

# Decay constants of heavy mesons from QCD sum rules

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**We present a sum-rule extraction of the decay constants of  $D$ ,  $D_s$ ,  $B$ , and  $B_s$  mesons from the two-point function of heavy-light pseudoscalar currents with the main emphasis on the uncertainties in these quantities, both related to the input QCD parameters and the intrinsic uncertainties of the method of sum rules. The latter are studied making use of a recently formulated algorithm based on the Borel-parameter-dependent effective continuum threshold.**

**A QCD sum-rule calculation of hadron parameters involves two steps:**

**I. one calculates the operator product expansion (OPE) series for a relevant correlator, and obtains the sum rule which relates this OPE to the sum over hadronic states.**

**II. one extracts the parameters of the ground state by a numerical procedure.**

### **I. The basic object**

$$\Pi(p^2) = i \int dx e^{ipx} \langle 0 | T (j_5(x) j_5(0)) | 0 \rangle, \quad j_5(x) = (m_Q + m) \bar{q} i \gamma_5 Q(x).$$

**Borel-transform ( $p^2 \rightarrow \tau$ ): Green function  $\Pi(p^2) \rightarrow$  evolution operator  $\Pi(\tau)$  in Euclidean space,  $\tau$  is related to Euclidean time. OPE series for  $\Pi(\tau)$ :**

$$\Pi(\tau) = \Pi_{\text{pert}}(\tau, \mu) + \Pi_{\text{power}}(\tau, \mu),$$

$$\Pi_{\text{pert}}(\tau, \mu) = \int_{(m_Q+m)^2}^{\infty} e^{-s\tau} \rho_{\text{pert}}(s, \mu) ds, \quad \rho_{\text{pert}}(s, \mu) = \rho_0(s, \mu) + \alpha_s(\mu) \rho_1(s, \mu) + \alpha_s(\mu)^2 \rho_2(s, \mu) + \dots$$

$$\Pi_{\text{power}}(\tau, \mu) = (m_Q + m)^2 e^{-m_Q^2 \tau} \left[ -m_Q \langle \bar{q}q \rangle \left\{ 1 + \frac{2C_F \alpha_s}{\pi} \left( 1 - m_Q^2 \tau / 2 \right) \right\} + \dots \right].$$

$\mu$  is renormalization point. Quark masses,  $\alpha_s$ -running  $\overline{\text{MS}}$  parameters;  $m_Q \equiv \bar{m}_Q(\bar{m}_Q)$ .

**The OPE for  $\Pi(\tau)$  is a double expansion (i) in powers of  $\alpha_s$  (ii) in powers of  $\tau$ , in terms of the condensates of increasing dimensions.**

**Only a few lowest-order *radiative* and *power* corrections are known, so we have to deal with *truncated* OPE.**

**Features of this *truncated* OPE:**

**(i) depends on  $\mu$**

**(ii) gives a good description of exact correlator at not very large  $\tau$ ,  $m_Q^2\tau \leq 2 \div 3$ .**

**The same correlator may be calculated in terms of hadron intermediate states:**

$$\Pi_{\text{hadr}}(\tau) = \Pi_{\text{g}}(\tau) + \text{contribution of excited states}, \quad \Pi_{\text{g}}(\tau) = f_Q^2 M_Q^4 e^{-M_Q^2 \tau}.$$

**where  $f_Q$  is the decay constant of heavy-light meson defined as  $(m_Q + m_u)\langle 0|\bar{q}i\gamma_5 Q|B\rangle = f_Q M_Q^2$ .**

$$\underline{\text{Sum rule:}} \quad \Pi_{\text{hadr}}(\tau) = \Pi_{\text{OPE}}(\tau)$$

**At large  $\tau$  the lowest-mass (ground) state dominates, but at large  $\tau$  OPE does not work.**

**We want to isolate the ground-state contribution at relatively low values of  $\tau$ .**

$$\text{Quark-hadron duality assumption: } \Pi_{\text{excited}}(\tau) \leftrightarrow \int_{s_{\text{eff}}}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s).$$

$$f_Q^2 M_Q^4 e^{-M_Q^2 \tau} \leftrightarrow \int_{(m_Q+m_u)^2}^{s_{\text{eff}}} e^{-s\tau} \rho_{\text{pert}}(s, \alpha, \mu) ds + \Pi_{\text{power}}(\tau, \mu) \equiv \Pi_{\text{dual}}(\tau, \mu, s_{\text{eff}})$$

Obviously, we need to know  $s_{\text{eff}}$ .

- Standard approximation:  $s_{\text{eff}} = \text{const.}$

Then the extracted  $f_Q$  depends on  $s_{\text{eff}}$  and  $\tau$  (unphysical dependence).

- Usual criterion to fix  $s_{\text{eff}}$ :

Stability of  $f_Q$ , i.e. its max independence of  $\tau$ .

- How to estimate the error of the obtained  $f_Q$ ?

Often: Variation of  $f_Q(\tau)$  over the “window”.

Where to test this algorithm?

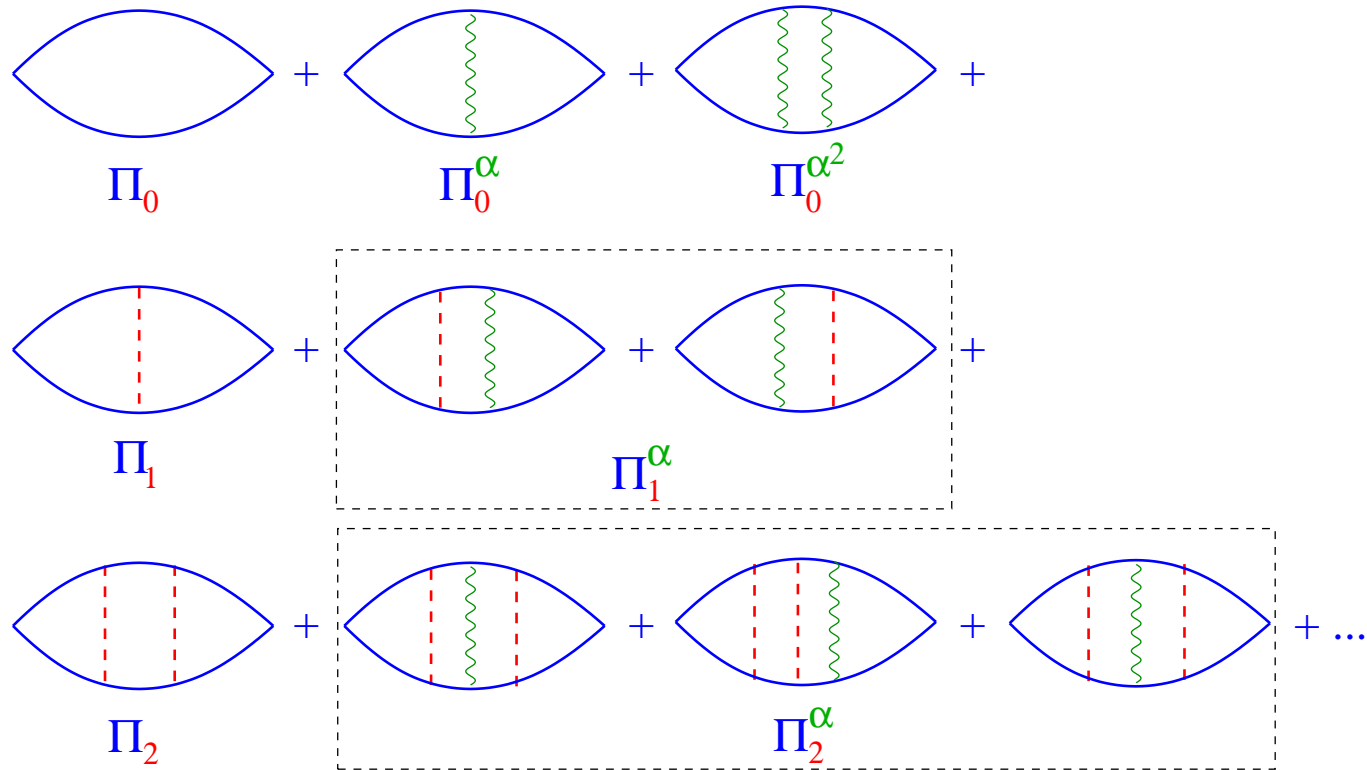
*In quantum mechanics*

Bound-state parameters are known from Schroedinger equation. Applying the sum-rule algorithms and comparing the outcomes with exact hadron parameters allows one to test these algorithms.

## Correlator and sum rule in quantum-mechanical potential model

**Confining + attractive Coulomb potential:**  $H = \frac{k^2}{2m} + V_{\text{conf}}(r) - \frac{\alpha}{r}$ .

**We construct OPE for this correlator keeping perturbative contributions up to  $\alpha^2$  terms (3 loops of the non-relativistic field theory) and 2 power corrections, including  $O(\alpha)$  corrections to them:**



**Our findings from quantum mechanics: Stability necessary but not sufficient condition, does not guarantee extraction of reliable value of hadron parameter.**

**In quantum mechanics one finds that  $s_{\text{eff}}$  depends on  $\tau$ .**

**Taking into account the dependence of  $s_{\text{eff}}$  on  $\tau$   
allows one to improve the accuracy of the duality approximation.**

**How to determine this  $\tau$ -dependent effective continuum threshold?**

**For those cases where the bound-state mass is known, it can be used for obtaining the  $\tau$ -dependent threshold.**

**Our algorithm:**

$$s_{\text{eff}}^{(n)}(\tau) = \sum_{j=0}^n s_j^{(n)}(\tau)^j.$$

**We fix its parameters as follows:**

**(i) calculate the “dual mass”:**

$$M_{\text{dual}}^2(\tau) = -\frac{d}{d\tau} \log \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)).$$

**(ii) Evaluate  $M_{\text{dual}}^2(\tau)$  at several values of  $\tau = \tau_i$  ( $i = 1, \dots, N$ ,  $N$  can be taken arbitrary large) chosen uniformly in the working region of  $\tau$  — the “window”. Minimize the squared difference between  $M_{\text{dual}}^2$  and the known value  $M_Q^2$ :**

$$\chi^2 \equiv \frac{1}{N} \sum_{i=1}^N [M_{\text{dual}}^2(\tau_i) - M_Q^2]^2.$$

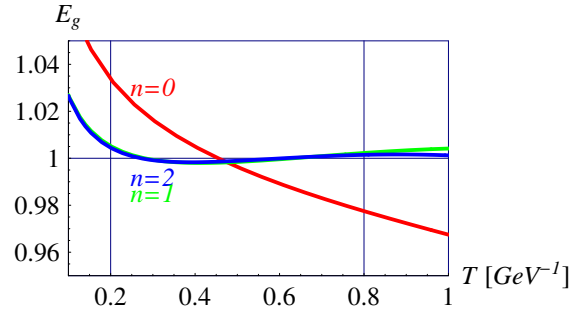
**This gives us the parameters of the effective continuum threshold. As soon as the latter is fixed, it is straightforward to calculate the decay constant.**

**Finally, take the band of values provided by the results corresponding to linear and quadratic effective thresholds as the characteristic of the intrinsic uncertainty of the extration procedure.**

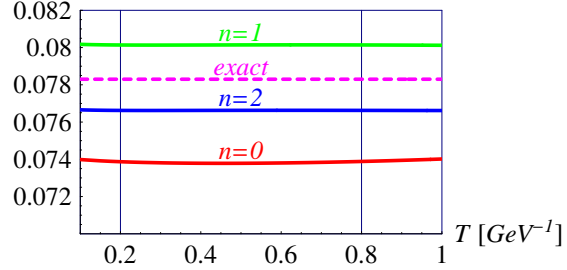
## Extraction of $f_P$ : QCD vs potential model

### Potential Model (HO + Coulomb)

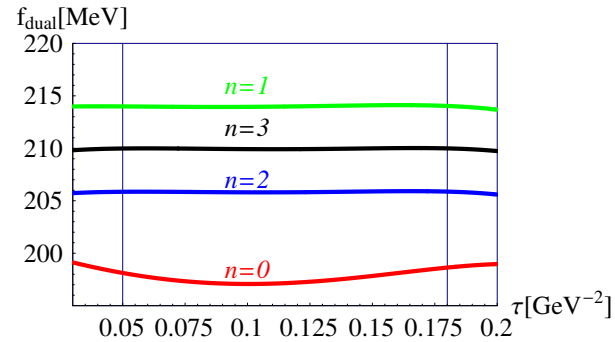
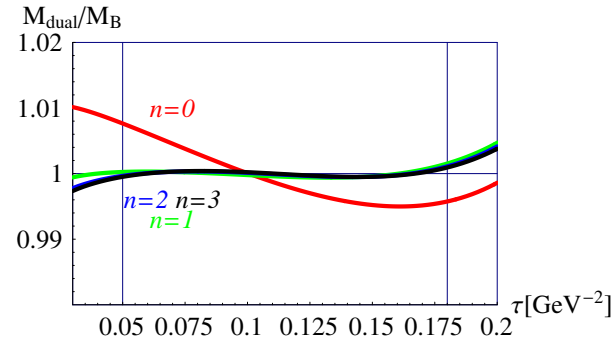
$$E_{\text{dual}}(T, z_{\text{eff}}(T))$$



$$f_{\text{dual}}(T, z_{\text{eff}}(T)) [\text{GeV}]$$



### QCD ( $f_B$ for $\bar{m}_b(\bar{m}_b) = 4.20 \text{ GeV}$ )

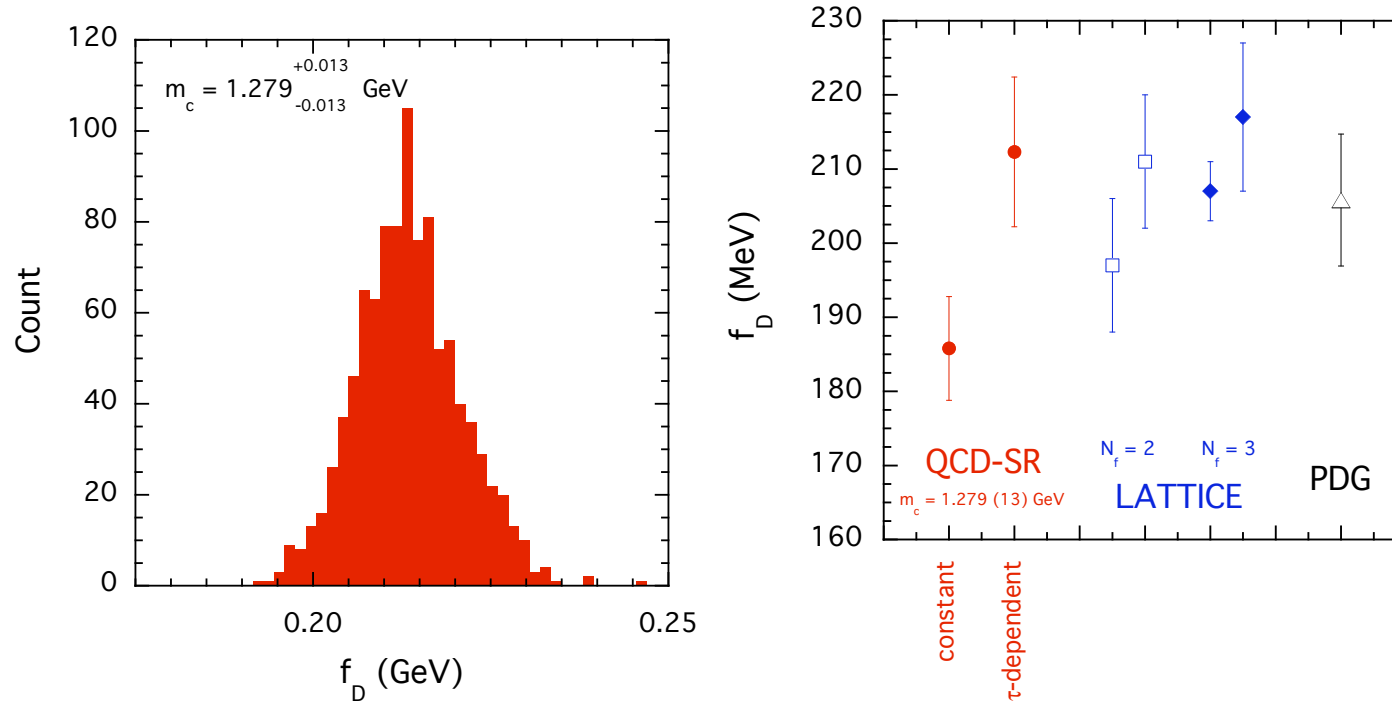


As soon as quark-hadron duality is implemented as a cut on the perturbative correlator, the extraction of the ground-state parameters in QCD and in potential model are very similar.



## Extraction of $f_D$ and $f_{D_s}$

$m(2\text{GeV}) = 3.5 \pm 0.5\text{MeV}$ ,  $m_s(2\text{GeV}) = 105 \pm 15\text{MeV}$ ,  $\langle\bar{q}q\rangle(2\text{GeV}) = -(267 \pm 17 \text{ MeV})^3$ ,  $\langle GG\rangle(2\text{GeV}) = (0.024 \pm 0.012) \text{ GeV}^4$ ,  $\alpha_S(M_Z) = 0.1176 \pm 0.0020$ ,  $m_c(m_c) = 1.27 \pm 0.10 \text{ GeV}$   $\mu = 1 - 3\text{GeV}$



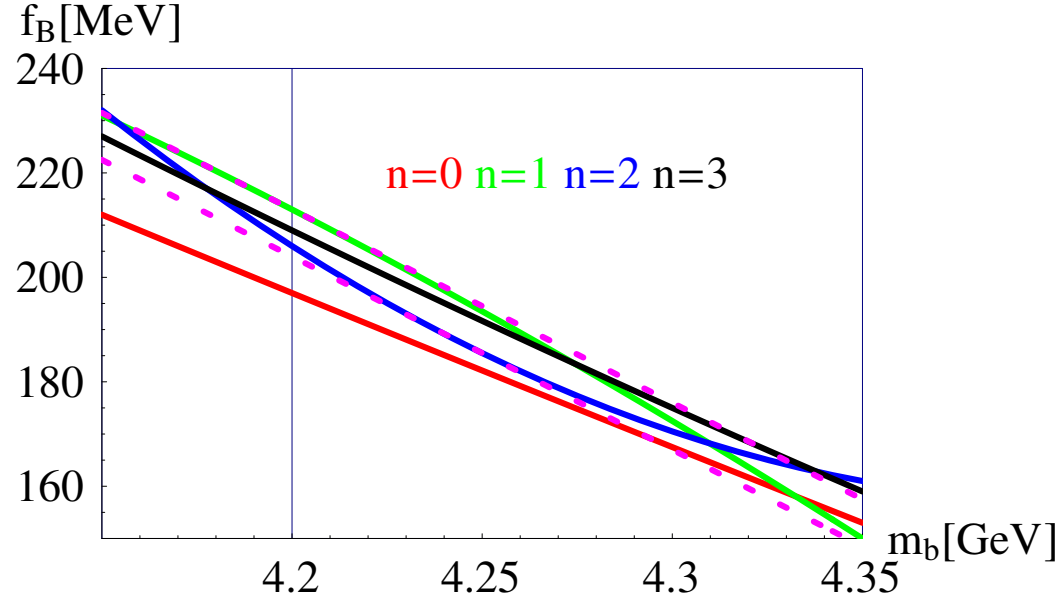
$$f_D = 212 \pm 10_{(\text{OPE})} \pm 7_{(\text{syst})} \text{ MeV} \quad (182 \pm 7_{(\text{OPE})} \text{ MeV (constant)})$$

$$f_{D_s} = 245 \pm 20_{(\text{OPE})} \pm 8_{(\text{syst})} \text{ MeV} \quad (215 \pm 15_{(\text{OPE})} \text{ MeV (constant)})$$

**The effect of  $\tau$ -dependent threshold is visible.**

## Extraction of $f_B$

### A strong sensitivity to $m_b(m_b)$



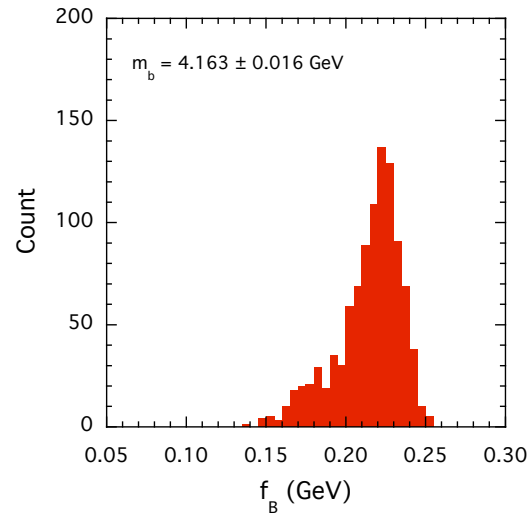
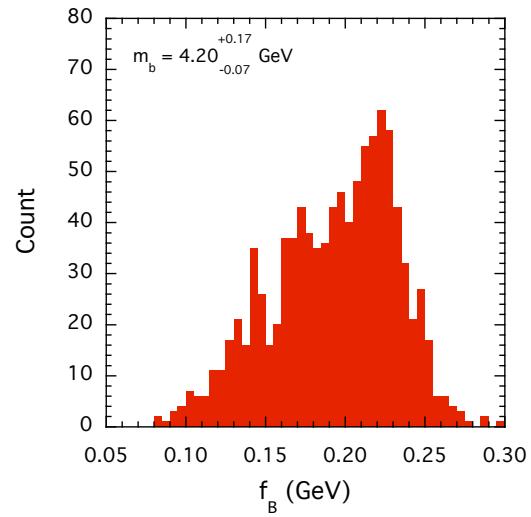
**$\tau$ -dependent effective threshold:**

$$f_B^{\text{dual}}(m_b, \langle \bar{q}q \rangle, \mu = m_b) = \left[ 208 \pm 4 - 37 \left( \frac{m_b - 4.2 \text{ GeV}}{0.1 \text{ GeV}} \right) + 4 \left( \frac{\langle \bar{q}q \rangle^{1/3} - 0.267 \text{ GeV}}{0.01 \text{ GeV}} \right) \right] \text{MeV},$$

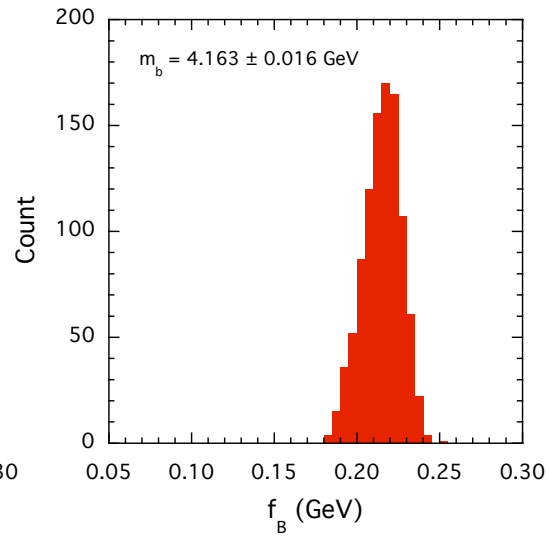
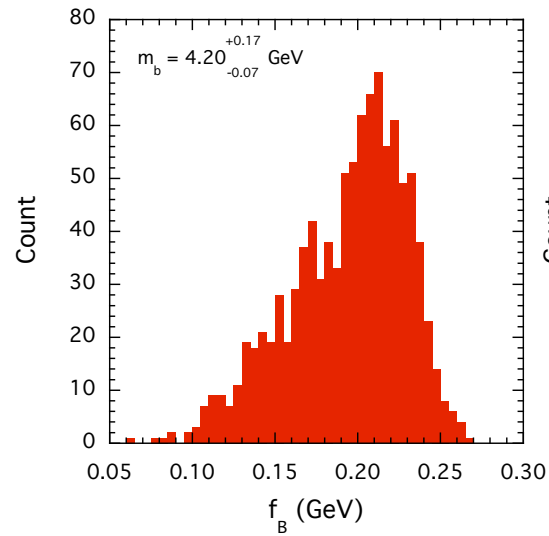
**$\tau$ -independent effective threshold:**

$$f_B^{[n=0]}(m_b, \langle \bar{q}q \rangle, \mu = m_b) = \left[ 196 - 30 \left( \frac{m_b - 4.2 \text{ GeV}}{0.1 \text{ GeV}} \right) + 4 \left( \frac{\langle \bar{q}q \rangle^{1/3} - 0.267 \text{ GeV}}{0.01 \text{ GeV}} \right) \right] \text{MeV},$$

$\mu=2-8$  GeV:

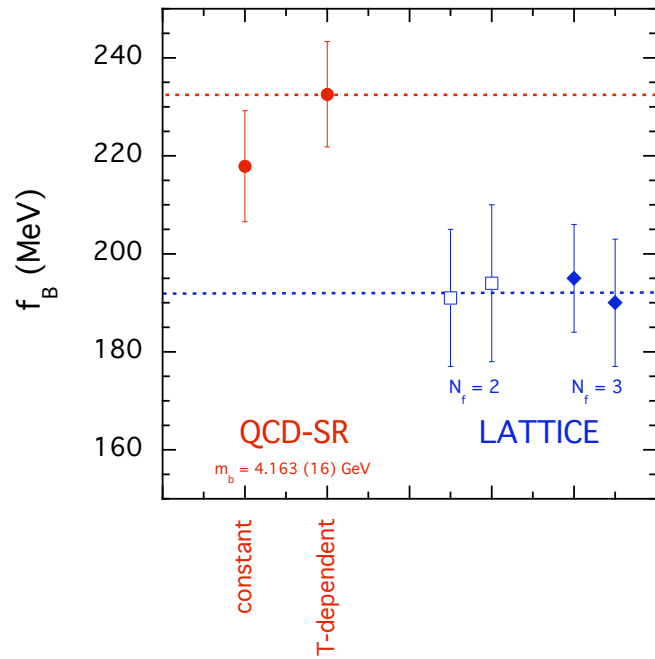


$\mu=3-6$  GeV:



The prediction for  $f_B$  seems not feasible without a very precise knowledge of  $m_b$

If the error on  $m_b(m_b)$  provided by Chetyrkin et al is trustable, then there may be tension between lattice and sum rules:



## Conclusions

The effective continuum threshold  $s_{\text{eff}}$  is an important ingredient of the method which determines to a large extent the numerical values of the extracted hadron parameters. Finding a criterion for fixing  $s_{\text{eff}}$  poses a problem in the method of sum rules.

- $\tau$ -dependence of  $s_{\text{eff}}$  emerges naturally when trying to make quark-hadron duality more accurate. For those cases where the ground-state mass  $M_Q$  is known, we proposed a new algorithm for fixing  $s_{\text{eff}}$ : We proposed an algorithm for fixing this  $s_{\text{eff}}(\tau)$ .
- $\tau$ -dependent  $s_{\text{eff}}$  is a useful concept as it allows one to probe realistic intrinsic uncertainties of the extracted parameters of the bound states. Although not rigorous in the mathematical sense, this estimate may be considered as a realistic educated guess supported by the results obtained in a model where the actual value of the decay constant is known.
- As first applications in QCD, we obtained estimates for decay constants of heavy mesons  $f_Q$  which include also systematic uncertainties.

