# QCD, MONOPOLES ON THE LATTICE AND GAUGE INVARIANCE

Adriano Di Giacomo, Pisa University and INFN

QCD@work, MARTINA FRANCA JUNE 2010

BASED ON THE PAPERS

- C.Bonati, A. Di Giacomo, L.Lepori, F. Pucci, Phys. Rev. D **81**, 085022, (2010)
  and
- C. Bonati, M. D'Elia, A Di Giacomo
  in preparation.

# PLAN OF THE TALK

- INTRODUCTION AND MOTIVATION.
- MONOPOLES AND NON-ABELIAN BIANCHI IDENTITIES.
- THE 't HOOFT POLYAKOV MONOPOLE REVISITED.
- GENERIC MONOPOLE CONFIGURATIONS
- MONOPOLE CONDENSATION AND CONFINEMENT
- CONCLUSIONS AND OUTLOOK.

Monopoles can condense in the QCD vacuum and produce dual superconductivity and confinement . Dual Abrikosov chromoelectric flux tubes give a linear rising potential


- Prototype monopole configuration a soliton configuration in the Higgs-broken phase of the $SO(3)$ Higgs model ['t Hooft 74, Polyakov 74].
  A non trivial mapping of the sphere $S_2$ at spatial infinity onto $SO(3)/U(1)$, or a non trivial $\pi_2$.
- A monopole in a gauge theory is an abelian object [Coleman 83] : the magnetic monopole term in a multipole expansion obeys abelian equations and identifies a $U(1)$ subgroup of the gauge group (modulo a global $\times$ an arbitrary gauge transformation $U(\vec{r})$ which tends to the identity as $r \rightarrow \infty$). In the model example this $U(1)$ is the direction of the Higgs breaking in the unitary representation.

# INTRODUCTION AND MOTIVATION$_2$

- In *QCD* there is no Higgs field and apparently no privileged residual U(1) symmetry. 't Hooft [81] : any operator $O$ in the adjoint representation can act as an effective Higgs field. Any choice of $O$ identifies an "effective" unitary representation, which is named an Abelian Projection.
- Monopoles in $U(1)$ lattice gauge theory [Degrand, Toussaint 80].
  $U_\mu(n) = \exp i\theta_\mu(n)$, $\theta_\mu(n) \approx eaA_\mu(n)$
  $\Pi_{\mu\nu}(n) = \exp i\theta_{\mu\nu}(n)$ , $\theta_{\mu\nu} = \Delta_\mu\theta_\nu - \Delta_\nu\theta_\mu \approx ea^2 F_{\mu\nu}$
  $n \equiv$ lattice site, $a \equiv$ lattice spacing.
  Compactified theory:
  $-\pi \leq \theta_\mu(n) \leq \pi$ , $-4\pi \leq \theta_{\mu\nu}(n) \leq 4\pi$.
  Def. $\theta_{\mu\nu} \equiv \bar{\theta}_{\mu\nu} + 2\pi n_{\mu\nu}$, with $-\pi \leq \bar{\theta}_{\mu\nu} \leq \pi$ and $n_{\mu\nu}$ an integer.
  If $\theta^*_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma}$ it is identically
  $$\Delta_\mu\theta^*_{\mu\nu} = 0 \text{ (naive Bianchi identities).}$$

- Integer multiples of $2\pi$ are unobservable, so $ea^2 F_{\mu\nu} \approx \bar{\theta}_{\mu\nu}$. The naive Bianchi identities read
$$ea^3 \partial_\mu F^*_{\mu\nu} + 2\pi \Delta_\mu n^*_{\mu\nu} = 0$$
$n^*_{\mu\nu}$ is the net # of Dirac magnetic strings crossing $(\mu, \nu)$. A non-zero magnetic current $J_\nu \equiv -\frac{2}{ea^3}\pi \Delta_\mu n^*_{\mu\nu}$ is a violation of the physical Bianchi identities.
For $\nu = 0$ $\qquad ea^3 \vec{\nabla} \vec{B} = -2\pi \bar{n}$.
( $\bar{n}$ the net # of Dirac strings coming out of the elementary spatial cube, or # of monopole charges in it).
Finally $e\Phi(\vec{B}) = e4\pi m = -2\pi \bar{n}$ or $|m| = \frac{\bar{n}}{2e}$ ( Dirac quantization condition).

The procedure is gauge-invariant and based on Dirac condition.

# INTRODUCTION AND MOTIVATION$_4$

- In a non abelian gauge theory, say $SU(2)$, the game is repeated by fixing the gauge (the abelian projection), and by working on the residual $U(1)$.
  $\Pi_{\mu\nu} = \exp(i\frac{\sigma_3}{2}\theta^3_{\mu\nu})\exp(i\frac{\vec{\sigma}^\perp}{2}\vec{\theta}^\perp_{\mu\nu})$
  $\theta^3_{\mu\nu}$ depends on the choice of the gauge and therefore the number and location of the monopoles are gauge dependent. Maximal abelian gauge preferred for years, because of abelian and monopole dominance [Suzuki et al. 90].
- Either monopoles can appear or disappear by gauge transformations, and then they are not physical objects, or it is not true that all abelian projections are equivalent, as suggested by 't Hooft, at least for detection.

WE INVESTIGATE THIS PROBLEM BY USING AS KEY TOOL THE NON-ABELIAN BIANCHI IDENTITIES.

# THE NON-ABELIAN BIANCHI IDENTITIES$_1$

- The $U(1)$ (abelian) Bianchi identities $\partial_\mu F^*_{\mu\nu} = 0$, or
$$\vec{\nabla}\vec{B} = 0 \ , \ \vec{\nabla} \wedge \vec{E} + \partial_t \vec{B} = 0$$
imply zero magnetic current. A violation has the form
$$\partial_\mu F^*_{\mu\nu} = j_\nu.$$
$j_\nu$ is a magnetic current and is conserved $\partial_\nu j_\nu = 0$ because of the antisymmetry of $F^*_{\mu\nu}$.
- The non-abelian Bianchi identities (NABI) read
$$D_\mu G^*_{\mu\nu} = J_\nu \tag{1}$$
A gauge covariant equation. It implies
$$D_\nu J_\nu = 0 \tag{2}$$
Indeed $D_\nu D_\mu G^*_{\mu\nu} = \frac{1}{2}\left[D_\nu, D_\mu\right] G^*_{\mu\nu}$
$\propto \left[G_{\mu\nu}, G^*_{\mu\nu}\right] = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left[G_{\mu\nu}, G_{\rho\sigma}\right] = 0$

# THE NON-ABELIAN BIANCHI IDENTITIES$_2$

- The four components of the magnetic current $J_\mu$ commute [Coleman,Mandula 67].
  To extract the physical (gauge-invariant) information from the NABI Eq.(1), diagonalize them by a gauge transformation and project on a complete set of diagonal matrices, $\phi_0^a$ $(a = 1, ..r)$, $r$ rank of the gauge group.
  A convenient choice for $\phi_0^a$ fundamental weights : there is one of them for each simple root of the Lie algebra $\vec{\alpha}^a$.
  $[\phi_0^a, H_i] = 0$ , $[\phi_0^a, E_{\pm\vec{\alpha}}] = \pm(\vec{c}^a\vec{\alpha})E_{\pm\vec{\alpha}}$, $(\vec{c}^a\vec{\alpha}^b) = \delta_{ab}$

$$Tr(\phi_0^a \left[D_\mu G^*_{\mu\nu}\right]_{diag}) = Tr(\phi_0^a \left[J_\nu\right]_{diag})$$

  In the generic gauge $\phi_I^a = U(x)\phi_0^a U^\dagger(x)$,

$$Tr(\phi_I^a D_\mu G^*_{\mu\nu}) = Tr(\phi_I^a J_\nu) \equiv j_\nu^a(x, I) \qquad (3)$$

# THE NON-ABELIAN BIANCHI IDENTITIES$_3$

- Define $\phi^a_V = V(x)\phi^a_I V^\dagger(x)$ , $V(x)$ any gauge transformation, and consider the generalization of Eq.(3), derived from Eq.(1)

$$Tr(\phi^a_V D_\mu G^*_{\mu\nu}) = Tr(\phi^a_V J_\nu) \equiv j^a_\nu(x, V) \qquad (4)$$

  $\phi^a_V$ is diagonal with $J_\nu$ only in the special case $V = I$, the identity.

- THEOREM : For any compact gauge group and any $V(x)$

$$\partial_\mu {F^a}^*_{\mu\nu}(x, V) = j^a_\nu(x, V) \qquad (5)$$

  with ${F^a}_{\mu\nu}(x, V)$ the 't Hooft tensor , i.e. the abelian field strength $\partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ in the gauge (abelian projection) in which $\phi^a_V$ is diagonal.
  From Eq.(5) it follows $\partial_\nu j^a_\nu(x, V) = 0$: a functional infinity of conservation laws! $j^a_\nu(x, V) \propto J_\nu$ for all $V$'s. Violation of ABI related to that of NABI's.

Check theorem on the soliton configuration of 't Hooft, Polyakov.
$L = -\frac{1}{4}\vec{G}_{\mu\nu}\vec{G}_{\mu\nu} + (D_\mu\phi)^\dagger(D_\mu\phi) - V(\phi^2)$

- Hedgehog gauge $\vec{\phi}(\vec{r}) = H(r)\hat{r}$ , $H(r)_{r\to\infty} \to v$
  $\vec{A}_0 = 0$ $\qquad A_i^a = -\epsilon_{iak}\frac{r^k}{gr^2}[1 - K(gvr)]$
  $[1 - K(x)]_{x\to 0} \propto x^2$ , $\qquad K(x)_{x\to\infty} \approx \exp(-x)$
  In this gauge
  $\partial_\mu A_\mu = \partial_i A_i = 0$ $\qquad \equiv$ Landau Gauge !
  't Hooft tensor : $\quad F_{ij} = \partial_i A_j^3 - \partial_j A_i^3 \quad b_i = \frac{1}{2}\epsilon ijk F_{jk}$.

$$\vec{b} \approx_{r\to\infty} \frac{2\hat{r}}{gr^2} cos(\theta) \tag{6}$$

$$Q_m = r^2 \int d\Omega \hat{r}\cdot\vec{b} = \frac{2}{g}\int d\Omega cos(\theta) = 0 \tag{7}$$

No magnetic charge !

- Unitary gauge: $\phi_i \sigma_i = v\sigma_3$. NABI: $J_i = 0$ (static)

$$J_0 = D_\mu B_\mu \equiv D_i B_i = \frac{2\pi}{g}\delta^3(\vec{r})\sigma_3 \qquad (8)$$

By direct calculation Maximal Abelian gauge condition satisfied!(Unitary $\equiv$ Max.Ab.)

$$\partial_\mu A^\pm_\mu \pm ig\left[A^3_\mu, A^\pm_\mu\right] = 0$$

One fundamental weight $\phi_0 = \frac{\sigma_3}{2}$ (Rank of $SU(2)$ =1).

Our theorem

$$Tr(\phi_0 \vec{\nabla} \cdot \vec{B}) = \vec{\nabla} \cdot \vec{b} = Tr\phi_0 J_0 = \frac{2\pi}{g}\delta^3(\vec{r}) \qquad (9)$$

or

$$Q_m = \frac{1}{g} \qquad (10)$$

To summarize

- Hedgehog gauge coincides with Landau gauge. Abelian magnetic charge zero.
- Unitary gauge coincides with maximal abelian gauge. Abelian magnetic charge 2.
  The abelian magnetic current is $j_\nu(\vec{r}, I)$ , and carries the correct magnetic charge. Monopole detectable a la Degrand -Toussaint.
- Same in any gauge obtained from the max. abel. by an arbitrary $W(\vec{r})$ if $W(\vec{r})_{r\to\infty} \to 1$.
- Interpolating gauges
  $U_\gamma(\theta,\phi) = \exp(i\phi\frac{\sigma_3}{2})\exp(i\theta\gamma\frac{\sigma_2}{2})\exp(-i\phi\frac{\sigma_3}{2})$ , $U_0(\theta,\phi) = 1$, $U_1(\theta,\phi)$ Max Ab $\to$ Landau. $\frac{Q_m(\gamma)}{Q_m(0)} = \frac{1+\cos(\gamma\pi)}{2} \leq 1$

# GENERIC MONOPOLE CONFIGURATIONS$_1$.

Extend to generic static configurations, by a theorem of Coleman[Erice Lectures 1983,The magnetic monopole 50 years later.Sec.3.3]:

The magnetic monopole term in the multipole expansion of a generic static field configuration is abelian: it obeys abelian Eqs. of motion and can be gauged along one direction in color space (modulo a global transformation).

For a generic configuration it holds for the superposition of the state with its time reflected, with the same consequences.

- The magnetic field at large $r$ in this gauge only depends on the total magnetic charge. $\vec{B} = \frac{m}{2}\frac{\vec{r}}{2gr^3}\sigma_3$ .

  Terms in $\sigma_\pm$ are non leading in $r$. $\rightarrow \vec{b} \approx_{r\rightarrow\infty} \frac{m}{2}\frac{\vec{r}}{2gr^3}$
- The gauge field at large distances obeys the Max. Abelian gauge condition. This gauge differs from the max abelian by a gauge transformation $W(\vec{r})$ , $W(\vec{r})_{r\rightarrow\infty} \rightarrow I$.

# GENERIC MONOPOLE CONFIGURATIONS$_2$.

- Put $\vec{b} \equiv \frac{m}{2}\frac{\vec{r}}{2gr^3} + \delta\vec{b}$. $\rightarrow j_0(\vec{r}, I) \equiv \vec{\nabla}\vec{b} = \frac{m\pi}{g}\delta^3(\vec{r}) + \delta j_0$ , $\int d^3 r \delta j_0 = 0$
- The t'Hooft tensor ( $\vec{b}$) is linear in $A_\mu \rightarrow A_\mu = A_\mu^0 + \delta A_\mu$. $A_\mu^0$ contains the monopole part, dominates at large distances and obeys the Maximal abelian gauge; $\delta A_\mu$ contains higher multipoles. $\vec{B} = \vec{B}_0 + \delta\vec{B}$ , $D_i B_i = \frac{m\pi}{g}\delta^3(\vec{r}) + \delta J_0$ and, projecting on the fundamental weight, $\delta j_0 = Tr(\frac{\sigma_3}{2}\delta J_0)$

A generic configuration looks as a point-like classical monopole carrying the total charge plus a globally magnetically neutral background. This is well known in the abelian case, but also holds in the non-abelian one due to our theorem relating the non-abelian to abelian Bianchi identities.

# GENERIC MONOPOLE CONFIGURATIONS$_3$.

- On the lattice monopoles are assumed to be point-like and to lie inside the elementary cubes . Their number density being finite, the probability of finding them on the border is zero. Looking at the flux through cubes of different sizes can help to eliminate ultraviolet noise [Del Debbio et al 95]. In the max abelian gauge the DG-T method detects all monopoles.
- Any gauge transformation which is trivial on the sphere at infinity (on the border of the cube) is irrelevant.
- A gauge transformation $U(\theta, \phi)$ on the sphere can change the number of observed monopoles. An example : $U(\theta, \phi) = \exp(i\phi\frac{\sigma_3}{2})\exp(-i\gamma\theta\frac{\sigma_2}{2})\exp(i\phi\frac{\sigma_3}{2})$ depending on one parameter $\gamma$ which is the identity for $\gamma = 0$ and the transformation from Max Abelian (unitary) to Landau for $\gamma = 1$ The corresponding magnetic charge can be easily computed to be $Q(\gamma) = Q(0)\frac{1+\cos(\gamma\pi)}{2}$ This prediction can be checked in lattice simulations.

# MONOPOLE CONDENSATION AND CONFINEMENT.$_1$

The magnetic charge density, in the max. ab. projection is

$$j_0(x, I) = Tr(\phi_I J_0(x))$$

with $J_0$ the violation of the NABI ,
$\phi_I$ the fundamental weight diagonal with it.
The E.T. commutator with any local $O(y)$ of magnetic charge $m$

$$[j_0(\vec{x}, x_0, I), O(\vec{y}, x_0)] = m\delta^3(\vec{x} - \vec{y})O(\vec{y}, x_0) + S.T.$$

or, after integration over $\vec{x}$ ,

$$[Q(I), O(y)] = mO(y)$$

If $m \neq 0$ and $\langle O \rangle \neq 0$ the magnetic $U(1)$ is Higgs broken (dual superconductivity).

Consider now a generic abelian projection $j_0(x,V) = Tr(V(x)\phi_I V^\dagger(x) J_0(x))$ and compute the current in the gauge in which $J_0(x)$ is diagonal and $\phi_I$ with it.
Since $\phi_I$ and $V(x)\phi_I V^\dagger(x)$ belong to the algebra it will be $V(x)\phi_I V^\dagger(x) = C(x,V)\phi_I + \sum_{\vec{\alpha}} E_{\vec{\alpha}} D^{\vec{\alpha}}(x,V)$
In the gauge chosen only the first term will contribute and $[Q(V), O(y)] = mO(y)C(y,V)$
Since $C(y,V)$ is generically non vanishing , $O$ will have a non-zero charge also in the new gauge and if $\langle O \rangle \neq 0$, also the new $U(1)$ is Higgs broken.

A consequence of the NABI is that the system behaves as a superconductor in any abelian projection.
Dual superconductivity is a physical property independent on the choice of the gauge. [Kanazawa 07]

# CONCLUSIONS AND OUTLOOK$_1$

- The abelian Bianchi identities for the residual $U(1)$ field in a generic abelian projections are related to the NON ABELIAN BIANCHI IDENTITIES : the abelian magnetic current is a projection of the non-abelian one.
- Any magnetically charged configuration identifies a gauge in which the asymptotic field is diagonal, modulo a global group transformation and a gauge transformation which is asymptotically $I$. The max abelian gauge belongs to that class of gauges. In these gauges the Dirac quantization condition is satisfied.
- In other abelian projections the total magnetic charge is generically smaller, so that the DGT method to detect monopoles can fail. This explains why different numbers and locations of monopoles are found in different abelian projections. No monopoles in the Landau gauge.

# CONCLUSIONS AND OUTLOOK$_2$

- Dual superconductivity is a projection independent statement. All the $U(1)$ residual gauge symmetries are Higgs broken or exact at the same time.
  In that sense the 't Hooft conjecture about the equivalence of abelian projections is correct. However the DGT method to detect monopoles only holds in the class of abelian projections defined above.

- For the sake of simplicity our presentation was based on $SU(2)$ gauge theory. However all we said holds for the monopoles of gauge theories with an arbitrary gauge group , with and without fermions [A.Di Giacomo, L.Lepori, F.Pucci, JHEP **10**, 2008, 096; C.Bonati, A. Di Giacomo, L.Lepori, F. Pucci, Phys. Rev. D **81**, 085022, (2010)] .