# Pauli–Villars regularization of field theories on the light front

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# Introduction

- need to regulate (3+1)-dimensional theories.
- attempt to preserve as many symmetries as possible.
- add enough Pauli–Villars (PV) fields to regulate perturbation theory & assume that the nonperturbative eigenproblem is also regulated.
- numerical methods then approximate a finite theory, just as was the case for (1+1)-dimensional superrenormalizable theories.
- compute wave functions as coefficients in Fock-state expansions.
- will compare sector-dependent and standard parameterizations.
- for QED, not meant to compete with perturbation theory.

# Outline

- summary of applications
- light-cone coordinates
- PV-regulated QED in Feynman gauge
- photon eigenstate
- dressed-electron eigenstate
  - one-photon truncation
  - anomalous magnetic moment
  - restoration of chiral symmetry
  - sector-dependent parameterization
- summary & future work

# Initial applications

- models with a heavy source:
  PRD **58**, 025005 (1998); PRD **60**, 054506 (1999).
- Yukawa theory without antifermions
  - dressed fermion state:
    - 3 PV bosons using DLCQ: PRD **64**, 114023(2001).
    - 1 PV fermion and 1 PV boson via Gauss-Legendre quadrature
      - one-photon truncation:
        Ann Phys **305**, 266 (2003).
      - two-boson truncation:
        Ann Phys **321**, 1240 (2006).
  - exact solutions: Ann Phys **296**, 406 (2002).
  - two-fermion state [in progress].

# Applications to gauge theories

- dressed electron in QED
  - one-photon truncation in Feynman gauge and light-cone gauge with 1 PV fermion and 1 PV photon: NPB **703**, 333 (2004).
  - restoration of chiral limit, with 1 PV fermion and 2 PV photons: PRD **79**, 114017 (2009).
  - sector-dependent vs standard parameterizations: Ann. Phys., arXiv:0911.3686[hep-ph].
  - two-photon truncation: PRD **81**, 074030 (2010).
- photon state in QED: arXiv:1006.1077[hep-ph].
- scheme proposed for QCD:
  Paston, Franke, and Prokhvatilov, Theor Math Phys **120**, 1164 (1999).

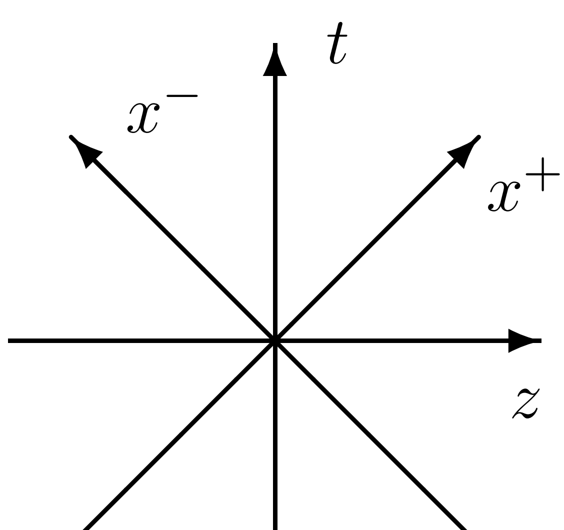
# Sector-dependent parameterization

- the bare parameters of the Lagrangian are allowed to depend on the Fock sector(s) on which the operators act.
- originally proposed by Perry, Harindranath, and Wilson, PRL **65**, 2959 (1990); PRD **43**, 4051 (1991).
- applied to QED: jrh and Brodsky, PRD **59**, 016006 (1998).
- more recently, investigated by Karmanov, Mathiot, and Smirnov, PRD **77**, 085028 (2008).
- a comparison with standard parameterization was needed to better understand how to proceed with high-order truncations.

# Light-cone coordinates

Dirac, RMP **21**, 392 (1949).
Brodsky, Pauli, and Pinsky, Phys. Rep. **301**, 299 (1997).

- time: $x^+ = t + z$
- space: $\underline{x} = (x^-, \vec{x}_\perp), \;\; x^- \equiv t - z, \;\; \vec{x}_\perp = (x, y)$
- energy: $p^- = E - p_z$
- momentum: $\underline{p} = (p^+, \vec{p}_\perp), \;\; p^+ \equiv E + p_z, \;\; \vec{p}_\perp = (p_x, p_y)$
- mass-shell condition: $p^2 = m^2 \;\; \Rightarrow \;\; p^- = \frac{m^2 + p_\perp^2}{p^+}$



# QED Lagrangian

$$\mathcal{L} = \sum_{i=0}^{2}(-1)^i\left[-\frac{1}{4}F_i^{\mu\nu}F_{i,\mu\nu}+\frac{1}{2}\mu_i^2A_i^\mu A_{i\mu}-\frac{1}{2}\left(\partial^\mu A_{i\mu}\right)^2\right]$$
$$+\sum_{i=0}^{2}(-1)^i\bar{\psi}_i(i\gamma^\mu\partial_\mu-m_i)\psi_i-e_0\bar{\psi}\gamma^\mu\psi A_\mu,$$

$$\psi=\sum_{i=0}^{2}\sqrt{\beta_i}\psi_i,\quad A_\mu=\sum_{i=0}^{2}\sqrt{\xi_i}A_{i\mu},\quad F_{i\mu\nu}=\partial_\mu A_{i\nu}-\partial_\nu A_{i\mu}.$$

$$\xi_0=1,\quad \sum_{i=0}^{2}(-1)^i\xi_i=0,\quad \beta_0=1,\quad \sum_{i=0}^{2}(-1)^i\beta_i=0,$$

and require chiral symmetry restoration and zero photon mass

# Dynamical fields

$$\psi_{i+} = \frac{1}{\sqrt{16\pi^3}} \sum_s \int d\underline{k} \chi_s \left[ b_{is}(\underline{k}) e^{-i\underline{k}\cdot\underline{x}} + d^\dagger_{i,-s}(\underline{k}) e^{i\underline{k}\cdot\underline{x}} \right],$$

$$A_{i\mu} = \frac{1}{\sqrt{16\pi^3}} \int \frac{d\underline{k}}{\sqrt{k^+}} \left[ a_{i\mu}(\underline{k}) e^{-i\underline{k}\cdot\underline{x}} + a^\dagger_{i\mu}(\underline{k}) e^{i\underline{k}\cdot\underline{x}} \right].$$

$$\{b_{is}(\underline{k}), b^\dagger_{i's'}(\underline{k}'\} = (-1)^i \delta_{ii'} \delta_{ss'} \delta(\underline{k} - \underline{k}'),$$

$$\{d_{is}(\underline{k}), d^\dagger_{i's'}(\underline{k}'\} = (-1)^i \delta_{ii'} \delta_{ss'} \delta(\underline{k} - \underline{k}'),$$

$$[a_{i\mu}(\underline{k}), a^\dagger_{i'\nu}(\underline{k}'] = (-1)^i \delta_{ii'} \epsilon^\mu \delta_{\mu\nu} \delta(\underline{k} - \underline{k}').$$

$$\epsilon^\mu = (-1, 1, 1, 1).$$

# Fermionic constraint

The nondynamical components of the fermion fields satisfy the constraints ($i = 0, 1, 2$)

$$i(-1)^i \partial_- \psi_{i-} + eA_- \sqrt{\beta_i} \sum_j \psi_{j-}$$

$$= \; (i\gamma^0\gamma^\perp) \left[ (-1)^i \partial_\perp \psi_{i+} - ieA_\perp \sqrt{\beta_i} \sum_j \psi_{j+} \right] - (-1)^i m_i \gamma^0 \psi_{i+}.$$

Multiply by $(-1)\sqrt{\beta_i}$ and sum over $i$

$$i\partial_- \psi_- = (i\gamma^0\gamma^\perp)\partial_\perp \psi_+ - \gamma^0 \sum_i \sqrt{\beta_i} m_i \psi_{i+},$$

same constraint as for free fermion field.

# Hamiltonian

$$\mathcal{P}^- = \sum_{i,s} \int d\underline{p} \frac{m_i^2 + p_\perp^2}{p^+} (-1)^i b_{i,s}^\dagger(\underline{p}) b_{i,s}(\underline{p})$$

$$+ \sum_{l,\mu} \int d\underline{k} \frac{\mu_l^2 + k_\perp^2}{k^+} (-1)^l \epsilon^\mu a_{l\mu}^\dagger(\underline{k}) a_{l\mu}(\underline{k})$$

$$+ \sum_{i,j,l,s,\mu} \sqrt{\beta_i \beta_j \xi_l} \int d\underline{p} d\underline{q} \left\{ b_{i,s}^\dagger(\underline{p}) \left[ b_{j,s}(\underline{q}) V_{ij,2s}^\mu(\underline{p}, \underline{q}) \right.\right.$$

$$\left.\left. + b_{j,-s}(\underline{q}) U_{ij,-2s}^\mu(\underline{p}, \underline{q}) \right] a_{l\mu}^\dagger(\underline{q} - \underline{p}) + H.c. \right\}$$

$+$ antifermion terms.

The instantaneous fermion and instantaneous photon terms cancel.

# Vertex functions - no spin flip

$$V^0_{ij\pm}(\underline{p},\underline{q}) = \frac{e_0}{\sqrt{16\pi^3}} \frac{\vec{p}_\perp \cdot \vec{q}_\perp \pm i\vec{p}_\perp \times \vec{q}_\perp + m_i m_j + p^+ q^+}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$V^3_{ij\pm}(\underline{p},\underline{q}) = \frac{-e_0}{\sqrt{16\pi^3}} \frac{\vec{p}_\perp \cdot \vec{q}_\perp \pm i\vec{p}_\perp \times \vec{q}_\perp + m_i m_j - p^+ q^+}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$V^1_{ij\pm}(\underline{p},\underline{q}) = \frac{e_0}{\sqrt{16\pi^3}} \frac{p^+(q^1 \pm iq^2) + q^+(p^1 \mp ip^2)}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$V^2_{ij\pm}(\underline{p},\underline{q}) = \frac{e_0}{\sqrt{16\pi^3}} \frac{p^+(q^2 \mp iq^1) + q^+(p^2 \pm ip^1)}{p^+ q^+ \sqrt{q^+ - p^+}},$$

Standard approach: $m_0$ same for all sectors and $e_0 = e$.

Sector-dependent: $m_0$ and $e_0$ depend on the Fock sector.

# Vertex functions - spin flip

$$U^0_{ij\pm}(\underline{p},\underline{q}) = \frac{\mp e_0}{\sqrt{16\pi^3}} \frac{m_j(p^1 \pm ip^2) - m_i(q^1 \pm iq^2)}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$U^3_{ij\pm}(\underline{p},\underline{q}) = \frac{\pm e_0}{\sqrt{16\pi^3}} \frac{m_j(p^1 \pm ip^2) - m_i(q^1 \pm iq^2)}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$U^1_{ij\pm}(\underline{p},\underline{q}) = \frac{\pm e_0}{\sqrt{16\pi^3}} \frac{m_i q^+ - m_j p^+}{p^+ q^+ \sqrt{q^+ - p^+}},$$

$$U^2_{ij\pm}(\underline{p},\underline{q}) = \frac{i e_0}{\sqrt{16\pi^3}} \frac{m_i q^+ - m_j p^+}{p^+ q^+ \sqrt{q^+ - p^+}}.$$

# Photon eigenstate

$$|\psi^{(\lambda)}(\underline{P})\rangle = \sum_l z_l^\lambda a_l^{(\lambda)\dagger}(\underline{P})|0\rangle$$
$$+\sum_{ijss'}\int d\underline{k} C^\lambda_{ijss'}(\underline{k}) b^\dagger_{is}(\underline{k}) d^\dagger_{js'}(\underline{P}-\underline{k})|0\rangle.$$

$$\frac{M_\lambda^2}{P^+} z_l^\lambda = \frac{\mu_l^2}{P^+} z_l^\lambda + \sum_{ijss'\mu}\int d\underline{k}(-1)^{i+j}\sqrt{\beta_i\beta_j\xi_l}\, C^\lambda_{ijss'}(\underline{k}) e_\mu^{(\lambda)}(\underline{P})$$
$$\times\left[\delta_{s's}\bar{V}^{\mu *}_{ij,2s}(\underline{k},\underline{P}-\underline{k}) + \delta_{s',-s}\bar{U}^{\mu *}_{ij,-2s}(\underline{k},\underline{P}-\underline{k})\right],$$

$$\frac{M_\lambda^2}{P^+} C^\lambda_{ijss'}(\underline{k}) = \left(\frac{m_i^2+k_\perp^2}{k^+} + \frac{m_j^2+k_\perp^2}{P^+-k^+}\right) C^\lambda_{ijss'}(\underline{k})$$
$$+\sum_{k\mu} z_k^\lambda (-1)^k \sqrt{\beta_i\beta_j\xi_k}\,\epsilon^\lambda e_\mu^{(\lambda)}(\underline{P})$$
$$\times\left[\delta_{s's}\bar{V}^{\mu}_{ij,2s}(\underline{k},\underline{P}-\underline{k}) + \delta_{s',-s}\bar{U}^{\mu}_{ij,-2s}(\underline{k},\underline{P}-\underline{k})\right].$$

# Equation in bare-photon sector

Eliminate two-body wave function

$$M_\lambda^2 z_l^\lambda = \mu_l^2 z_l^\lambda + m_e^2 \sqrt{\xi_l}\epsilon^\lambda I(M_\lambda^2)\sum_k (-1)^k \sqrt{\xi_k} z_k^\lambda,$$

with $m_e$ the physical mass of the electron and

$$I(M^2) = \frac{e_0^2}{8\pi^3}\sum_{ij}(-1)^{i+j}\frac{\beta_i\beta_j}{m_e^2}\int\frac{dx\,d^2k_\perp}{x(1-x)} \times\frac{(1-2x)^2k_1^2+k_2^2+(m_i(1-x)+m_jx)^2}{\left[M^2x(1-x)-(m_i^2+k_\perp^2)(1-x)-(m_j^2+k_\perp^2)x\right]}.$$

# Analytic solution

$H\vec{z}^\lambda = \frac{M^2}{m_e^2}\vec{z}^\lambda$, where $\vec{z}^\lambda = (z_0^\lambda, z_1^\lambda, z_2^\lambda)^T$ and

$$H = \begin{pmatrix} \mu_0^2/m_e^2 + \xi_0 I(M^2) & -\sqrt{\xi_0\xi_1}I(M^2) & \sqrt{\xi_0\xi_2}I(M^2) \\ \sqrt{\xi_0\xi_1}I(M^2) & \mu_1^2/m_e^2 - \xi_1 I(M^2) & \sqrt{\xi_1\xi_2}I(M^2) \\ \sqrt{\xi_0\xi_2}I(M^2) & -\sqrt{\xi_1\xi_2}I(M^2) & \mu_2^2/m_e^2 + \xi_2 I(M^2) \end{pmatrix}$$

When the bare photon mass $\mu_0$ is zero,

$$\mathrm{det} H = \xi_0 \frac{\mu_1^2\mu_2^2}{m_e^4} I(M^2).$$

Then require $I(0) = 0$ to obtain $\beta_2$.

# Dressed-electron eigenstate

In the one-electron/one-photon truncation, the Fock-state expansion for the total $J_z = \pm\frac{1}{2}$ is

$$|\psi^{\pm}(\underline{P})\rangle = \sum_i z_i b_{i\pm}^{\dagger}(\underline{P})|0\rangle + \sum_{ijs\mu}\int d\underline{k} C_{ijs}^{\mu\pm}(\underline{k}) b_{is}^{\dagger}(\underline{P}-\underline{k}) a_{j\mu}^{\dagger}(\underline{k})|0\rangle .$$

Can remove second PV fermion flavor: $m_2 \to 0$ and $\beta_2 \to 0$.

Project on positively normed Fock states

$$|\psi_{\rm phys}^{\pm}(\underline{P})\rangle = \sum_i (-1)^i z_i b_{0\pm}^{\dagger}(\underline{P})|0\rangle + \sum_{s\mu}\int d\underline{k} \sum_{i=0}^{1} \sum_{j=0,2} \sqrt{\xi_j} \times \sum_{k=j/2}^{j/2+1} \frac{(-1)^{i+k}}{\sqrt{\xi_k}} C_{iks}^{\mu\pm}(\underline{k}) b_{0s}^{\dagger}(\underline{P}-\underline{k}) a_{j\mu}^{\dagger}(\underline{k})|0\rangle .$$

# Coupled equations

$$[M^2 - m_i^2] z_i = \int (P^+)^2 dy d^2 k_\perp \sum_{j,l,\mu} \sqrt{\xi_l} (-1)^{j+l} \epsilon^\mu$$

$$\times \left[ V^{\mu *}_{ji\pm}(\underline{P} - \underline{k}, \underline{P}) C^{\mu\pm}_{jl\pm}(\underline{k}) + U^{\mu *}_{ji\pm}(\underline{P} - \underline{k}, \underline{P}) C^{\mu\pm}_{jl\mp}(\underline{k}) \right],$$

$$\left[ M^2 - \frac{m_j^2 + k_\perp^2}{1-y} - \frac{\mu_l^2 + k_\perp^2}{y} \right] C^{\mu\pm}_{jl\pm}(\underline{k})$$
$$= \sqrt{\xi_l} \sum_{i'} (-1)^{i'} z_{i'} P^+ V^\mu_{ji'\pm}(\underline{P} - \underline{k}, \underline{P}),$$

$$\left[ M^2 - \frac{m_j^2 + k_\perp^2}{1-y} - \frac{\mu_l^2 + k_\perp^2}{y} \right] C^{\mu\pm}_{jl\mp}(\underline{k})$$
$$= \sqrt{\xi_l} \sum_{i'} (-1)^{i'} z_{i'} P^+ U^\mu_{ji'\pm}(\underline{P} - \underline{k}, \underline{P}).$$

An index of $i$ corresponds to the one-electron sector and $j$ to the one-electron/one-photon sector.

# Equation in one-electron sector

$$(M^2 - m_i^2)z_i = 2e_0^2 \sum_{i'} (-1)^{i'} z_{i'} \left[\bar{J} + m_i m_{i'} \bar{I}_0 - 2(m_i + m_{i'})\bar{I}_1\right],$$

with

$$\bar{I}_n(M^2) = \int \frac{dy dk_\perp^2}{16\pi^2} \sum_{jl} \frac{(-1)^{j+l}\xi_l}{M^2 - \frac{m_j^2+k_\perp^2}{1-y} - \frac{\mu_l^2+k_\perp^2}{y}} \frac{m_j^n}{y(1-y)^n},$$

$$\bar{J}(M^2) = \int \frac{dy dk_\perp^2}{16\pi^2} \sum_{jl} \frac{(-1)^{j+l}\xi_l}{M^2 - \frac{m_j^2+k_\perp^2}{1-y} - \frac{\mu_l^2+k_\perp^2}{y}} \frac{m_j^2+k_\perp^2}{y(1-y)^2}.$$

Can show $\bar{J} = M^2 \bar{I}_0$.

Fix $\xi_2$ by requiring $M = 0$ for $m_0 = 0$.

# Analytic solution

$$\alpha_{0\pm} = \frac{(M \pm m_0)(M \pm m_1)}{8\pi(m_1 - m_0)(2\bar{I}_1 \pm M\bar{I}_0)}, \quad z_1 = \frac{M \pm m_0}{M \pm m_1} z_0,$$

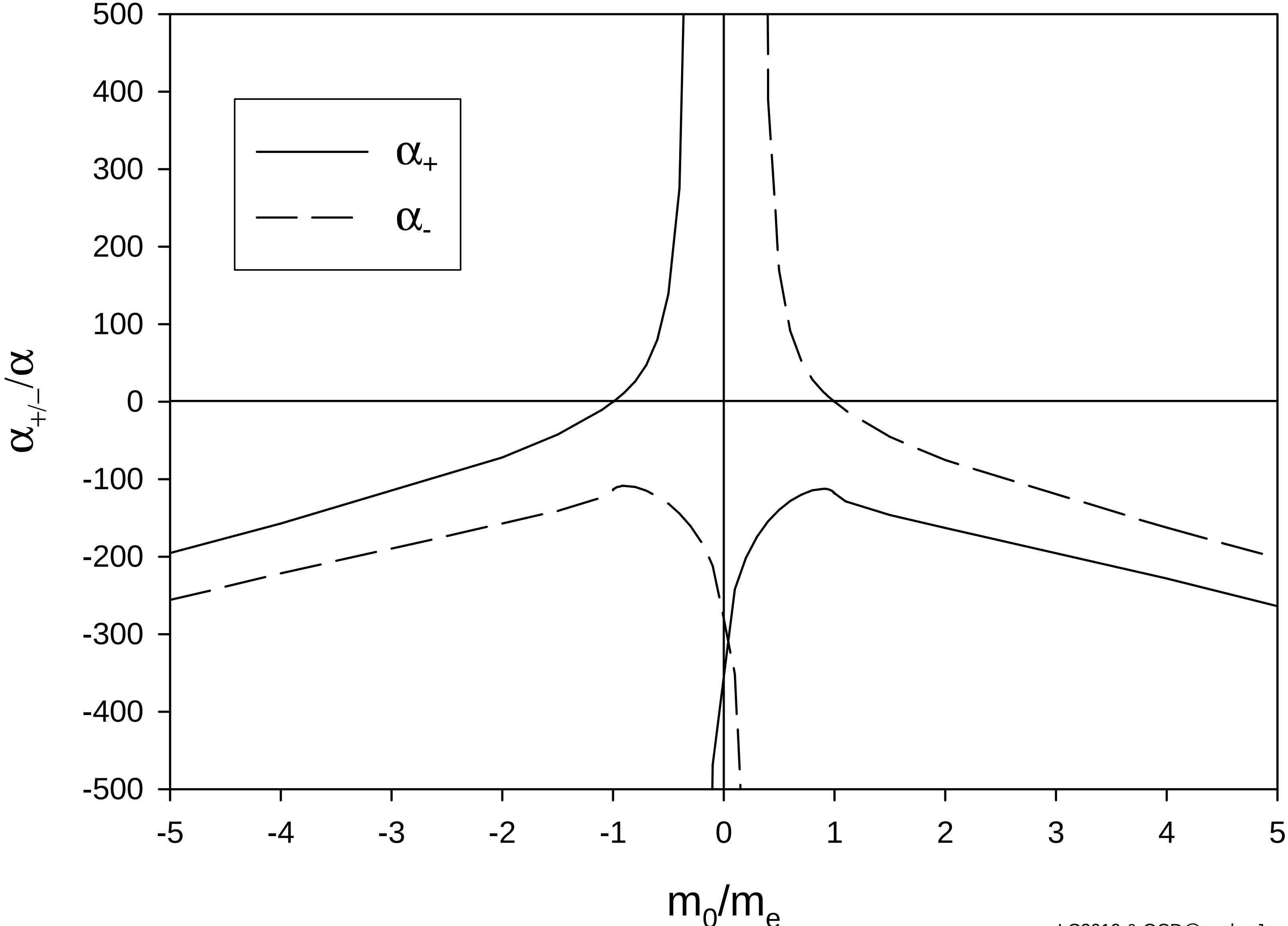
Can safely take the $m_1 \to \infty$ limit, where $z_1 = 0$, $m_1 z_1 \to \pm(M - m_0)z_0$,

$$\alpha_{0\pm} = \pm \frac{M(M \pm m_0)}{8\pi(2\bar{I}_1 \pm M\bar{I}_0)},$$

and the second PV photon flavor can be discarded. In the sector-dependent approach, $\bar{I}_1$ and $\bar{I}_0$ are independent of $m_0$, and the solution for $\alpha_0$ can be written as an explicit expression for $m_0$

$$m_0 = \mp M + 8\pi \frac{\alpha_{0\pm}}{M}(2\bar{I}_1 \pm M\bar{I}_0).$$

# Graphical solution



# Sector-dependent bare coupling

- for the coupling between the bare-electron and one-electron/one-photon sectors, the bare coupling is given by $e_0 = e/z_0$, where $z_0$ is the amplitude for the bare-electron Fock state computed without projection onto the physical subspace.
- in general, the bare coupling would be $e_0 = Z_1 e/\sqrt{Z_{2i} Z_{2f} Z_3}$; this includes the truncation effect that splits the usual $Z_2$ into a product of different $\sqrt{Z_2}$ from each fermion leg.
- no fermion-antifermion loop $\rightarrow Z_3 = 1$.
- only one photon $\rightarrow$ no vertex correction and $Z_1 = 1$.
- also, only the fermion leg with no photon spectator will be corrected by $\sqrt{Z_2}$
- therefore, $\sqrt{Z_{2i} Z_{2f}} = z_0$.

# Sector-dependent bare amplitude

In the infinite-$m_1$ limit, the bare-electron amplitude without projection is determined by the normalization $1 = z_0^2 + e_0^2 z_0^2 \tilde{J}_2$, with

$$\tilde{J}_2 = \frac{1}{8\pi^2} \int y\, dy dk_\perp^2 \sum_{k=0}^{1} (-1)^k \frac{(y^2+2y-2)m_e^2 + k_\perp^2}{[k_\perp^2 + (1-y)\mu_k^2 + y^2 m_e^2]^2}.$$

Replace $e_0$ by $e/z_0$ and solve for $z_0$ as $z_0 = \sqrt{1 - e^2 \tilde{J}_2}$ and find $e_0 = e/\sqrt{1-e^2\tilde{J}_2}$.

Because $\tilde{J}_2 \simeq \frac{1}{8\pi^2}\left(\ln \frac{\mu_1 \mu_0^2}{m_e^3} + \frac{9}{8}\right)$ for large $\mu_1$, $e_0$ can become imaginary and Fock-sector probabilities range outside $[0,1]$ due to IR and UV divergences.

$\Rightarrow$ limits on $\mu_0$ and $\mu_1$.

# Standard bare amplitude

The projected normalization condition is $1 = z_0^2 + e^2 z_0^2 J_2$, with

$$J_2 = \frac{1}{8\pi^2} \int y\, dy dk_\perp^2 [m_0^2 - 4m_0 m_e(1-y) + m_e^2(1-y)^2 + k_\perp^2] \times \left( \sum_{k=0}^{1} (-1)^k \frac{1}{[k_\perp^2 + (1-y)\mu_k^2 + y m_0^2 - y(1-y)m_e^2]} \right)^2 .$$

Thus the bare amplitude is $z_0 = 1/\sqrt{1+e^2 J_2}$, which is driven to zero as $\mu_1 \to \infty$ and causes most expectation values also to go to zero.

$\Rightarrow$ limit on $\mu_1$, but $\mu_0 = 0$ OK.

These limits may be difficult to detect in a basis-function calculation without separate regularization.

# Anomalous magnetic moment

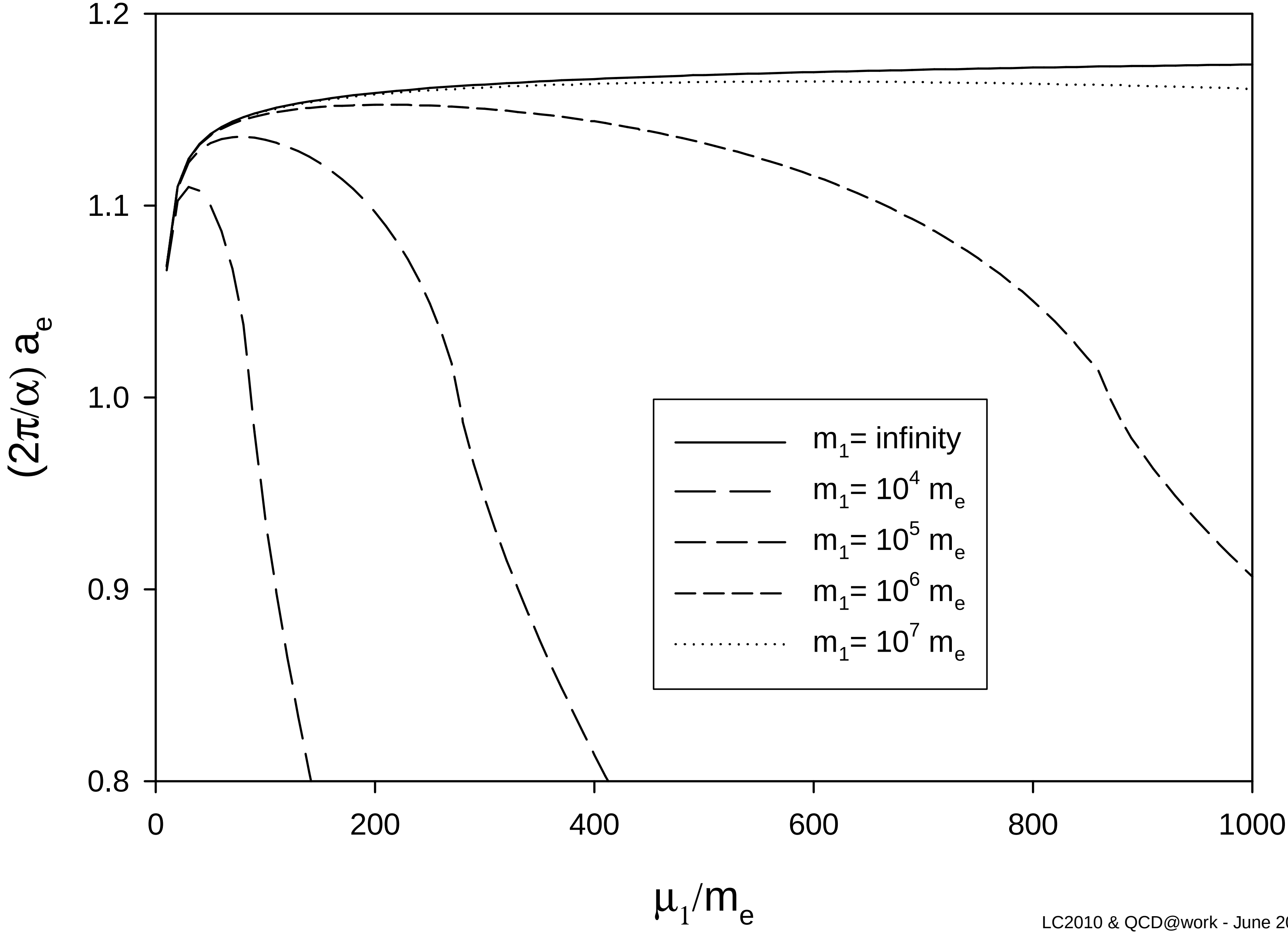
Computed from the spin-flip matrix element of the electromagnetic current $J^+$

$$-\left(\frac{Q_x - iQ_y}{2M}\right) F_2(Q^2) = \pm\frac{1}{2}\langle\psi^\pm(\underline{P}+\underline{Q})|\frac{J^+(0)}{P^+}|\psi^\mp(\underline{P})\rangle_{\rm phys},$$

where $a_e = F_2(0)$.
In the one-photon truncation, it reduces to

$$a_e = m_e \sum_{s\mu}\int d\underline{k}\epsilon^\mu \sum_{j=0,2}\xi_j \left(\sum_{i'=0}^{1}\sum_{k'=j/2}^{j/2+1}\frac{(-1)^{i'+k'}}{\sqrt{\xi_{k'}}}C^{\mu+}_{i'k's}(\underline{k})\right)^*$$
$$\times y\left(\frac{\partial}{\partial k_x}+i\frac{\partial}{\partial k_y}\right)\left(\sum_{i=0}^{1}\sum_{k=j/2}^{j/2+1}\frac{(-1)^{i+k}}{\sqrt{\xi_k}}C^{\mu-}_{iks}(\underline{k})\right).$$

# Without second PV photon flavor



# Substitution of wave functions

In the limit where the PV electron mass $m_1$ is infinite, the expression for the anomalous moment is

$$a_e = \frac{\alpha_0}{\pi} m_e^2 z_0^2 \int y^2(1-y)\,dy\,dk_\perp^2 \times \left( \sum_{k=0}^{1} \frac{(-1)^k}{ym_0^2 + (1-y)\mu_k^2 + k_\perp^2 - m_e^2 y(1-y)} \right)^2 .$$

For the sector-dependent parameterization, the product $\alpha_0 z_0^2$ is just $\alpha$, and the bare mass $m_0$ in the denominator is replaced by the physical mass $m_e$.

# Sector-dependent case

To be consistent with Karmanov et al, we eliminate the projection, which does not affect the result significantly, and obtain

$$a_e = \frac{\alpha}{\pi} m_e^2 \int y^2(1-y)\,dy\,dk_\perp^2 \sum_{k=0}^{1}(-1)^k \left(\frac{1}{ym_e^2 + (1-y)\mu_k^2 + k_\perp^2 - m_e^2 y(1-y)}\right)^2$$

In the $\mu_1 \to \infty$, $\mu_0 \to 0$ limit, this becomes exactly the Schwinger result

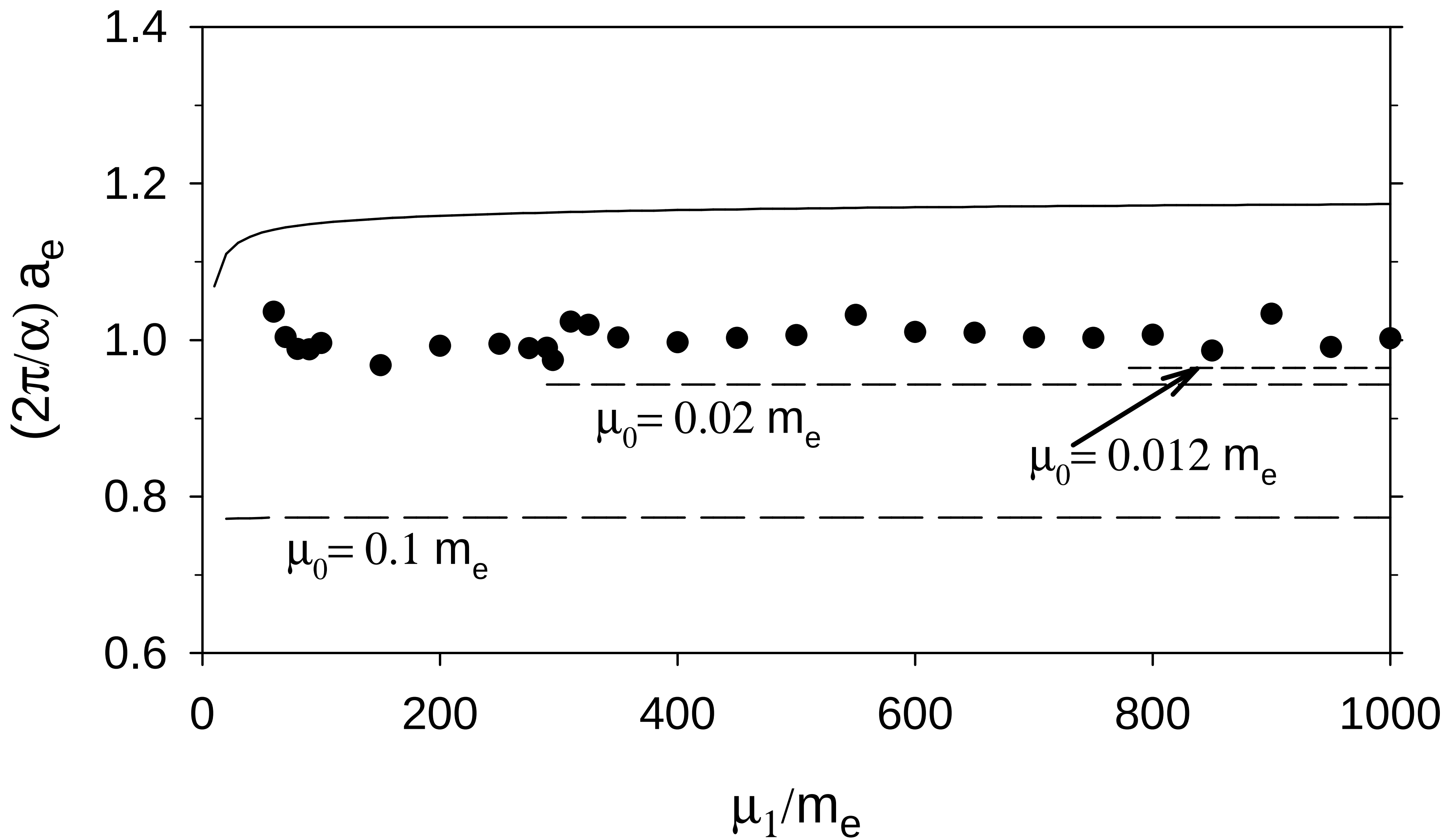
$$a_e = \frac{\alpha}{\pi} m_e^2 \int \frac{dy\,dq_\perp^2/(1-y)}{\left[\frac{m_e^2+q_\perp^2}{1-y} + \frac{q_\perp^2}{y} - m_e^2\right]^2} = \frac{\alpha}{2\pi}.$$

# Standard parameterization case

- $\alpha_0 = \alpha$
- normalization $z_0^2 = 1/(1+e^2 J_2)$
- anomalous moment

$$
\begin{aligned}
a_e &= \frac{\alpha m_e^2}{\pi(1+e^2 J_2)} \int y^2(1-y)\,dy\,dk_\perp^2 \\
&\times \left( \sum_{k=0}^{1} \frac{(-1)^k}{y m_0^2 + (1-y)\mu_k^2 + k_\perp^2 - m_e^2 y(1-y)} \right)^2 .
\end{aligned}
$$

# Comparison plot



# Summary

- important to maintain symmetries, which can be done with additional PV fields.
- can extract meaningful results at finite PV masses (see talk by Chabysheva on two-photon truncation).
- the standard parameterization leads to well-defined wave functions.
- the sector-dependent approach is at least as difficult, and probably worse due to the IR problem.
- best to regulate before applying numerical methods to clearly separate limits.
- PV fields add to numerical load but also reduce, by eliminating instantaneous fermion and instantaneous photon interactions.

# Future work

- electron-positron pair contribution to
  - dressed-electron eigenstate
  - charge renormalization
  - current covariance
- large-$\alpha$ calculation to compare with higher-order perturbation theory.
- electron in magnetic field and induced magnetic moment.
- two-fermion bound state in Yukawa theory and QED.