_A(x-y/2) : \rangle$$

$$\langle \widehat{\mathcal{S}}^{\lambda\mu\nu}\rangle = \frac{i}{8}\langle \bar{\Psi}\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}\Psi\rangle = \frac{i}{8}\int \mathrm{d}^4k\; \mathrm{tr}_4(\{\gamma^\lambda[\gamma^\mu,\gamma^\nu]\}W(x,k))$$

This is dual to the axial current:

$$j_A^\mu = \langle \bar{\Psi}\gamma^\mu\gamma^5\Psi\rangle = \int \mathrm{d}^4k\; \mathrm{tr}_4(\gamma^\mu\gamma^5 W(x,k))$$

NOTE the normal ordering in the definition of the mean values

An approximate solution of the Wigner function can be obtained by neglecting second order gradients in the equation, so that it can be written as an on-shell solution:
(De Groot, *Relativistic kinetic theory*)

$$W^+(x,k) \equiv \theta(k^0) W(x,k) = \frac{1}{2} \int \frac{\mathrm{d}^3\mathrm{p}}{\varepsilon}\, \delta^4(k-p) \sum_{r,s} u_r(p) f_{rs}(x,p) \bar{u}_s(p)$$

$$W^-(x,k) \equiv \theta(-k^0) W(x,k) = -\frac{1}{2} \int \frac{\mathrm{d}^3\mathrm{p}}{\varepsilon}\, \delta^4(k+p) \sum_{r,s} v_s(p) \bar{f}_{rs}(x,p) \bar{v}_r(p)$$

The *u, v* spinors are the usual solution of the free Dirac equation, with all of their well known properties (orthogonality and completeness).

Thus, the distribution function for free spin ½ particles becomes effectively a 2x2 matrix

What about the density operator?

Global quantum-relativistic equilibrium

General covariant expression of an equilibrium density operator

$$\widehat{\rho} = \frac{1}{Z} \exp \left[- \int_\Sigma \mathrm{d}\Sigma_\mu \ \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

Zubarev 1979
Weert 1982

Obtained by maximizing the entropy $S = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$ with respect to $\widehat{\rho}$ with the constraints of fixed energy, momentum and charge density.

Global equilibrium requires:

$$\boxed{\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0}$$

Killing equation

$$\partial_\mu \zeta = 0$$

Solution of the Killing equation in Minkowski spacetime:

$$\beta^\nu = b^\nu + \varpi^{\nu\mu} x_\mu$$

constants

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)$$

Thermal vorticity
Adimensional in natural units

Density operator becomes:

$$\widehat{\rho} = \frac{1}{Z} \exp \left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right]$$

Local thermodynamic equilibrium

local thermodynamical equilibrium quantum density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \ \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

i.e. without enforcing β to be a Killing vector and ζ to be a constant

$$W(x,k) = \mathrm{tr}(\widehat{\rho}\,\text{Combination of quantum fields})$$

Taylor expansion of the β and ζ fields around the point *x* where *W* is to be calculated (hydrodynamic limit), so that *W* is determined by the local values of *T*,*u*,μ and their local derivatives (antisymmetric part: local thermal vorticity):

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\beta(x) \cdot \widehat{P} + \zeta(x) \widehat{Q} + \frac{1}{2} \varpi_{\mu\nu}(x) \widehat{J}^{\mu\nu}_x \right.$$
$$\left. + \text{terms vanishing at equilibrium} \right]$$

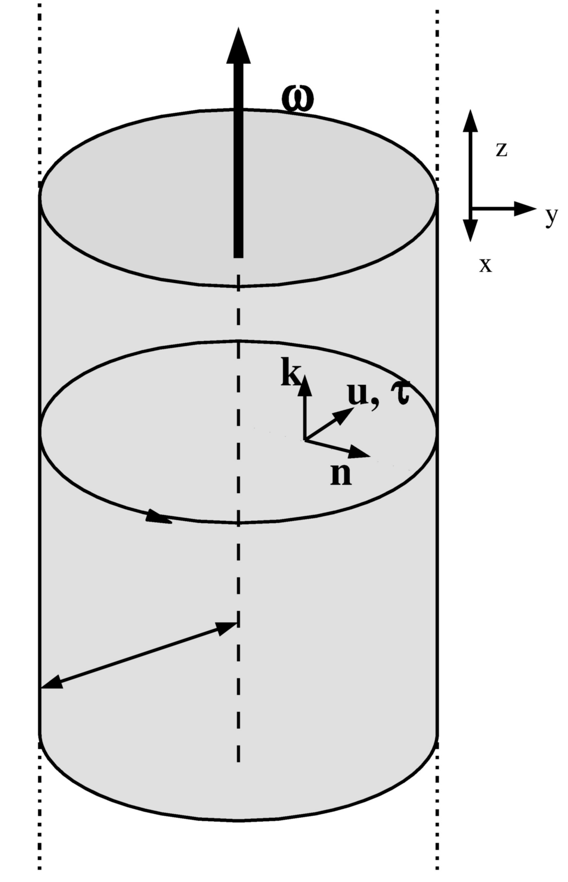
F. B., L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75 (2015) 191 (β frame)

Single particle distribution function at global thermodynamical equilibrium with rotation

In the Boltzmann limit, for an ideal relativistic gas, this calculation can be done without quantum field theory, just with quantum statistical mechanics and group theory (F. B., L. Tinti, Ann. Phys. 325, 1566 (2010)).
Spelled out: maximal entropy principle (equipartition), angular momentum conservation and Lorentz group representation theory.

$$f(x,p)_{rs} = \mathrm{e}^{\xi}\, \mathrm{e}^{-\beta\cdot p}\ \frac{1}{2}\left(D^S([p]^{-1}\mathsf{R}_{\hat{\omega}}(i\omega/T)[p]) + D^S([p]^{\dagger}\mathsf{R}_{\hat{\omega}}(i\omega/T)[p]^{\dagger -1})\right)_{rs}$$

$\mathsf{R}_{\hat{\omega}}(i\omega/T) = \exp[D^S(J_3)\omega/T]$ = SL(2,C) matrix representing a rotation around **ω** axis (z or 3) by an imaginary angle iω/T.



Particle density in phase space

$$\mathrm{tr}_{2S+1} f = \mathrm{e}^{\xi}\, \mathrm{e}^{-\beta\cdot p}\mathrm{tr}_{2S+1}\mathsf{R}_{\hat{\omega}}(i\omega/T) = \mathrm{e}^{\xi}\, \mathrm{e}^{-\beta\cdot p} \sum_{\sigma=-S}^{S} \mathrm{e}^{-\sigma\omega/T} \equiv \mathrm{e}^{\xi}\, \mathrm{e}^{-\beta\cdot p}\chi\left(\frac{\omega}{T}\right)$$

As a consequence, particles with spin get polarized in a rotating gas

$$\mathbf{\Pi}_0 = \frac{\sum_{n=-S}^{S} n \mathrm{e}^{n\omega/T}}{\sum_{n=-S}^{S} \mathrm{e}^{n\omega/T}} \left[\frac{\varepsilon}{m}\hat{\boldsymbol{\omega}} - \frac{\hat{\boldsymbol{\omega}}\cdot\mathbf{p}\mathbf{p}}{m(\varepsilon+m)}\right]$$

F. B., F. Piccinini, Ann. Phys. 323 (2008)

Ansatz for the LTE distribution function with FD statistics

The explicit calculation of *W(x,k)* and the extraction of *f* in the most general case is difficult. One can make a reasonable ansatz extending previous special cases.

The general solution must:

- reduce to the global equilibrium solution with rotation in the Boltzmann limit
- reduce to the known Fermi-Juttner or Bose-Juttner formulae at the LTE in the non-rotating case

$$f(x,p) = \frac{1}{2m}\bar{U}(p)\left(\exp[\beta(x)\cdot p - \xi(x)]\exp[-\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} U(p)$$

$$\bar{f}(x,p) = -\frac{1}{2m}(\bar{V}(p)\left(\exp[\beta(x)\cdot p + \xi(x)]\exp[\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} V(p))^T$$

U, V 4x2 Dirac spinors and Σ the generators of the Lorentz transformation in the fundamental representation

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Polarization four-vector for spin ½ particles

At first order in the gradients

$$n_F = (e^{\beta\cdot p - \xi} + 1)^{-1}$$

$$S^\mu(x,p) = -\frac{1}{8m}(1-n_F)\epsilon^{\mu\rho\sigma\tau} p_\tau \varpi_{\rho\sigma}$$

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)$$

$$S^\mu(p) = \frac{\int \mathrm{d}\Sigma_\lambda \, p^\lambda n_F S^\mu(x,p)}{\int \mathrm{d}\Sigma_\lambda \, p^\lambda n_F}$$

Same formula obtained with a perturbative expansion of the solution of the Wigner function e.o.m. in

R. Fang, L.G. Pang, Q. Wang, X.N. Wang, arXiv:1604.04036

Polarization is induced by acceleration and vorticity

The thermal vorticity

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)$$

It can be readily shown that at *global* equilibrium (β = Killing vector) the thermal vorticity has the following decomposition

$$\varpi^{\mu\nu} = \frac{1}{T}\left(a^\mu u^\nu - a^\nu u^\mu + \varepsilon^{\mu\nu\rho\sigma} \omega_\rho u_\sigma\right)$$

where T is the local proper temperature and:

$$a^\mu = u^\nu \partial_\nu u^\mu \qquad\qquad \omega^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu u_\rho u_\sigma$$

are the properly called *acceleration* and *vorticity*.

In general, in *local* thermodynamical equilibrium, there are terms involving gradients of T

Λ polarization in relativistic heavy ion collisions

Distribution of protons in the Λ rest frame

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}\Omega} = \frac{1}{4\pi}(1 + \alpha \mathbf{P}_0 \cdot \hat{\mathbf{p}}^*) \qquad \mathbf{P}_0(p) = \mathbf{P}(p) - \frac{\mathbf{p}}{\varepsilon(\varepsilon + m)}\mathbf{P}(p)\cdot\mathbf{p}$$

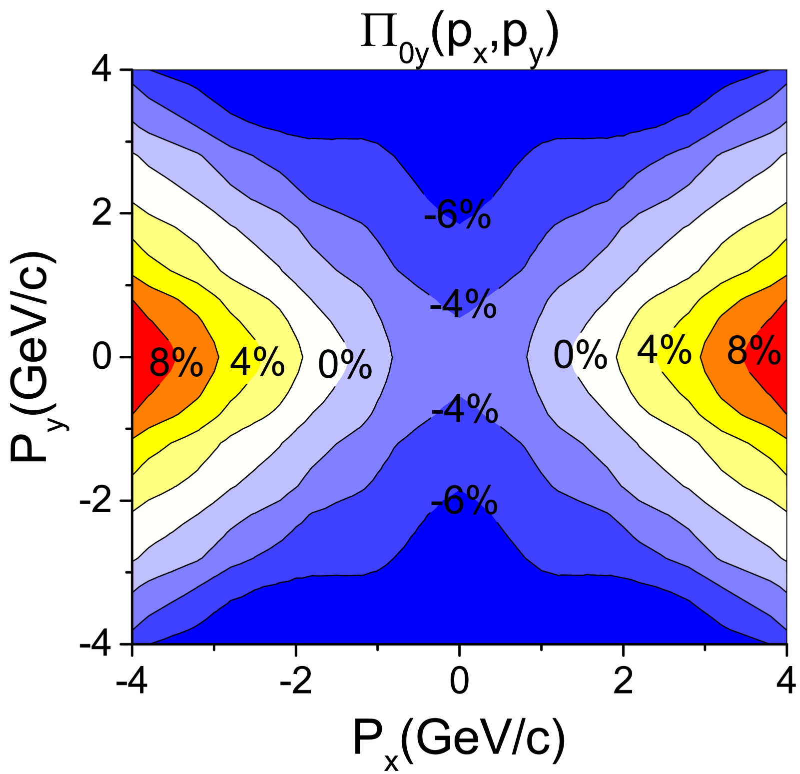
A. Ayala et al., Phys. Rev. C 65 024902 (2002)
Z. T. Liang, X. N. Wang, Phys. Rev. Lett. 94 102301 (2005) and others
B. Betz, M. Gyulassy and G. Torrieri, Phys. Rev. C 76 044901 (2007)
F. B., F. Piccinini and J. Rizzo, Phys. Rev. C 77 024906 (2008)
C. C. Barros, Y. Hama, Phys. Lett. B 699 (2011) 74
M. Baznat, K. Gudima, A. Sorin and O. Teryaev, Phys. Rev. C 93 (2016), 031902
L. G. Pang, H. Petersen, Q. Wang, X.N. Wang, arXiv:1605.04024 [hep-ph]
...

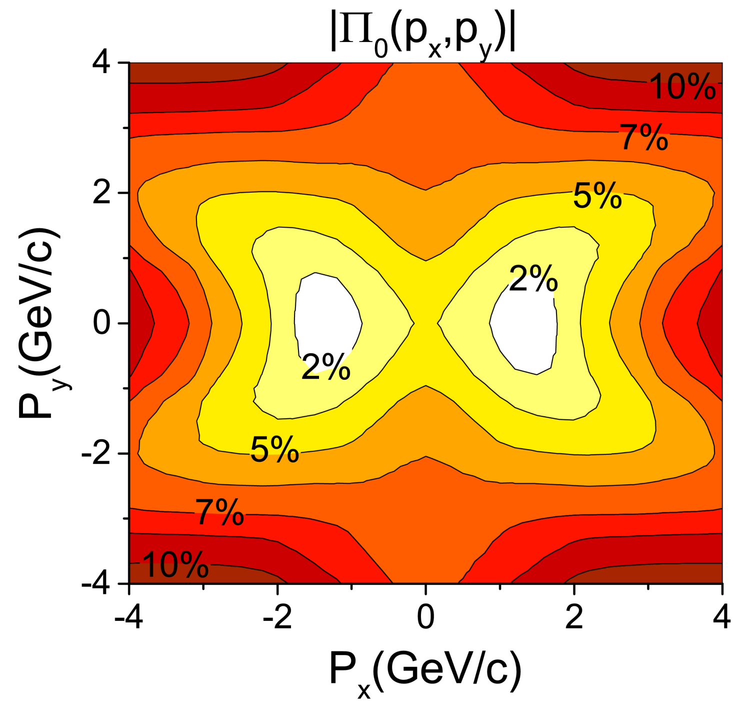
DISTINCTIVE FEATURE OF THERMODYNAMIC POLARIZATION:
particle and antiparticle polarization have the same orientation, unlike in a magnetic field

The Λ polarization strongly depends on initial conditions

Λ polarization with "fire-streak" initial conditions (initial spatial vorticity ≠ 0)

L. Csernai et al.





F.B., L. Csernai, D.J. Wang, Phys. ReV. C 88 (2013) 034905. Erratum Phys. Rev. C 93 (2016) 069901

Λ polarization with "conservative" Bjorken-like initial conditions

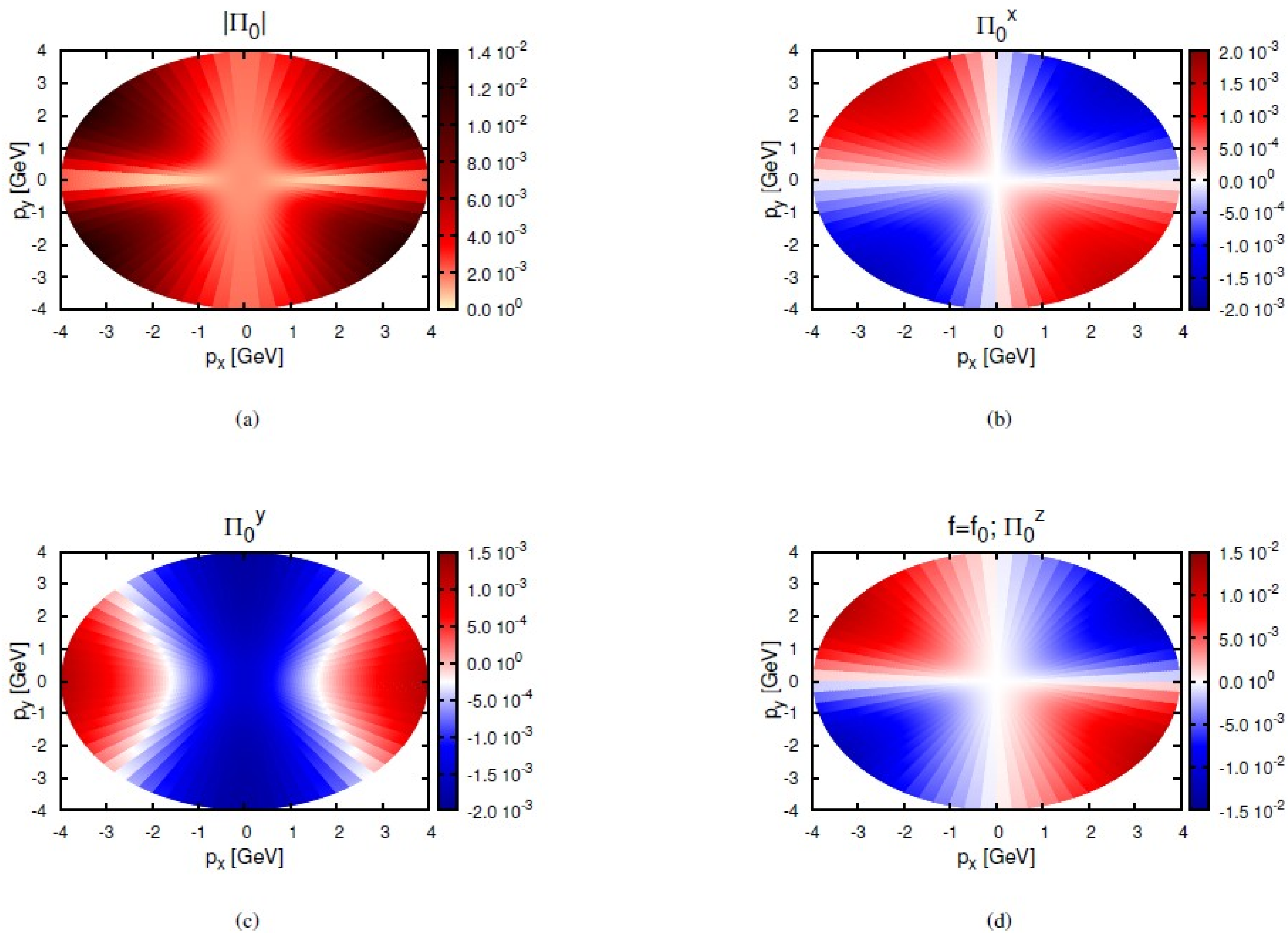
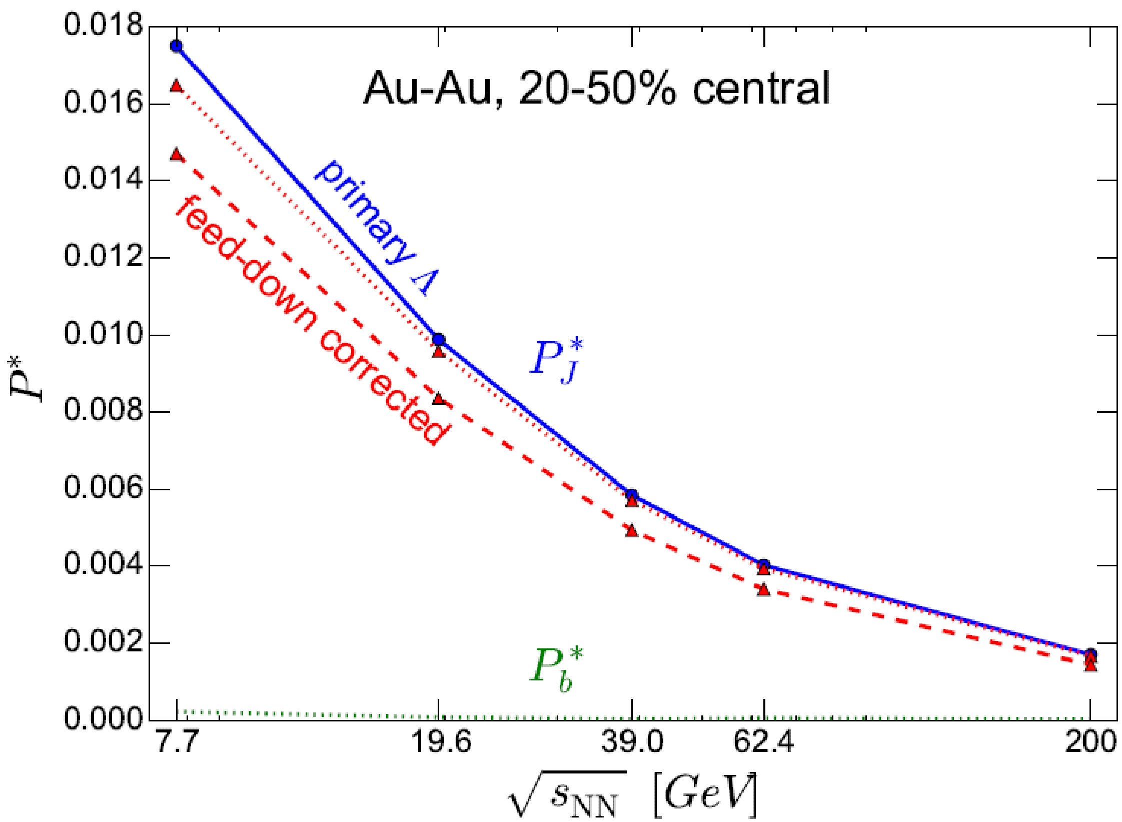


Figure 14: (color online) Magnitude (panel a) and components (panels b,c,d) of the polarization vector of the Λ hyperon in its rest frame.

F. B., G. Inghirami, V. Rolando, A. Beraudo, L. Del Zanna, A. De Pace, M. Nardi, G. Pagliara, V. Chandra
Eur. Phys. J C 75 (2015) 46

Λ polarization calculation at lower energy

I. Karpenko, F. B., arXiv:1610.04717



Lower than measured!
Revision of initial conditions necessary

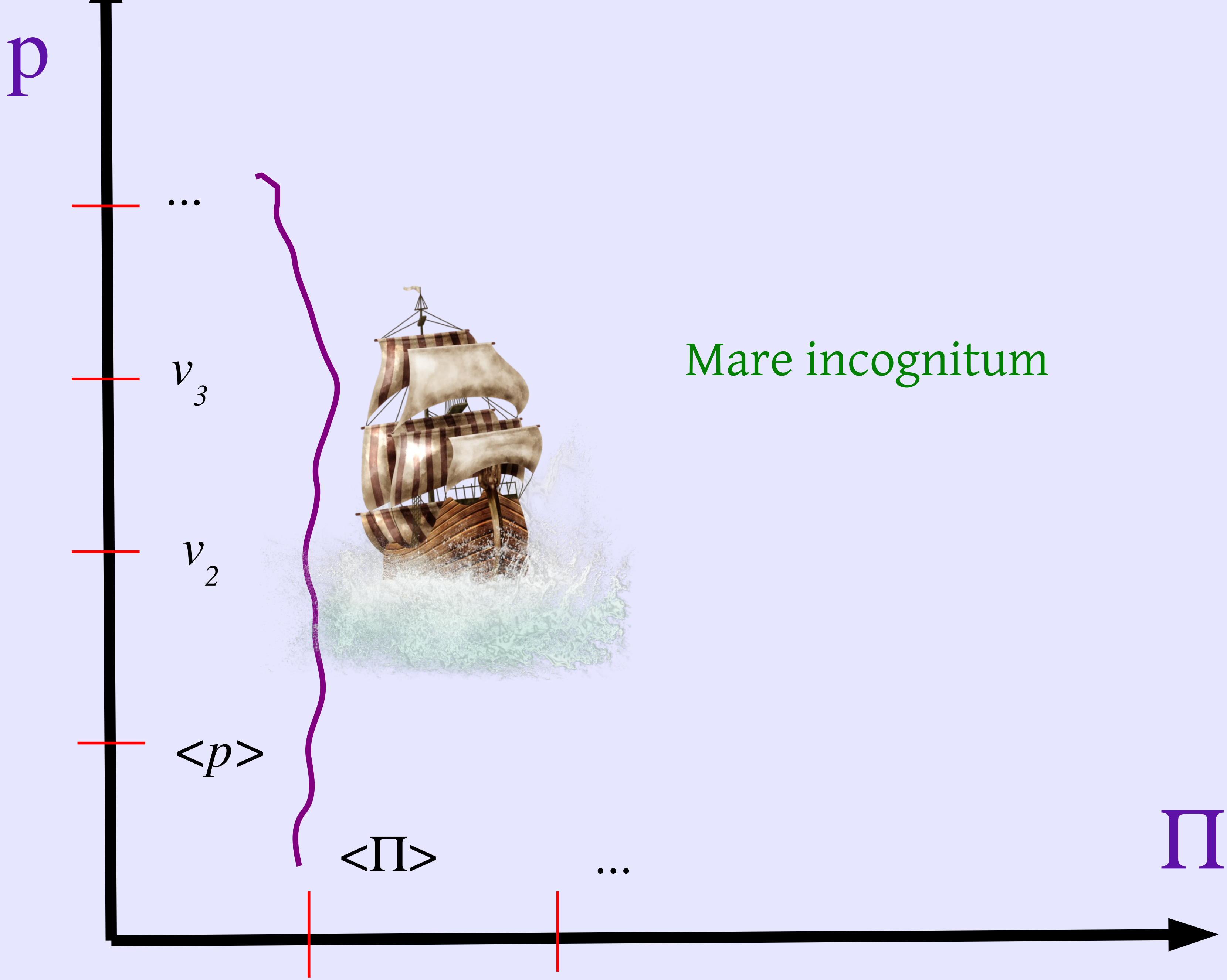
F. B., I. Karpenko, M. Lisa, I. Upsal, S. Voloshin, arXiv:1610.02506

Inclusion of decay chain effects (2-body)

$$\mathbf{S}^*_{\rm daughter} = C\mathbf{S}^*_{\rm parent}$$

$$C = \sum_{\lambda_A,\lambda_B,\lambda'_A} T^J(\lambda_A,\lambda_B)T^J(\lambda'_A,\lambda_B)^* \sum_{n=-1}^{1} \langle \lambda'_A|\widehat{S}_{A,-n}|\lambda_A\rangle \times \frac{c_n}{\sqrt{J(J+1)}}\langle J\lambda|J1|\lambda' n\rangle \left(\sum_{\lambda_A,\lambda_B} |T^J(\lambda_A,\lambda_B)|^2\right)^{-1}$$

A new direction in Heavy Ion phase space



Conclusions and Outlook

- Relativistic distribution function of particles with spin ½ at local thermodynamical equilibrium. Particles get polarized, in general, if the fluid has a local thermal vorticity (exterior derivative of β), comprising acceleration and vorticity.

- Hydrodynamic first order quantum effect

- Distinctive feature of thermodynamic polarization: C-even effect, particle and antiparticle have the same polarization (unlike E.M. induced polarization).

- Λ polarization in relativistic heavy ion collisions: from some permille to some percent, a sensitive probe of the initial conditions and a stringent test of the hydrodynamic picture.

- Can we find an EXACT formula for the polarization in a (locally) rotating-accelerating fluid, for ANY spin ?

Why “vorticous” ?

LATIN	ITALIAN	FRENCH	ENGLISH
gratiosus	grazioso	gracieux	gracious
deliciosus	delizioso	delicieux	delicious
copiosus	copioso	copieux	copious
vorticosus	vorticoso	?	vorticous

Λ polarization correlations

L. G. Pang, H. Petersen, Q. Wang, X.N. Wang, arXiv:1605.04024 [hep-ph].

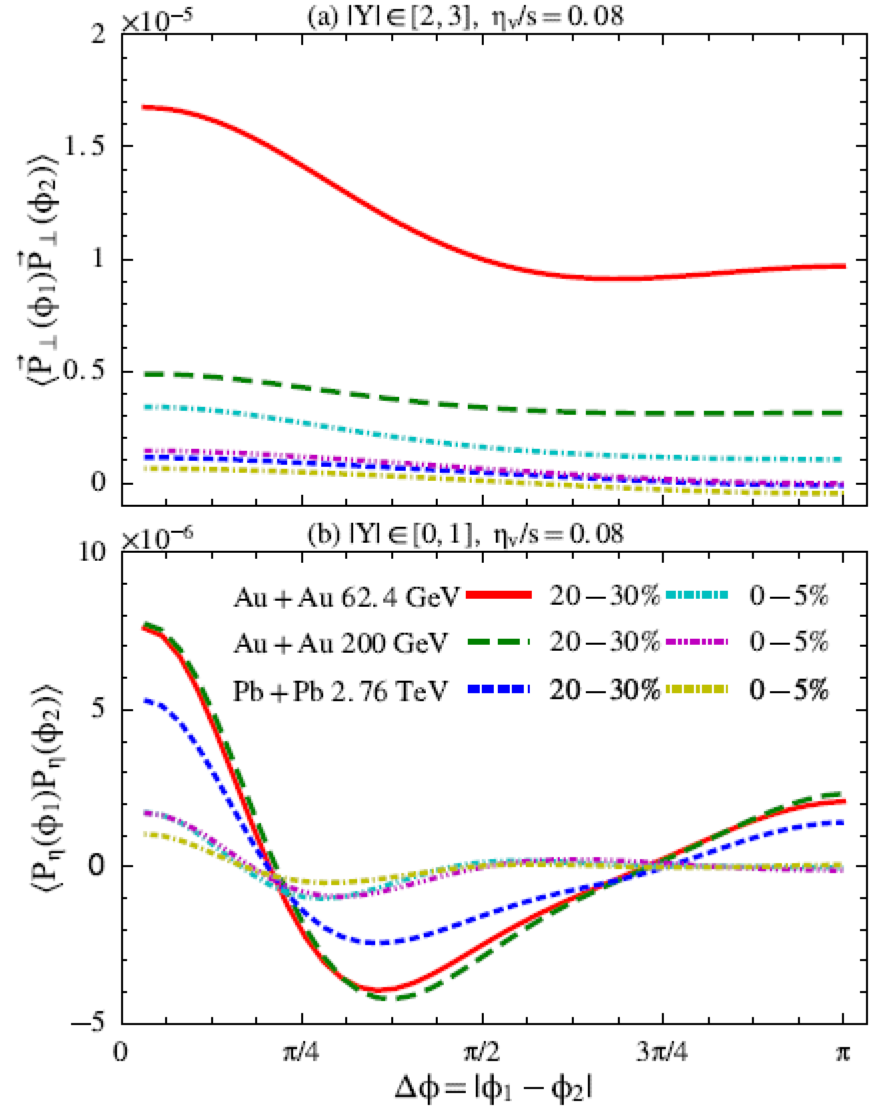


Figure 3: (color online) (a) Transverse ($|Y| \in [2,3]$) and (b) longitudinal ($|Y| \in [0,1]$) spin correlation of two Λ's as a function of the azimuthal angle difference (of their momenta) in semi-peripheral (20-30%) and central (0-5%) Au+Au collisions at $\sqrt{s}_{\rm NN} = 62.4, 200$ GeV and Pb+Pb collisions at $\sqrt{s}_{\rm NN} = 2.76$ TeV with $\eta_v/s = 0.08$.

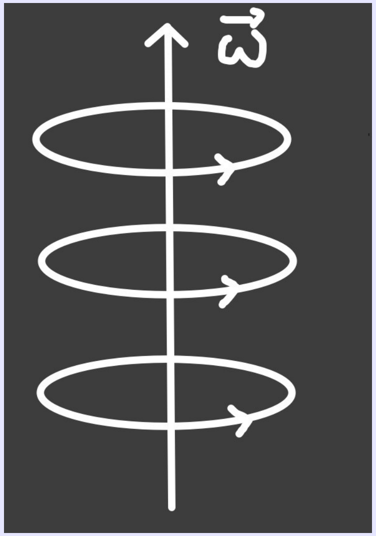
Special cases

Pure rotation (just seen)

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (\omega/T_0)(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu})$$

$$\beta^\mu = \frac{1}{T_0}(1, \boldsymbol{\omega} \times \mathbf{x})$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + \omega \widehat{J}_z/T_0]$$

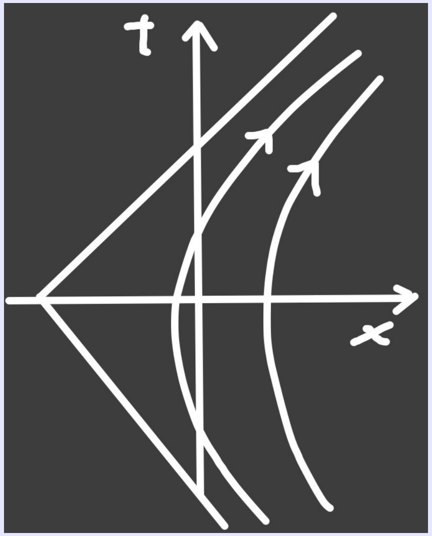


Pure acceleration

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (a/T_0)(g_{0\mu}g_{3\nu} - g_{3\mu}g_{0\nu})$$

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at)$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + a\widehat{K}_z/T_0]$$



Polarization in global equilibrium: rotation

Density operator (see e.g. Landau, *Statistical physics*; A. Vilenkin, Phys. Rev. D 21 2260)

$$\widehat{\rho} = \frac{1}{Z_\omega} \exp\left[-\frac{\widehat{H}}{T_0} + \frac{\mu_0 \widehat{Q}}{T_0} + \frac{\omega \widehat{J}_z}{T_0} \right]$$

Grand-canonical rotational partition function

Obtained by maximizing the entropy $S = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$ with respect to $\widehat{\rho}$
with the constraints of total mean energy, mean momentum and mean angular momentum
Fixed (equivalent to exact conservation for a *large* system)

ω/T is the Lagrange multiplier of the angular momentum conservation constraint and its physical meaning is that of an *angular velocity*

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$$

Polarization in a relativistic fluid

Definition:

$$\Pi_\mu = -\frac{1}{2}\epsilon_{\mu\rho\sigma\tau} S^{\rho\sigma}\frac{p^\tau}{m}$$

also known as Pauli-Lubanski vector

should be the total angular momentum vector of the particle

Mean polarization in *x*:

$$\langle \Pi_\mu(x,p)\rangle = -\frac{1}{2}\frac{1}{\mathrm{tr}_2 f}\epsilon_{\mu\rho\sigma\tau}\frac{\mathrm{d}\mathcal{J}^{0,\rho\sigma}(x,p)}{\mathrm{d}^3 p}\frac{p^\tau}{m}$$

Total angular momentum tensor

$$\mathcal{J}^{\lambda,\rho\sigma}(x) = x^\rho T^{\lambda\sigma}(x) - x^\sigma T^{\lambda\tau}(x) + \mathcal{S}^{\lambda,\rho\sigma}(x)$$

$$\frac{\mathrm{d}\mathcal{J}^{0,\rho\sigma}(x)}{\mathrm{d}^3 p} = (x^\rho p^\sigma - x^\sigma p^\rho)\mathrm{tr}_2 f(x,p) + \frac{\mathrm{d}\mathcal{S}^{\lambda,\rho\sigma}(x)}{\mathrm{d}^3 p}$$

vanished by the Levi-Civita symbol

Canonical spin tensor

By using the Wigner function and the distribution function definitions, it can be shown that

$$\mathcal{S}^{\lambda,\mu\nu}(x) \equiv \frac{1}{2}\langle :\overline{\Psi}(x)\{\gamma^\lambda,\Sigma^{\mu\nu}\}\Psi(x):\rangle = \frac{1}{2}\int \frac{\mathrm{d}^3\mathrm{p}}{2\varepsilon}\mathrm{tr}_2\left(f(x,p)\bar{U}(p)\{\gamma^\lambda,\Sigma^{\mu\nu}\}U(p)\right) - \mathrm{tr}_2\left(\bar{f}^T(x,p)\bar{V}(p)\{\gamma^\lambda,\Sigma^{\mu\nu}\}V(p)\right)$$

...tracing the γ's, expanding in $\varpi(x)$ which is usually a small number (at global equilibrium $\hbar\omega/KT \ll 1$)...

$$\boxed{\frac{\mathrm{d}\mathcal{S}^{\lambda,\rho\sigma}(x)}{\mathrm{d}^3p} \simeq \frac{1}{2\varepsilon}\left(p^\lambda n_F(1-n_F)\varpi^{\rho\sigma} + \text{rotation of indices}\right)}$$

NOTE: the above spin tensor is the dual of the axial current

$$n_F = \frac{1}{\mathrm{e}^{\beta(x)\cdot p-\xi(x)}+1}$$

Spinorization of f

For the case S=1/2 the formulae can be rewritten using Dirac spinors (Weyl repr'n)

$$f(x,p) = \mathrm{e}^{\xi}\,\mathrm{e}^{-\beta\cdot p}\;\frac{1}{2m}\bar{U}(p)\exp[(\omega/T)\,\Sigma_z]U(p) \qquad \Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$\bar{f}(x,p) = -\mathrm{e}^{-\xi}\,\mathrm{e}^{-\beta\cdot p}\;\frac{1}{2m}\left[\bar{V}(p)\exp[-(\omega/T)\,\Sigma_z]V(p)\right]^T$$

They can be also rewritten in a fully covariant form by using the thermal vorticity tensor:

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu\beta_\mu - \partial_\mu\beta_\nu)$$

being

$$\beta = \frac{1}{T}(1, \boldsymbol{\omega}\times\mathbf{x}) = \frac{1}{T_0}(\gamma, \gamma\mathbf{v})$$

and the generators of the Lorentz group in the Dirac representation

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$f(x,p) = \mathrm{e}^{\xi}\,\mathrm{e}^{-\beta\cdot p}\;\frac{1}{2m}\bar{U}(p)\exp\left[\frac{1}{2}\varpi^{\mu\nu}\Sigma_{\mu\nu}\right]U(p) \qquad \bar{f}(x,p) = -\mathrm{e}^{-\xi}\,\mathrm{e}^{-\beta\cdot p}\;\frac{1}{2m}\left[\bar{V}(p)\exp\left[-\frac{1}{2}\varpi^{\mu\nu}\Sigma_{\mu\nu}\right]V(p)\right]^T$$

The latter equation can be checked explicitely, but it is form is indeed a deeper consequence of relativity coupled with thermodynamics

Equilibrium in relativity can be achieved only if the inverse four-temperature field is a Killing vector

$$\boxed{\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0}$$

$$\Rightarrow \quad \beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \qquad b \text{ and } \varpi \text{ constants}$$

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

$\Rightarrow$ ***"Thermal vorticity"***

If deviations from equilibrium are *small*, we know that the tensor $\varpi(x)$ should differ from the above expression only by terms which vanish at equilibrium, i.e. second-order terms in the gradients of the β field

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) + \mathcal{O}(\partial^2 \beta)$$

This is what we need for leading-order hydrodynamics!

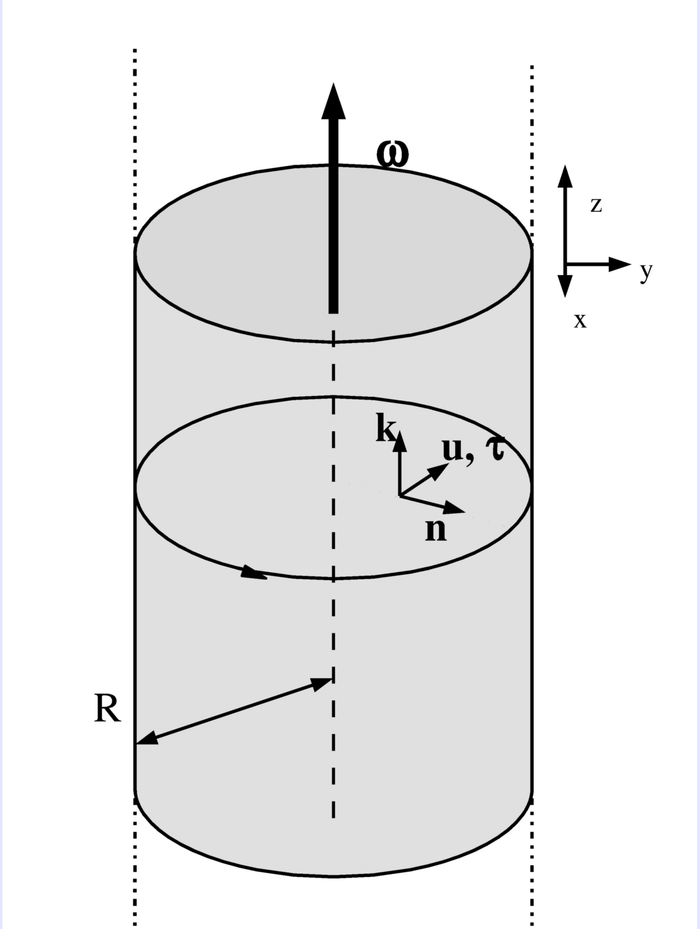
What is $\varpi(x)$?

This is a crucial issue to calculate polarization

At global equilibrium with rotation:

$$\varpi_{\mu\nu} = (\omega/T)(\delta^1_\mu \delta^2_\nu - \delta^1_\nu \delta^2_\mu) = \sqrt{\beta^2}\Omega_{\mu\nu}$$

$$\sqrt{\beta^2} = \frac{1}{T_0} \qquad \Omega^{\mu\nu} = \sum_{i=1}^{4} \frac{De^\mu_i}{\mathrm{d}\tau} e^{i\nu}$$



$$e_0 = u \quad e_1 = (0, \hat{\mathbf{n}}) \quad e_2 = (0, \hat{\mathbf{k}}) \quad e_3 = \tau$$

At the same time,we have seen that, at global equilibrium

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

being:

Converse: Einstein-De Haas effect

the only experiment by Einstein

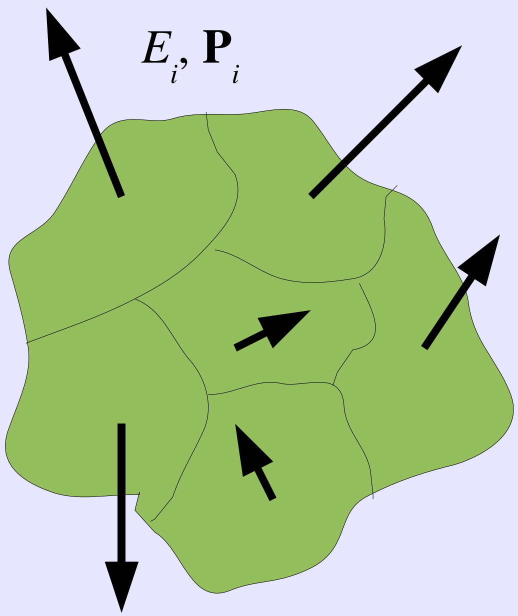
A. Einstein, W. J. de Haas, Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings, 18 I, 696-711 (1915)

Rotation of a ferromagnet originally at rest when put into an external H field

An effect of angular momentum conservation:
spins get aligned with H (irreversibly) and this must be compensated by a on overall orbital angular momentum

Global thermodynamical equilibrium with angular momentum

non-quantum Landau's argument

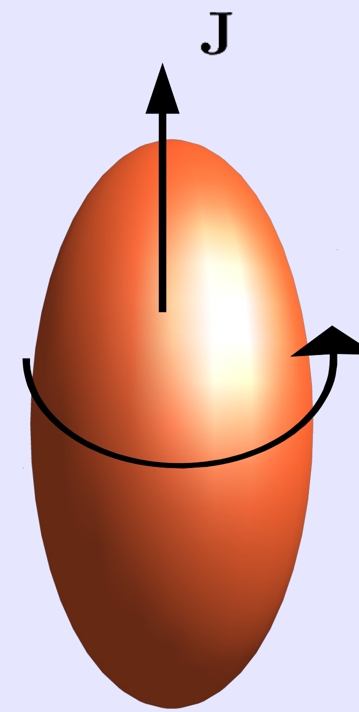


$$S = \sum_i S_i(\sqrt{E_i^2 - \mathbf{P}_i^2}) \qquad \frac{\partial S_i}{\partial E_i} = \frac{E_i}{M_i}\frac{\partial S_i}{\partial M_i} = \frac{\gamma_i}{T_i}$$

Maximize entropy with constraints

$$\sum_i S_i - \frac{\boldsymbol{\beta}}{T}\cdot\sum_i \mathbf{P}_i - \frac{1}{T}(\sum_i E_i - E_0) - \frac{\boldsymbol{\omega}}{T}\cdot(\sum_i \mathbf{x}_i \times \mathbf{P}_i - \mathbf{J})$$

$$\frac{\gamma_i}{T_i} = \frac{1}{T} \quad \forall i \qquad \mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{x}_i \quad \forall i$$



Local temperature

$$T_i = \frac{T}{\sqrt{1 - (\boldsymbol{\omega} \times \mathbf{x}_i)^2}}$$

The role of the spin tensor in relativistic hydrodynamics

F. B., L. Tinti, Phys. Rev. D 84 (2011) 025013
F. B., L. Tinti, Phys. Rev. D 87 (2013) 025029

In Minkowski space-time, from translational and Lorentz invariance one obtains two conserved Noether currents:

$$\widehat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^a)} \partial^\nu \Psi^a - g^{\mu\nu} \mathcal{L}$$

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Psi^a)} D^A (J^{\mu\nu})^a_b \Psi^b$$

It is very important to stress that these are <u>operators</u> (henceforth denoted with a hat)

$$\partial_\mu \widehat{T}^{\mu\nu} = 0$$

$$\partial_\lambda \widehat{\mathcal{J}}^{\lambda,\mu\nu} = \partial_\lambda \left(\widehat{\mathcal{S}}^{\lambda,\mu\nu} + x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right) = \partial_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu} + \widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu} = 0$$

Relation between macroscopic (classical) and quantum observables

$$\mathcal{O}_{\rm cl.} = {\rm tr}\left(\widehat{\rho} : \widehat{O} :\right)$$

Therefore we define as the (macroscopic) stress-energy and spin tensors:

$$T^{\mu\nu}(x) = {\rm tr}\left(\widehat{\rho} : \widehat{T}^{\mu\nu}(x) :\right)$$

$$\mathcal{S}^{\lambda,\mu\nu}(x) = {\rm tr}\left(\widehat{\rho} : \widehat{\mathcal{S}}^{\lambda,\mu\nu}(x) :\right)$$

Pseudo-gauge transformations with a *superpotential* $\widehat{\Phi}$

F.W. Hehl, Rep. Mat. Phys. 9 (1976) 55

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu} \right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu} + \partial_\alpha \widehat{\Xi}^{\alpha\lambda,\mu\nu}$$

With (we confine ourselves to $\Xi = 0$):

$$\int_{\partial\Omega} \mathrm{d}S \, \left(\widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) n_i = 0$$

$$\int_{\partial\Omega} \mathrm{d}S \, \left[x^\mu \left(\widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) - x^\nu \left(\widehat{\Phi}^{i,0\mu} - \widehat{\Phi}^{0,i\mu} - \widehat{\Phi}^{\mu,i0} \right) \right] n_i = 0$$

They leave the conservation equations and spacial integrals (=generators, or total energy, momentum and angular momentum) invariant.

This seems to be enough for a quantum relativistic field theory. It is not in gravity but, as long as we disregard it, different couples of tensors related by a pseudo-gauge transformation cannot be distinguished

Example: Belinfante symmetrization procedure

Just take $\widehat{\Phi} = \widehat{\mathcal{S}}$

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\mathcal{S}}^{\alpha,\mu\nu} - \widehat{\mathcal{S}}^{\mu,\alpha\nu} - \widehat{\mathcal{S}}^{\nu,\alpha\mu} \right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

This is a way of getting rid of the spin tensor, whose physical meaning seems to be thus very limited in QFT (eliminated by a pseudo-gauge transformation). The (mean value of the) above symmetrized Belinfante tensor is commonly assumed to be the source of the gravitational field, at least in GR.

Nevertheless, if we are interested in local mean values, that is mean energy-momentum density and angular momentum density or polarization, the equivalence may be broken. Thermodynamics and hydrodynamics make a difference!