A Nonperturbative Calculation in Light-Front QED

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Bound – state problem

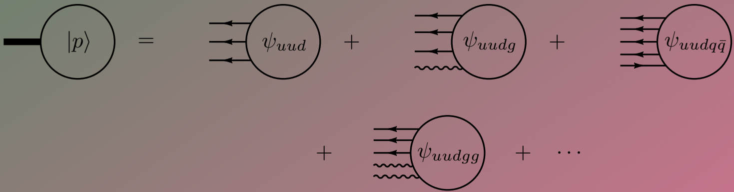
The purpose is to explore a nonperturbative method that can be used to solve for the bound states of quantum field theories, in particular QCD. The problem is notoriously difficult and there are only a few approaches.

- ✠ lattice gauge theory
- ✠ transverse lattice
- ✠ Dyson–Schwinger equations
- ✠ Bethe–Salpeter equation
- ✠ sector-dependent renormalization
- ✠ PV-regulated light-front Hamiltonian

Fock – state expansion

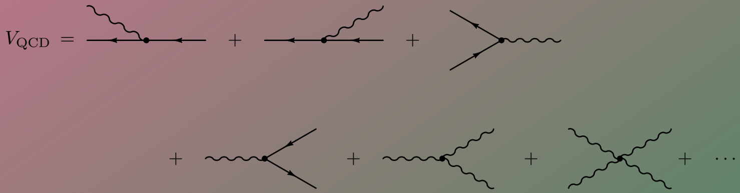
$$|p\rangle \;=\; \int \psi_{uud}|uud\rangle + \int \psi_{uudg}|uudg\rangle + \int \psi_{uudq\bar{q}}|uudq\bar{q}\rangle + \int \psi_{uudgg}|uudgg\rangle + \cdots$$

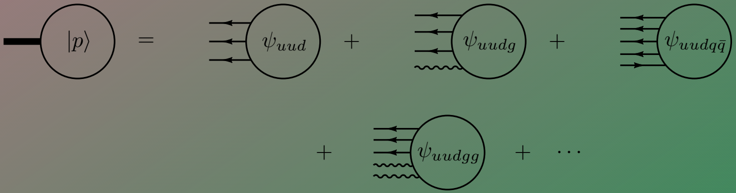
or graphically



Proton eigenvalue problem

$$H_{\mathrm{LC}}|p\rangle = \left(K + V_{\mathrm{QCD}}\right)|p\rangle = M^2|p\rangle$$



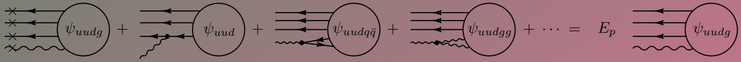


Coupled equations

$$\sum_i \frac{m_i^2 + k_i^2}{k_i^+/p^+}\psi_{uud} + \int d\underline{k}_g V_{qg\to q}(\underline{k}_g)\psi_{uudg}(\underline{k}_g) = M^2\psi_{uud}$$

$$\sum_i \frac{m_i^2 + k_i^2}{k_i^+/p^+}\psi_{uudg} \quad + V_{q\to qg}(\underline{k}_g)\psi_{uud} + \int d\underline{k}_q V_{q\bar{q}\to g}(\underline{k}_q)\psi_{uudq\bar{q}}(\underline{k}_q)$$

$$+ \int d\underline{k}_g V_{gg\to g}(\underline{k}_g)\psi_{uudgg}(\underline{k}_g) = M^2\psi_{uudg}$$



Finding wave functions

- convert coupled system of integral equations to matrix eigenvalue problem: $H\vec{c} = E_p\vec{c}$
by discretization: $p \to p_i$ and $\psi \to c_{ij\ldots} \equiv \psi(p_{1i}, p_{2j}, \ldots)$
- large matrix cannot be diagonalized by standard techniques $\to$ use iterative Lanczos process
- the eigenvector of the matrix yields the wave functions $\to$ from this can calculate any physical observable from an expectation value

History

Simple models with a heavy fermion which can emit and absorb bosons:
S.J. Brodsky, J.R. Hiller, and G. McCartor, Pauli–Villars as a nonperturbative ultraviolet regulator in discretized light-cone quantization, Phys. Rev. D **58**, 025005 (1998).

S.J. Brodsky, J.R. Hiller, and G. McCartor, Application of Pauli–Villars regularization and discretized light-cone quantization to a (3+1)-dimensional model, Phys. Rev. D **60**, 054506 (1999).

Yukawa theory (QED with scalar bosons as exchanged fields):
S.J. Brodsky, J.R. Hiller, and G. McCartor, Application of Pauli–Villars regularization and discretized light-cone quantization to a single-fermion truncation of Yukawa theory, Phys. Rev. D **64**, 114023 (2001).

S.J. Brodsky, J.R. Hiller, and G. McCartor, The mass renormalization of nonperturbative light-front Hamiltonian theory: An illustration using truncated, Pauli–Villars-regulated Yukawa interactions, Ann. Phys. **305**, 266 (2003).

S.J. Brodsky, J.R. Hiller, and G. McCartor, Two-boson truncation of Pauli–Villars-regulated Yukawa theory, Ann. Phys. **321**, 1240 (2006).

History

Equal-mass PV particles yields unphysical limit but check for numerical calculation; also, shows complexity of equal-time states that correspond to simple light-front Fock states:
S.J. Brodsky, J.R. Hiller, and G. McCartor, Exact solutions to Pauli–Villars-regulated field theories, Ann. Phys. **296**, 406 (2002).

First application to QED; one-photon truncation:
S.J. Brodsky, V.A. Franke, J.R. Hiller, G. McCartor, S.A. Paston, and E.V. Prokhvatilov, A nonperturbative calculation of the electron's magnetic moment, Nucl. Phys. B **703**, 333 (2004).

Importance of preservation of symmetries:
S.S. Chabysheva and J.R. Hiller, Restoration of the chiral limit in Pauli–Villars-regulated light-front QED, Phys. Rev. D **79**, 114017 (2009).

Contribution of zero-momentum modes to eigenstates:
S.S. Chabysheva and J.R. Hiller, Zero momentum modes in discrete light-cone quantization, Phys. Rev. D **79**, 096012 (2009).

History

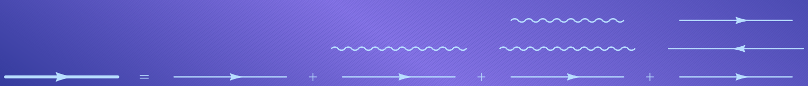
Truncation extended to two photons:
S.S. Chabysheva and J.R. Hiller, A nonperturbative calculation of the electron's magnetic moment with truncation extended to two photons, to appear in Phys. Rev. D, April 2010, arXiv:0911.4455[hep-ph].

Comparison of two parameterizations:
S.S. Chabysheva and **J.R. Hiller**, On the nonperturbative solution of Pauli–Villars-regulated light-front QED: A comparison of the sector-dependent and standard parameterizations, submitted to Annals of Physics, 2010, arXiv:0911.3686[hep-ph].

Electron Fock – state expansion

$$|\text{electron}\rangle = \int \psi_e |e\rangle + \int \psi_{e\gamma} |e\gamma\rangle + \int \psi_{e\gamma\gamma} |e\gamma\gamma\rangle$$
$$+ \int \psi_{eee^+} |ee^+\rangle + \cdots$$

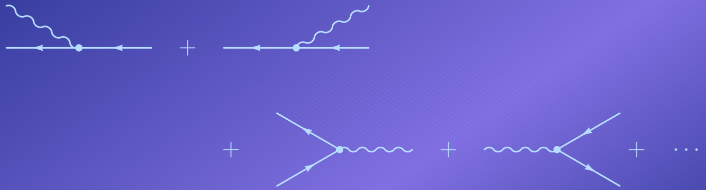
or graphically

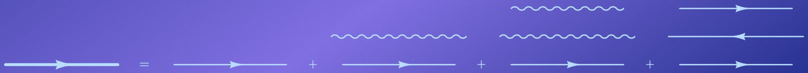


Electron eigenvalue problem

$$H_{\mathrm{LC}}|\mathrm{electron}\rangle = \left(K + V_{\mathrm{QED}}\right)|\mathrm{electron}\rangle = M^2|\mathrm{electron}\rangle$$

$V_{\mathrm{QED}} =$



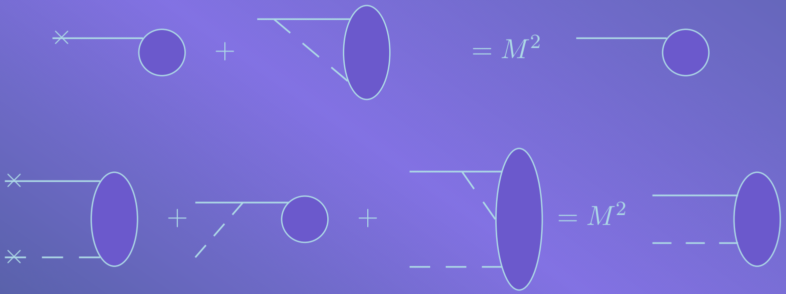


Coupled equations

$$m_0^2\psi_e + \int d\underline{k}_\gamma V_{e\gamma\to e}(\underline{k}_\gamma)\psi_{e\gamma}(\underline{k}_\gamma) = M^2\psi_e$$

$$\sum_i \frac{m_i^2 + k_i^2}{k_i^+/p^+}\psi_{e\gamma} \qquad + V_{e\to e\gamma}(\underline{k}_\gamma)\psi_e$$

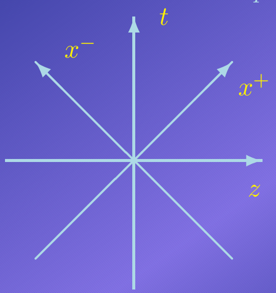
$$+\int d\underline{k}_\gamma V_{e\gamma\to e}(\underline{k}_\gamma)\psi_{e\gamma\gamma}(\underline{k}_\gamma) = M^2\psi_{e\gamma}$$



Light – cone coordinates

Chosen in order to have well-defined Fock-state expansions and a simple vacuum ($p^+ \equiv \sqrt{m^2 + p_z^2 + p_\perp^2} + p_z > 0$)

- time: $x^+ = t + z$
- space: $\underline{x} = (x^-, \vec{x}_\perp)$, $x^- \equiv t - z$, $\vec{x}_\perp = (x, y)$
- energy: $p^- = E - p_z$
- momentum: $\underline{p} = (p^+, \vec{p}_\perp)$, $p^+ \equiv E + p_z$, $\vec{p}_\perp = (p_x, p_y)$
- mass-shell condition: $p^2 = m^2 \quad \Rightarrow \quad p^- = \frac{m^2 + p_\perp^2}{p^+}$



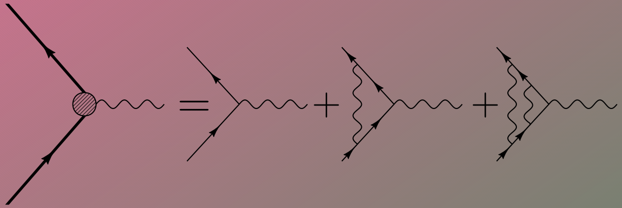
Pauli – Villars regularization

The basic idea is to subtract from each integral a contribution of the same form but of a PV particle with a much larger mass. This can be done by adding negative metric particles to the Lagrangian. For example, for free scalars

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2}\mu_0^2\phi_0^2\right] - \left[\frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}\mu_1^2\phi_1^2\right]$$

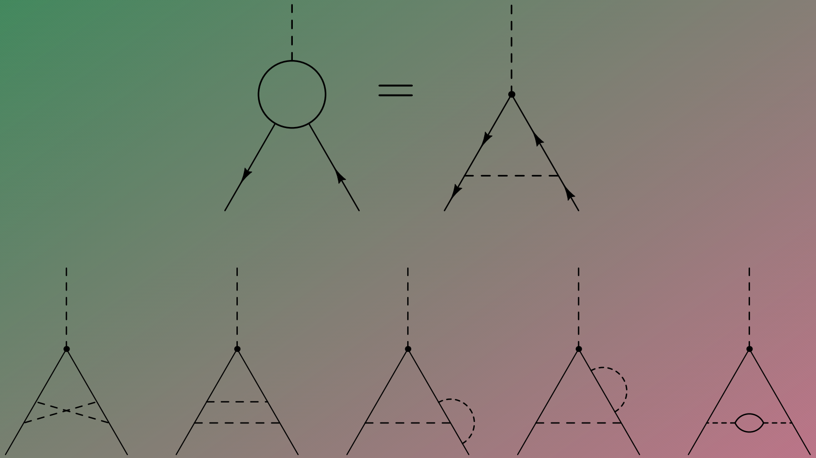
$$\longrightarrow \int \left[\frac{1}{p^2 - \mu_0^2} - \frac{1}{p^2 - \mu_1^2}\right] d^4p$$

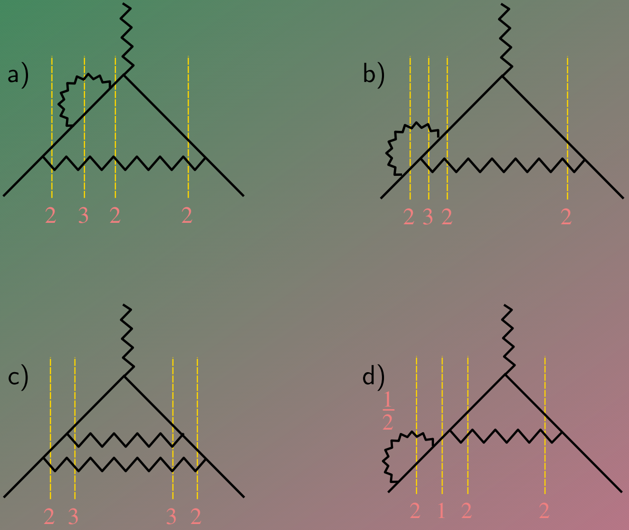
PV regularization is automatically relativistically covariant.



Theoretical prediction, computed perturbatively up to order α^4, is

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} - (0.328\ 478\ 965\ \ldots) \times \left(\frac{\alpha}{\pi}\right)^2 + (1.176\ 11\ \ldots) \times \left(\frac{\alpha}{\pi}\right)^3 - (1.434\ \ldots) \times \left(\frac{\alpha}{\pi}\right)^4 = 0.001\ 159\ 652\ 140\ \ldots$$




a)
2 3 2 2
b)
2 3 2 2
c)
2 3 3 2
d)
1/2
2 1 2 2

Light – front QED in Feynman gauge

The Feynman-gauge QED Lagrangian, regulated by two PV photons and one PV electron, is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\sum_{i=0}^{2}(-1)^i F_i^{\mu\nu}F_{i,\mu\nu} - \frac{1}{2}\sum_{i=0}^{2}(-1)^i\left(\partial^\mu A_{i\mu}\right)^2 \\ &\quad + \sum_{i=0}^{1}(-1)^i\bar{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - e\bar{\psi}\gamma^\mu\psi A_\mu,\end{aligned}$$

where A^μ and ψ are zero-norm fields:

$$A_\mu = \sum_{i=0}^{2}\sqrt{\xi_i}A_{i\mu}, \quad \psi = \sum_{i=0}^{1}\psi_i, \quad F_{i\mu\nu} = \partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}.$$

Coupling constraints

The ξ_l must satisfy constraints:

- physical charge, $\xi_0 = 1$
- cancellation of the log divergence, $\sum_{l=0}^{2}(-1)^l \xi_l = 0$
- correct chiral limit,

$$\sum_{l=0}^{2}(-1)^l \xi_l \frac{\mu_l^2/m_1^2}{1-\mu_l^2/m_1^2} \ln(\mu_l^2/m_1^2) = 0$$

The second constraint guarantees a zero norm for the sum of the boson fields.

The third constraint is trivially satisfied in the limit of infinite PV mass m_1.

Dynamical fields

$$
\begin{aligned}
\psi_{i+} &= \frac{1}{\sqrt{16\pi^3}} \sum_s \int d\underline{k}\chi_s \left[b_{is}(\underline{k}) e^{-i\underline{k}\cdot\underline{x}} + d^\dagger_{i,-s}(\underline{k}) e^{i\underline{k}\cdot\underline{x}} \right] , \\
A_{i\mu} &= \frac{1}{\sqrt{16\pi^3}} \int \frac{d\underline{k}}{\sqrt{k^+}} \left[a_{i\mu}(\underline{k}) e^{-i\underline{k}\cdot\underline{x}} + a^\dagger_{i\mu}(\underline{k}) e^{i\underline{k}\cdot\underline{x}} \right] ,
\end{aligned}
$$

The creation and annihilation operators satisfy (anti)commutation relations

$$
\begin{aligned}
\{b_{is}(\underline{k}), b^\dagger_{i's'}(\underline{k}'\} &= (-1)^i \delta_{ii'}\delta_{ss'}\delta(\underline{k}-\underline{k}'), \\
\{d_{is}(\underline{k}), d^\dagger_{i's'}(\underline{k}'\} &= (-1)^i \delta_{ii'}\delta_{ss'}\delta(\underline{k}-\underline{k}'), \\
[a_{i\mu}(\underline{k}), a^\dagger_{i'\nu}(\underline{k}'] &= (-1)^i \delta_{ii'}\epsilon^\mu\delta_{\mu\nu}\delta(\underline{k}-\underline{k}'), \\
[a_\mu(\underline{k}), a^\dagger_\nu(\underline{k}')] &= \left[\sum_i (-1)^i \xi_i \right] \epsilon^\mu \delta_{\mu\nu}\delta(\underline{k}-\underline{k}') = 0.
\end{aligned}
$$

with $\epsilon^\mu = (-1,1,1,1)$

Use of Feynman gauge

The coupling of the two zero-norm fields A^μ and ψ as the interaction term reduces the fermionic constraint equation to a solvable equation without forcing the gauge field $A_- = A^+$ to zero.

$$i(-1)^i\partial_-\psi_{i-} + eA_-\sum_j \psi_{j-}$$

$$= (i\gamma^0\gamma^\perp)\left[(-1)^i\partial_\perp\psi_{i+} - ieA_\perp\sum_j\psi_{j+}\right] - (-1)^i m\gamma^0\psi_{i+}.$$

For the null combination $\psi_0+\psi_1$ that couples to A^+, the constraint reduces to

$$i\partial_-(\psi_{0-}+\psi_{1-}) = (i\gamma^0\gamma^\perp)\partial_\perp(\psi_{0+}+\psi_{1+}) - m\gamma^0(\psi_{0+}+\psi_{1+}).$$

QED Hamiltonian

Without antifermion terms, the Hamiltonian is

$$\mathcal{P}^- = \sum_{i,s}\int d\underline{p}\frac{m_i^2+p_\perp^2}{p^+}(-1)^i b_{i,s}^\dagger(\underline{p})b_{i,s}(\underline{p})$$

$$+\sum_{l,\mu}\int d\underline{k}\frac{\mu_l^2+k_\perp^2}{k^+}(-1)^l\epsilon^\mu a_{l\mu}^\dagger(\underline{k})a_{l\mu}(\underline{k})$$

$$+\sum_{i,j,l,s,\mu}\int d\underline{p}d\underline{q}\left\{b_{i,s}^\dagger(\underline{p})\left[b_{j,s}(\underline{q})V_{ij,2s}^\mu(\underline{p},\underline{q})\right.\right.$$

$$\left.\left.+b_{j,-s}(\underline{q})U_{ij,-2s}^\mu(\underline{p},\underline{q})\right]\sqrt{\xi_l}a_{l\mu}^\dagger(\underline{q}-\underline{p})+h.c.\right\},$$

Note absence of instantaneous fermion contributions.

Vertex functions

$$V^0_{ij\pm}(\underline{p},\underline{q}) = \frac{e}{\sqrt{16\pi^3}} \frac{\vec{p}_\perp \cdot \vec{q}_\perp \pm i\vec{p}_\perp \times \vec{q}_\perp + m_i m_j + p^+ q^+}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$V^3_{ij\pm}(\underline{p},\underline{q}) = \frac{-e}{\sqrt{16\pi^3}} \frac{\vec{p}_\perp \cdot \vec{q}_\perp \pm i\vec{p}_\perp \times \vec{q}_\perp + m_i m_j - p^+ q^+}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$V^1_{ij\pm}(\underline{p},\underline{q}) = \frac{e}{\sqrt{16\pi^3}} \frac{p^+(q^1 \pm iq^2) + q^+(p^1 \mp ip^2)}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$V^2_{ij\pm}(\underline{p},\underline{q}) = \frac{e}{\sqrt{16\pi^3}} \frac{p^+(q^2 \mp iq^1) + q^+(p^2 \pm ip^1)}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$U^0_{ij\pm}(\underline{p},\underline{q}) = \frac{\mp e}{\sqrt{16\pi^3}} \frac{m_j(p^1 \pm ip^2) - m_i(q^1 \pm iq^2)}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$U^3_{ij\pm}(\underline{p},\underline{q}) = \frac{\pm e}{\sqrt{16\pi^3}} \frac{m_j(p^1 \pm ip^2) - m_i(q^1 \pm iq^2)}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$U^1_{ij\pm}(\underline{p},\underline{q}) = \frac{\pm e}{\sqrt{16\pi^3}} \frac{m_i q^+ - m_j p^+}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$U^2_{ij\pm}(\underline{p},\underline{q}) = \frac{ie}{\sqrt{16\pi^3}} \frac{m_i q^+ - m_j p^+}{p^+ q^+ \sqrt{q^+ - p^+}}.$$

One – Photon Truncation

Can solve analytically [NPB **703**, 333 (2004)].
The dressed-electron state with total $J_z = \pm\frac{1}{2}$ is

$$|\psi^\pm(\underline{P})\rangle = \sum_i z_i b^\dagger_{i\pm}(\underline{P})|0\rangle + \sum_{ijs\mu}\int d\underline{k} C^{\mu\pm}_{ijs}(\underline{k}) b^\dagger_{is}(\underline{P}-\underline{k}) a^\dagger_{j\mu}(\underline{k})|0\rangle,$$

Project onto the physical subspace by expressing Fock states in terms of positively normed creation operators and the null combinations $a^\dagger_\mu = \sum_i \sqrt{\xi_i} a^\dagger_{i\mu}$ and $b^\dagger_s = b^\dagger_{0s} + b^\dagger_{1s}$ that are dropped:

$$\begin{aligned}|\psi^\pm_{\rm phys}(\underline{P})\rangle = {} & \textstyle\sum_i (-1)^i z_i b^\dagger_{0\pm}(\underline{P})|0\rangle \\ & + \textstyle\sum_{s\mu}\int d\underline{k} \sum_{i=0}^1 \sum_{j=0,2}\sqrt{\xi_j} \\ & \times \textstyle\sum_{k=j/2}^{j/2+1} \frac{(-1)^{i+k}}{\sqrt{\xi_k}} C^{\mu\pm}_{iks}(\underline{k}) b^\dagger_{0s}(\underline{P}-\underline{k}) a^\dagger_{j\mu}(\underline{k})|0\rangle\end{aligned}$$

Coupled integral equations

From $H_{\rm LC}|\psi\rangle = M^2|\psi\rangle$ we have

$$[M^2 - m_i^2]z_i = \int P^+ dy d^2k_\perp \sum_{j,l,\mu} \sqrt{\xi_l}(-1)^{j+l}\epsilon^\mu P^+ \\ \times \left[V_{ji+}^{\mu *}(\underline{P}-\underline{k},\underline{P})C_{jl+}^{\mu +}(\underline{k}) + U_{ji+}^{\mu *}(\underline{P}-\underline{k},\underline{P})C_{jl-}^{\mu +}(\underline{k})\right],$$

$$\left[M^2 - \frac{m_i^2+k_\perp^2}{(1-y)} - \frac{\mu_l^2+k_\perp^2}{y}\right] C_{il\pm}^{\mu\pm}(\underline{k}) \\ = \sqrt{\xi_l}\sum_j (-1)^j z_j P^+ V_{ij\pm}^{\mu}(\underline{P}-\underline{k},\underline{P}),$$

$$\left[M^2 - \frac{m_i^2+k_\perp^2}{(1-y)} - \frac{\mu_l^2+k_\perp^2}{y}\right] C_{il\mp}^{\mu\pm}(\underline{k}) \\ = \sqrt{\xi_l}\sum_j (-1)^j z_j P^+ U_{ij\pm}^{\mu}(\underline{P}-\underline{k},\underline{P}).$$

Reduction to one – electron sector

$$(M^2 - m_i^2) z_i = 2e^2 \sum_{i'} (-1)^{i'} z_{i'} \left[\bar{J} + m_i m_{i'} \bar{I}_0 - 2(m_i + m_{i'}) \bar{I}_1 \right],$$

with

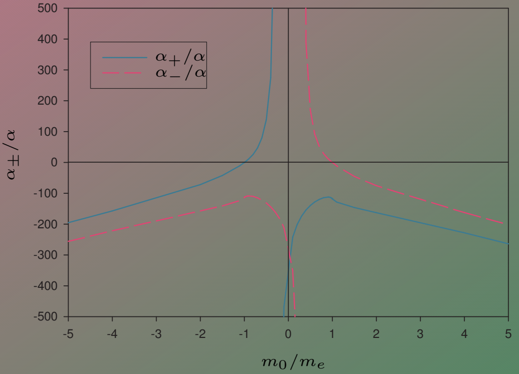
$$\begin{aligned} \bar{I}_n(M^2) &= \int \frac{dy dk_\perp^2}{16\pi^2} \sum_{jl} \frac{(-1)^{j+l} \xi_l}{M^2 - \frac{m_j^2 + k_\perp^2}{1-y} - \frac{\mu_l^2 + k_\perp^2}{y}} \frac{m_j^n}{y(1-y)^n}, \\ \bar{J}(M^2) &= \int \frac{dy dk_\perp^2}{16\pi^2} \sum_{jl} \frac{(-1)^{j+l} \xi_l}{M^2 - \frac{m_j^2 + k_\perp^2}{1-y} - \frac{\mu_l^2 + k_\perp^2}{y}} \frac{m_j^2 + k_\perp^2}{y(1-y)^2}. \end{aligned}$$

The integrals $\bar{I}_0$ and $\bar{J}$ satisfy an identity, $\bar{J} = M^2 \bar{I}_0$.

Solution of the Eigenvalue Problem

$$\alpha_\pm = \frac{(M \pm m_0)(M \pm m_1)}{8\pi(m_1 - m_0)(2\bar{I}_1 \pm M\bar{I}_0)}, \quad z_1 = \frac{M \pm m_0}{M \pm m_1} z_0$$

Require $\alpha_\pm$ to be equal to the physical value of α to fix m_0.

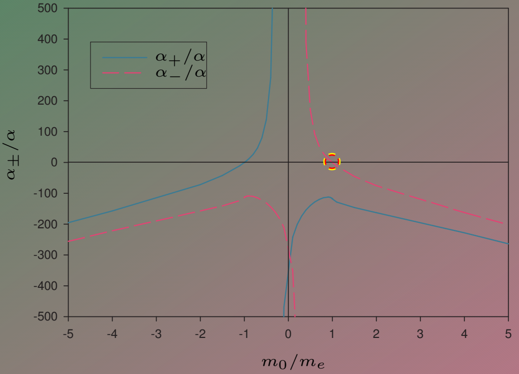


$m_1 = 1000 m_e$, $\mu_1 = 10 m_e$, $\mu_2 = \infty$.

Solution of the Eigenvalue Problem

$$\alpha_\pm = \frac{(M \pm m_0)(M \pm m_1)}{8\pi(m_1 - m_0)(2\bar{I}_1 \pm M\bar{I}_0)}, \quad z_1 = \frac{M \pm m_0}{M \pm m_1} z_0$$

Require $\alpha_\pm$ to be equal to the physical value of α to fix m_0.



$m_1 = 1000 m_e$, $\mu_1 = 10 m_e$, $\mu_2 = \infty$.

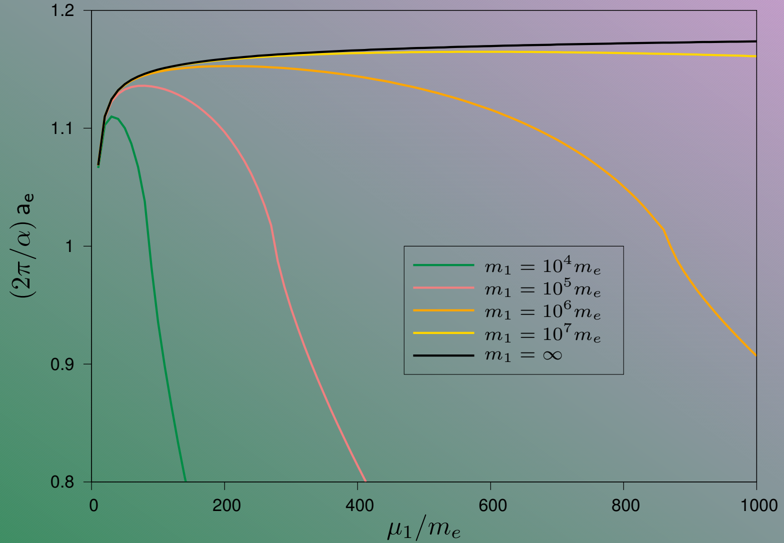
Anomalous Magnetic Moment

From the Brodsky–Drell formula [PRD **22**, 2236 (1980)] for the spin-flip matrix element of the electromagnetic current

$$a_e = m_e \sum_{s\mu} \int d\underline{k} \epsilon^\mu \sum_{j=0,2} \xi_j \\ \times \left(\sum_{i'=0}^{1} \sum_{k'=j/2}^{j/2+1} \frac{(-1)^{i'+k'}}{\sqrt{\xi_{k'}}} C^{\mu+}_{i'k's}(\underline{k}) \right)^* \\ \times y \left(\frac{\partial}{\partial k_x} + i \frac{\partial}{\partial k_y} \right) \left(\sum_{i=0}^{1} \sum_{k=j/2}^{j/2+1} \frac{(-1)^{i+k}}{\sqrt{\xi_k}} C^{\mu-}_{iks}(\underline{k}) \right).$$

$$= \frac{\alpha}{\pi} m_e \int y^2(1-y) dy dk_\perp^2 \sum_{l,l'} (-1)^{l+l'} z_l z_{l'} m_l \sum_{j=0,2} \xi_j \\ \times \left(\sum_{i=0}^{1} \sum_{k=j/2}^{j/2+1} \frac{(-1)^{i+k}}{y m_i^2 + (1-y)\mu_k^2 + k_\perp^2 - m_e^2 y(1-y)} \right)^2.$$

The anomalous moment of the electron in units of the Schwinger term $(\alpha/2\pi)$ plotted versus the PV photon mass, μ_1, for a few values of the PV electron mass, m_1, with the second PV photon mass, μ_2, set to infinity



The strong dependence on m_1, in the $\mu_2 \to \infty$ limit, arises from the sensitivity of the anomalous moment integral to the masses of the constituents, in particular the bare electron mass m_0. The leading finite-m_1 correction to the bare electron mass is of the form $\frac{\mu_1^2 \ln(\mu_1/m_1)}{8\pi^2 m_e m_1}$. This requires m_1 to be much larger than μ_1^2/m_e. Such behavior comes from the contribution of $\tilde{I}_1$ to the relationship between m_0 and α in the one-photon truncated solution.

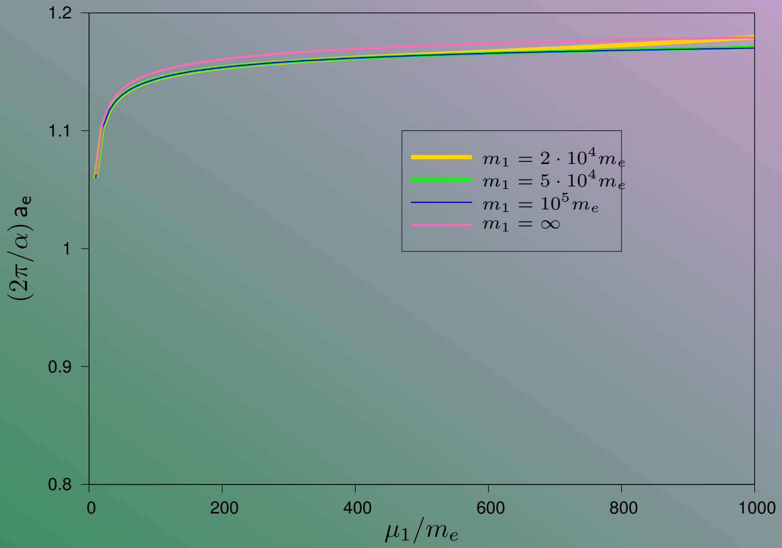
$$\tilde{I}_1 \simeq \tilde{I}_1(m_1 = \infty) - \frac{\mu_1^2 \ln(\mu_1/m_1)}{8\pi^2 m_e m_1}.$$

If we now add the second PV photon, we obtain

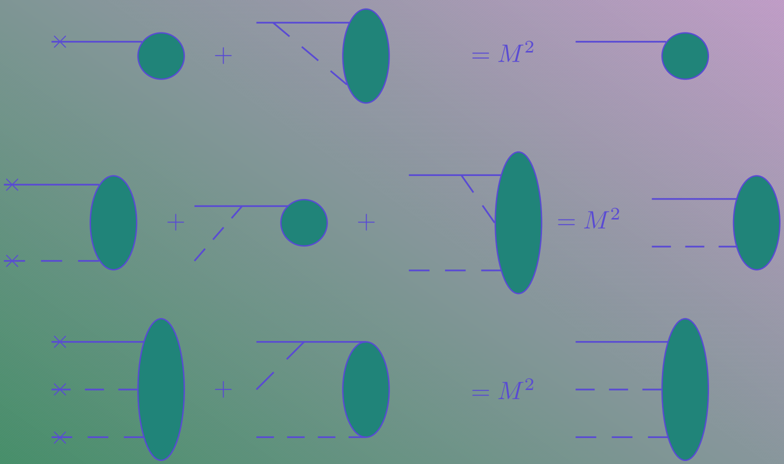
$$\tilde{I}_1 \simeq \tilde{I}_1(m_1 = \infty) + \sum_{j=1}^{2} \xi_j (-1)^j \frac{\mu_j^2 \ln(\mu_j/m_1)}{8\pi^2 m_e m_1}.$$

We then choose ξ_j such that the second term is zero.

The anomalous moment of the electron in units of the Schwinger term $(\alpha/2\,\pi)$ plotted versus the PV photon mass, μ_1 for $\mu_2 = \sqrt{2}\mu_1$. This ratio held fixed as μ_1 and μ_2 are varied



$$P^+P^-|\Phi_+\rangle = M^2|\Phi_+\rangle$$



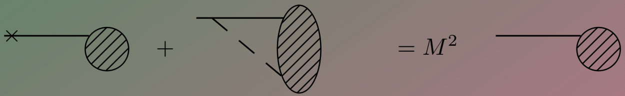
Three – body wave function

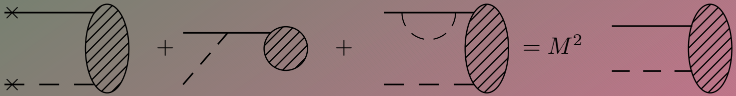
The equation in the three-body sector can be solved for the three-body wave functions in terms of the two-body wave functions.

$$
\begin{aligned}
C^{\mu\nu\pm}_{ijls}(\underline{q}_1,\underline{q}_2) = {} & \frac{1}{M^2 - \frac{m_i^2+(\vec{q}_{1\perp}+\vec{q}_{2\perp})^2}{(1-y_1-y_2)} - \frac{\mu_j^2+q_{1\perp}^2}{y_1} - \frac{\mu_l^2+q_{2\perp}^2}{y_2}} \frac{\sqrt{1+\delta_{jl}\delta^{\mu\nu}}}{2} \\
& \times \sum_a (-1)^a \left\{ \sqrt{\xi_j} \left[V^{\mu}_{ias}(\underline{P}-\underline{q}_1-\underline{q}_2, \underline{P}-\underline{q}_2) C^{\nu\pm}_{als}(\underline{q}_2) \right.\right. \\
& \qquad \left. + U^{\mu}_{ia,-s}(\underline{P}-\underline{q}_1-\underline{q}_2, \underline{P}-\underline{q}_2) C^{\nu\pm}_{al,-s}(\underline{q}_2) \right] \\
& \qquad + \sqrt{\xi_l} \left[V^{\nu}_{ias}(\underline{P}-\underline{q}_1-\underline{q}_2, \underline{P}-\underline{q}_1) C^{\mu\pm}_{ajs}(\underline{q}_1) \right. \\
& \qquad \left.\left. + U^{\nu}_{ia,-s}(\underline{P}-\underline{q}_1-\underline{q}_2, \underline{P}-\underline{q}_1) C^{\mu\pm}_{aj,-s}(\underline{q}_1) \right] \right\}.
\end{aligned}
$$

Self – energy contribution

Substitution of this solution into the two-body equation eliminates the three-body wave functions. Retain only the self-energy contributions, where the emitted photon is immediately reabsorbed by the electron, and omit the remaining two-photon contributions, where one photon is emitted and the other absorbed.





Self − energy eigenvalue problem

$$\left[M^2 - \frac{m_i^2 + q_\perp^2}{1-y} - \frac{\mu_j^2 + q_\perp^2}{y}\right] C_{ijs}^{\mu\pm}(y, q_\perp)$$

$$= S_{ijs}^{\mu\pm} + \frac{\alpha}{2\pi}\sum_{i'} \frac{I_{iji'}(y, q_\perp)}{1-y} C_{i'js}^{\mu\pm}(y, q_\perp),$$

$$S_{ils}^{\mu\pm} = \sqrt{\xi_l}\sum_j (-1)^j z_j P^+ [\delta_{s,\pm 1/2} V_{ijs}^{\mu}(\underline{P} - \underline{q}, \underline{P}) + \delta_{s,\mp 1/2} U_{ij,-s}^{\mu}(\underline{P} - \underline{q}, \underline{P})]$$

$$[M^2 - m_i^2] z_i \;=\; \int d\underline{q} \sum_{j,l,\mu} \sqrt{\xi_l}(-1)^{j+l}\epsilon^\mu P^+ \Big[V_{ji\pm}^{\mu *}(\underline{P} - \underline{q}, \underline{P}) C_{jl\pm}^{\mu\pm}(\underline{q})$$

$$+ U_{ji\pm}^{\mu *}(\underline{P} - \underline{q}, \underline{P}) C_{jl\mp}^{\mu\pm}(\underline{q}) \Big]$$

Two – body wave functions

The two-body equation can be expressed compactly as

$$\begin{aligned} A_{0j} C^{\mu\pm}_{0js} - B_j C^{\mu\pm}_{1js} &= -S^{\mu\pm}_{0js} \\ B_j C^{\mu\pm}_{0js} + A_{1j} C^{\mu\pm}_{1js} &= -S^{\mu\pm}_{1js}, \end{aligned}$$

with the solution

$$C^{\mu\pm}_{ijs} = -\frac{A_{1-i,j} S^{\mu\pm}_{ijs} + (-1)^i B_j S^{\mu\pm}_{1-i,js}}{A_{0j} A_{1j} + B_j^2}$$

and the definitions

$$A_{ij} = \tfrac{m_i^2 + q_\perp^2}{1-y} + \tfrac{\mu_j^2 + q_\perp^2}{y} + \tfrac{\alpha}{2\pi} \tfrac{I_{iji}}{1-y} - M^2 \ ,$$
$$B_j = \tfrac{\alpha}{2\pi} \tfrac{I_{1j0}}{1-y} = -\tfrac{\alpha}{2\pi} \tfrac{I_{0j1}}{1-y}.$$

One − body equation

Substitute the expressions for the two-body wave functions into the one-body equation

$$[M^2 - m_i^2]z_i = 2e^2 \sum_l (-1)^l z_l [m_i m_l \tilde{I}_0 - 2(m_i + m_l)\tilde{I}_1 + \tilde{J}],$$

where

$$\tilde{I}_0 = \int \frac{dy dq_\perp^2}{16\pi^2} \sum_j (-1)^j \xi_j \frac{A_{0j} - A_{1j} - 2B_j}{y[A_{0j}A_{1j} + B_j^2]},$$

$$\tilde{I}_1 = \int \frac{dy dq_\perp^2}{16\pi^2} \sum_j (-1)^j \xi_j \frac{m_1 A_{0j} - m_0 A_{1j} - (m_0 + m_1) B_j}{y(1-y)[A_{0j}A_{1j} + B_j^2]},$$

$$\tilde{J} = \int \frac{dy dq_\perp^2}{16\pi^2} \sum_j (-1)^j \xi_j \frac{(m_1^2 + q_\perp^2) A_{0j} - (m_0^2 + q_\perp^2) A_{1j} - 2(m_0 m_1 + q_\perp^2) B_j}{y(1-y)^2[A_{0j}A_{1j} + B_j^2]} .$$

2×2 eigenvalue problem

$$G\vec{z} = \frac{1}{2e^2}\vec{z}, \quad \vec{z} = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \quad G = \begin{pmatrix} G_{00} & G_{01} \\ G_{10} & G_{11} \end{pmatrix},$$

with $G_{il} = \frac{(-1)^l}{M^2 - m_i^2}[m_i m_l \tilde{I}_0 - 2(m_i + m_l)\tilde{I}_1 + \tilde{J}]$

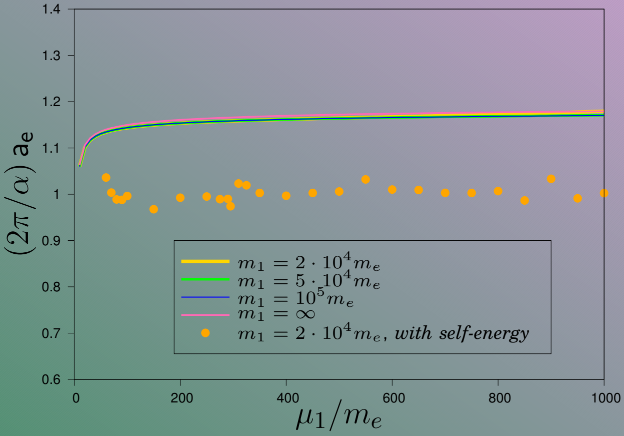
and solution

$$\alpha_\pm = \frac{G_{00} + G_{11} \pm \sqrt{(G_{00} - G_{11})^2 - 4G_{10}G_{01}}}{16\pi[G_{00}G_{11} - G_{10}G_{01}]},$$

$$\frac{z_1}{z_0} = \frac{[G_{11} - G_{00}]/2 \mp \sqrt{(G_{00} - G_{11})^2 - 4G_{10}G_{01}}}{G_{01}}.$$

Yields α as a function of m_0 and the PV masses. Then find m_0 such that α takes the physical value.

Self – energy effect



One-photon result $\longrightarrow$ differs by $\sim 17\%$.

With self-energy $\longrightarrow$ consistent with perturbative QED.

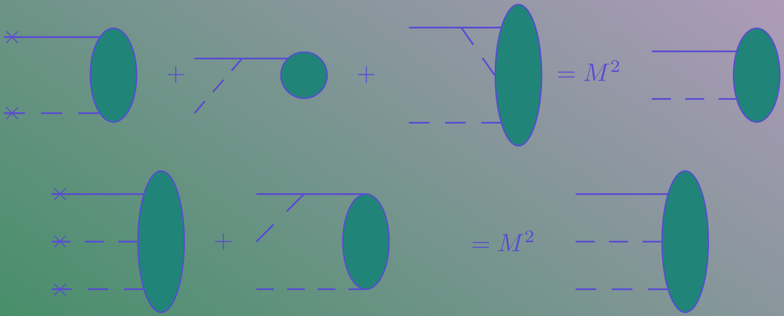
Results

That the self-energy contribution brings the result so close to the leading Schwinger contribution can be understood.

The denominator of the integral that yields the anomalous moment contains m_0^2 plus the self-energy correction, which for the dominant contribution near zero photon momentum becomes just m_e^2.

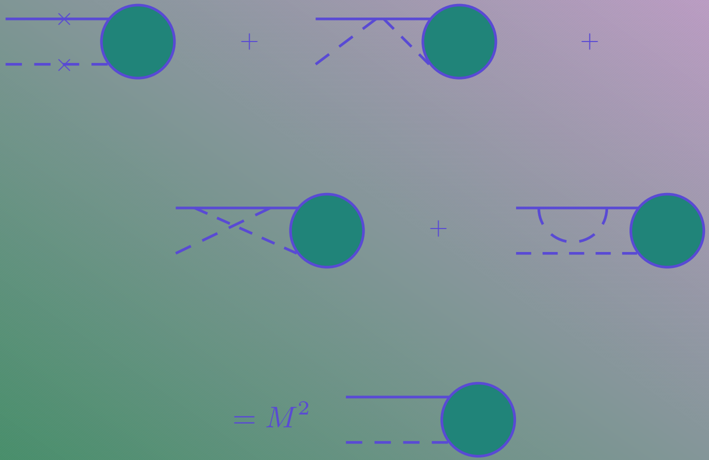
This is the mass that appears in the Schwinger expression.

$$P^+P^-|\Phi_+\rangle = M^2|\Phi_+\rangle$$



THE EQUATION FOR TWO-PARTICLE AMPLITUDES ONLY

48 coupled integral equations



Equations for two − body only

$$\left[M^2 - \frac{m_i^2 + q_\perp^2}{1-y} - \frac{\mu_j^2 + q_\perp^2}{y}\right] C_{ijs}^{\mu\pm}(y, q_\perp)$$
$$= \frac{\alpha}{2\pi} \sum_{i'} \frac{I_{iji'}(y, q_\perp)}{1-y} C_{i'js}^{\mu\pm}(y, q_\perp)$$
$$+ \frac{\alpha}{2\pi} \sum_{i'j's'\nu} \epsilon^\nu \int_0^1 dy' dq_\perp'^2 J_{ijs,i'j's'}^{(0)\mu\nu}(y, q_\perp; y', q_\perp') C_{i'j's'}^{\nu\pm}(y', q_\perp')$$
$$+ \frac{\alpha}{2\pi} \sum_{i'j's'\nu} \epsilon^\nu \int_0^{1-y} dy' dq_\perp'^2 J_{ijs,i'j's'}^{(2)\mu\nu}(y, q_\perp; y', q_\perp') C_{i'j's'}^{\nu\pm}(y', q_\perp').$$

Total of 48 coupled equations,
with $i = 0, 1;\quad j = 0, 1, 2;\quad s = \pm\frac{1}{2};$ and $\mu = \pm, (\pm)$.

Fermion flavor mixing

Flavor changing self-energies leads naturally to a fermion flavor mixing of the two-body wave functions.

$$\begin{aligned} A_{0j}C^{\mu\pm}_{0js} - B_j C^{\mu\pm}_{1js} &= -\frac{\alpha}{2\pi} J^{\mu\pm}_{0js}, \\ B_j C^{\mu\pm}_{0js} + A_{1j} C^{\mu\pm}_{1js} &= -\frac{\alpha}{2\pi} J^{\mu\pm}_{1js}, \end{aligned}$$

where A_{ij} and B_j are defined as before and

$$\begin{aligned} J^{\mu\pm}_{ijs} &= \sum_{i'j's'\nu} \epsilon^{\nu} \int_0^1 dy' dq'^2_{\perp} J^{(0)\mu\nu}_{ijs,i'j's'}(y, q_{\perp}; y', q'_{\perp}) C^{\nu\pm}_{i'j's'}(y', q'_{\perp}) \\ &\quad + \sum_{i'j's'\nu} \epsilon^{\nu} \int_0^{1-y} dy' dq'^2_{\perp} J^{(2)\mu\nu}_{ijs,i'j's'}(y, q_{\perp}; y', q'_{\perp}) C^{\nu\pm}_{i'j's'}(y', q'_{\perp}). \end{aligned}$$

Mixed – flavor wave functions

The wave functions that diagonalize the left-hand side are

$$\tilde{f}_{ijs}^{\mu\pm} = A_{ij} C_{ijs}^{\mu\pm} + (-1)^i B_j C_{1-i,js}^{\mu\pm}.$$

The eigenvalue problem is

$$J_{ijs}^{\mu\pm}[\tilde{f}] = -\frac{2\pi}{\alpha} \tilde{f}_{ijs}^{\mu\pm}.$$

$J_{ijs}^{\mu\pm}$ is implicitly a functional of these new wave functions.

The original wave functions are recovered as

$$C_{ijs}^{\mu\pm} = \frac{A_{1-i,js} \tilde{f}_{ijs}^{\mu\pm} + (-1)^i B_j \tilde{f}_{1-i,js}^{\mu\pm}}{A_{0j} A_{1j} + B_j^2}.$$

Matrix form

$$\begin{pmatrix} J^{\mu\pm}_{0js} \\ J^{\mu\pm}_{1js} \end{pmatrix} = \int dy' dq'^2_\perp \sum_{j's'\nu} (-1)^{j'} \epsilon^\nu \begin{pmatrix} J^{\mu\nu}_{0js,0j's'} & J^{\mu\nu}_{0js,1j's'} \\ J^{\mu\nu}_{1js,0j's'} & J^{\mu\nu}_{1js,1j's'} \end{pmatrix}$$

$$\times \begin{pmatrix} A_{1j'} & B_{j'} \\ B_{j'} & -A_{0j'} \end{pmatrix} \begin{pmatrix} \tilde{f}^{\nu\pm}_{0j's'} \\ \tilde{f}^{\nu\pm}_{1j's'} \end{pmatrix}$$

where

$$J^{\mu\pm}_{ijs} = \int dy' dq'^2_\perp \sum_{i'j's'\nu} (-1)^{i'+j'} \epsilon^\nu J^{\mu\nu}_{ijs,i'j's'}(y, q_\perp; y', q'_\perp) C^{\nu\pm}_{i'j's'}(y', q'_\perp)$$

and

$$J^{\mu\nu}_{ijs,i'j's'} = J^{(0)\mu\nu}_{ijs,i'j's'} + J^{(2)\mu\nu}_{ijs,i'j's'}.$$

Two − photon eigenvalue problem

$$\int dy' dq_\perp'^2 \sum_{i'j's'\alpha\beta i''} J^{\mu\alpha}_{ijs,i'j's'}(y, q_\perp; y', q_\perp') \eta_{j',\alpha\beta,i'i''} \tilde{f}^{\beta\pm}_{i''j's'} = -\frac{2\pi}{\alpha} \tilde{f}^{\mu\pm}_{ijs}.$$

$$\eta_{j',\alpha\beta} = (-1)^{j'} \lambda_{\alpha\beta} \left(\begin{array}{cc} A_{1j'} & B_{j'} \\ B_{j'} & -A_{0j'} \end{array} \right)$$

$$\lambda = \left(\begin{array}{cccc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right),$$

Again α is a function of m_0 and PV masses. Find iteratively a value for m_0 such that α takes its physical value.

Numerical Methods

- Discretization of integral equations through variable transformation and Gauss-Legendre quadrature
- Cubic-spline interpolation for approximation of wave function and its derivative
- Matrix eigenvalue problem is solved for the lowest physical state via the Lanczos diagonalization algorithm
- Nonlinear equations for the bare mass and quadrature poles are solved with use of the Müller algorithm

Numerical methods II

Convert integral equations to matrix eigenvalue problem by discretization. Solve matrix problem by Lanczos iteration:
$\vec{u}_n \to \vec{u}_{n+1}$ such that

$$H\vec{u}_n = b_{n-1}\vec{u}_{n-1} + a_n\vec{u}_n + b_n\vec{u}_{n+1},$$

$$H \to T \equiv \begin{pmatrix} a_1 & b_1 & 0 & \dots \\ b_1 & a_2 & b_2 & \dots \\ 0 & b_2 & a_3 & \dots \\ . & . & . & \dots \end{pmatrix}$$ and diagonalize T.

Search for m_0 with iterative Müller algorithm, which uses quadratic fit and interpolation for next guess.

Alternate method of solution

- The Lanczos algorithm is unstable in finding the correct eigenstate.
- Had worked well for Yukawa theory, but that was for strong coupling.
- For the weak coupling of QED, considered alternate method that takes advantage of the weak coupling.

Coupled equations

One-body amplitudes coupled to two-body wave functions

$$(M^2 - m_a^2) z_a/z_0 = \sqrt{\frac{\alpha}{2}} \sum_{i'j's'\alpha\beta i''} \int dy' dq_\perp'^2 V^{(0)\alpha *}_{i'j'as'} \eta_{j',\alpha\beta,i'i''} \tilde{f}^{\beta\pm}_{i''j's'}/z_0,$$

Two-body wave functions coupled to one-body amplitudes and self-coupled through two-photon intermediates

$$\begin{aligned} \tilde{f}^{\mu\pm}_{ijs}/z_0 &= -\sqrt{\frac{\alpha}{2\pi^2}} \sum_a (-1)^a V^{(0)\mu}_{ijas} z_a/z_0 \\ &\quad -\frac{\alpha}{2\pi} \int dy' dq_\perp'^2 \sum_{i'j's'\alpha\beta i''} J^{(2)\mu\alpha}_{ijs,i'j's'} \eta_{j',\alpha\beta,i'i''} \tilde{f}^{\beta\pm}_{i''j's'}/z_0, \end{aligned}$$

Iterative solution

Update bare mass and one-body amplitude ratio

$$m_0 = +\sqrt{M^2 - \sqrt{\frac{\alpha}{2}} \sum_{i'j's'\alpha\beta i''} \int dy' dq_\perp'^2 V^{(0)\alpha *}_{i'j'0s'} \eta_{j',\alpha\beta,i'i''} \tilde{f}^{\beta\pm}_{i''j's'}/z_0},$$

$$z_1/z_0 = \frac{1}{M^2 - m_1^2}\sqrt{\frac{\alpha}{2}} \sum_{i'j's'\alpha\beta i''} \int dy' dq_\perp'^2 V^{(0)\alpha *}_{i'j'1s'} \eta_{j',\alpha\beta,i'i''} \tilde{f}^{\beta\pm}_{i''j's'}/z_0.$$

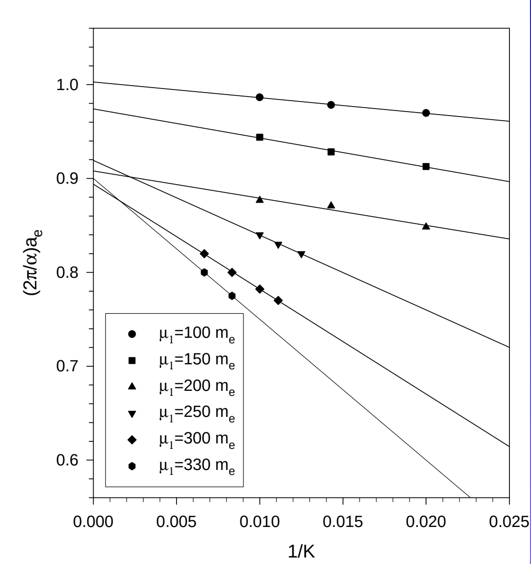
Jacobi iteration of linear system for two-body wave function

$$\tilde{f}^{\mu\pm}_{ijs}/z_0 \quad + \quad \frac{\alpha}{2\pi}\int dy' dq_\perp'^2 \sum_{i'j's'\alpha\beta i''} J^{(2)\mu\alpha}_{ijs,i'j's'} \eta_{j',\alpha\beta,i'i''} \tilde{f}^{\beta\pm}_{i''j's'}/z_0$$
$$= -\sqrt{\frac{\alpha}{2\pi^2}} \sum_a (-1)^a V^{(0)\mu}_{ijas} z_a/z_0.$$

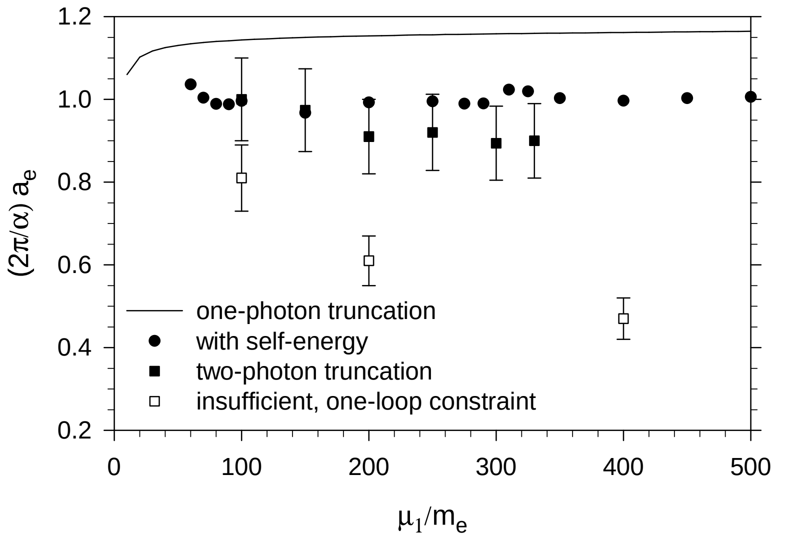
Need for high resolution

- This method converges if the resolution is high enough to accurately approximate the action of the two-photon kernel.
- The necessary resolution is much higher than was considered in the Lanczos method, with $K \geq 50$.
- Such high resolution does not allow storage of the matrix representing the two-photon kernel.
- Instead, compute matrix elements as needed in matrix multiplication.
- Alternate method converges rapidly enough for this to not require too much time, $\sim$1 cpu-day.
- There remains some resolution dependence $\longrightarrow$ extrapolate.

Extrapolation in longitudinal resolution



Final results for anomalous moment



Summary

The two-photon results with the correct chiral constraint are consistent with the Schwinger result, and therefore with experiment, to within the estimated numerical error of 10%.

The systematic deviation below the Schwinger result is expected to be due to the absence of the two-electron/one-positron Fock sector and the three-photon self-energy contributions.

In perturbation theory, cancellations exist between different types of contributions, such as between photon loops and electron-positron loops which are the same order in α, and, therefore, it is not surprising for the present two-photon calculation, which does not also include electron-positron loops, to have a somewhat worse result than the one-photon calculation with just the two-photon self-energy contribution.

Future applications

Open questions for QED:

- inclusion of an electron-positron pair would be very interesting. The renormalization of the electron charge would need to be re-examined, both because of vacuum polarization contributions and because covariance of the current may be restored, at least partially.
- analysis of true bound states, such as positronium, would also be interesting as further tests of the method.

In none of these cases is the nonperturbative analysis likely to produce results competitive with high-order perturbation theory; the numerical errors are large compared to the tiny perturbative corrections in a weakly coupled theory such as QED, but in a strongly coupled theory, such as QCD, the method will be more quantitative.

Application to QCD

- the PV-regulated formulation by Paston *et al.*
- the analog of the dressed-electron problem does not exist and the minimum truncation that would include non-Abelian effects would be to include at least two gluons. The smallest calculation would then be in the glueball sector.
- in the meson sector, the minimum truncation would be a quark-antiquark pair plus two gluons, which as a four-body problem would require discretization techniques beyond what are discussed here. One would discretize the coupled integral equations directly and diagonalize a very large but very sparse matrix.
- as an intermediate step, model the meson sector with effective interactions, particularly with an interaction to break chiral symmetry.