Collective modes in the color-flavor-locked phase of quantum chromodynamics

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Scope and motivation

A study of the low energy effective action for collective modes of the color-flavor-locked phase of QCD from a *microscopic* point of view.

Focus on *Higgs mode* and Nambu-Goldstone (*phonon*) mode of the diquark condensate and determination of ***interaction*** terms.

To pave the way for further studies on interaction processes, e.g. inelastic phonon-vortex interaction and **possible influences on physics of compact stars**

Compact stars ID

$R \sim 10 \div 15$ km

indirectly, fitting model atmospheres or redshift

$M \sim 1.25 \div 1.45\, M_\odot$

from bynary systems using Kepler's third law

$B \sim 10^{11} \div 10^{13}$ G

from evidence of cyclotron lines in the spectra

$T \sim 10^5 \div 10^6$ K

indirect, model dependent with cooling down

$P \sim 10^{-3} \div 10$ s

signals from radio-pulsars

$P' \sim 10^{-20} \div 10^{-10}$ s

slowing down with derivative P'

$M_\odot = 2\times10^{33}$ g

$B_E \sim 4\times10^{-1}$ G

10^7 K ~ 862 eV

Signals

Theoretical and experimental studies on

Mass-Radius relation

Frequency evolution

Thermal evolution

Oscillation modes

can provide informations about the behaviour of matter under extreme conditions (that we cannot reproduce)

Equation of state

Density pattern beyond the crust

Possible existence of ***superfluid*** matter

Possible existence of ***deconfined*** matter

... and uncovering larger parts of the QCD phase diagram

“The importance of being collective”

Collective modes can be present in the star,
influencing its macroscopic and microscopic properties.

Superfluid matter can be found in the *inner crust* of neutron stars

$T_{core} \sim 10^7 \div 10^9$ K
$T_{crit} \sim 10^9 \div 10^{10}$ K

Color superconductive quark matter ***might*** be found in the *core* of neutron stars

$\rho \gg \rho_0$ & $T_{crit} \sim 10 \div 50$ MeV

Thermal: cooling rate, heat capacity, conductivity

Magnetic: thermal dissipation due to superfluid neutrons that can rotate faster than charged protons

Hydrodynamic: *quantized vortex lines*, possible interplay with oscillation modes and rotation frequency of the star

Non-radial oscillation

r-modes are non-radial oscillations that can provide ***severe limitation on the star's rotation frequency*** [1] through coupling to gravitational radiation emission.

If a given dissipative phenomenon is "efficient" enough, it can damp *r*-modes and the star can rotate without losing angular momentum to gravitational radiation.

Hence, dissipative processes acting in rotating neutron stars ***are essential to prevent the growth of the r-mode instability***.

The study of *r*-modes can be useful for *costraining stellar parameters* and ***to rule out*** [2] ***some structure patterns***, i.e. certain color superconductive phase.

>>> Mannarelli and Alford's talks

[1] Anderson, Kokkotas (2001) [2] Madsen, PRL (2000)

Mutual friction in CFL stars

The **elastic** vortex-phonon interaction has been studied, **in [1],** as possible dissipative phenomenon responsible for damping r-modes.

The model considers a star *entirely made up* of **CFL** quark matter.

The E-V-P-I **does not contribute** to the damping of r-mode oscillations unless the star rotates slower than 1 Hz.

>>> Mannarelli's talk

Cases not yet studied:

[X] hybrid CFL + superfluid neutrons;

[X] ***inelastic processes***

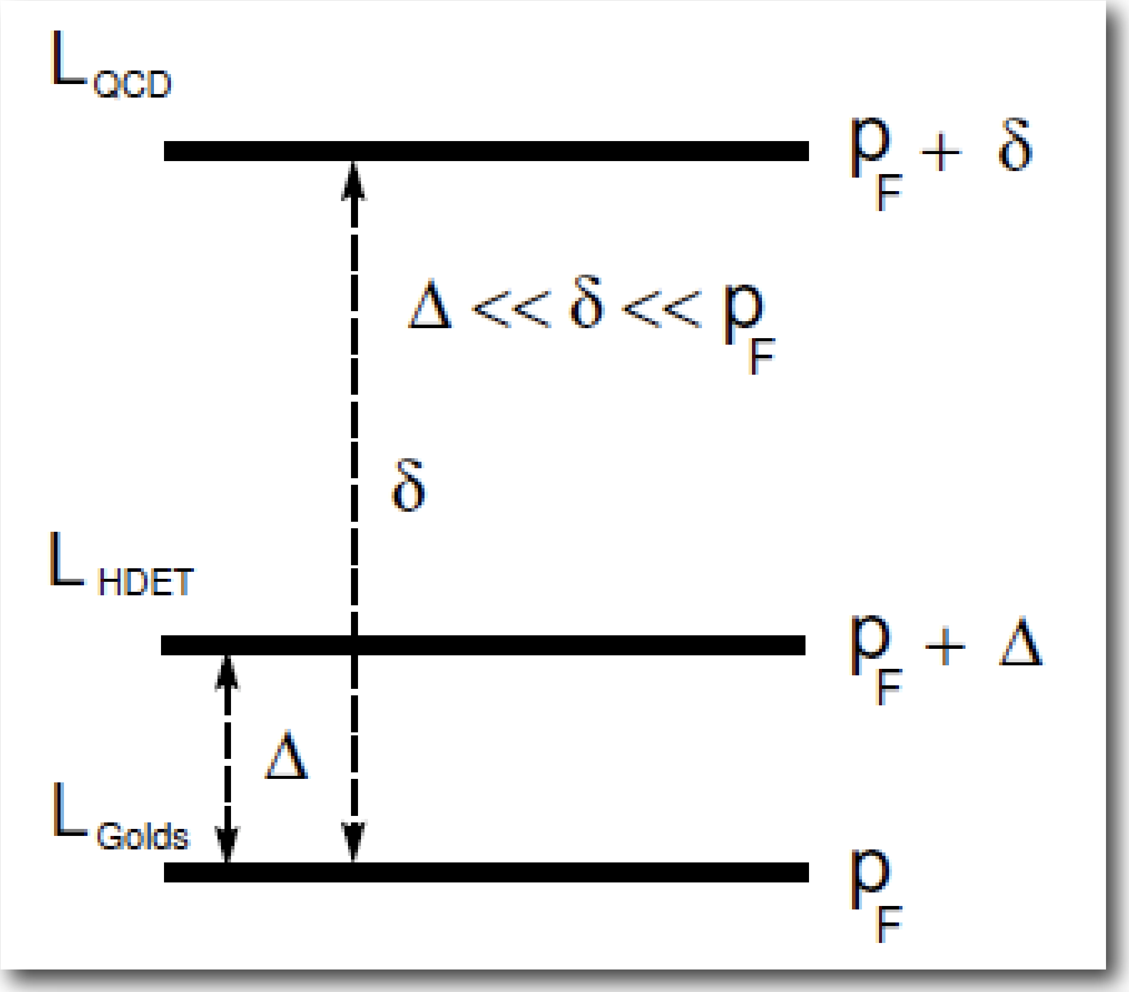
(one would need microscopic approach)

[1] Mannarelli, Manuel and Sa'd, **PRL** (2008)

The model

QCD at very high density can be easily studied through a hierarchy of effective field theories.

For energies close to Fermi energy, the four-fermi interaction that couples pair of fermions with opposite momenta is **the most relevant**: i.e. BCS-like.



This physics can be described using the High Density Effective Theory (HDET) [1]

Main idea: the observation that quarks participating in the dynamics have large momenta ($\sim\mu$)

[1] Beane et al., (2000); Casalbuoni and Nardulli, (2001); Hong, (2000).

The Higgs and phonon modes

The structure of CFL condensate is $\langle \psi^T_{\alpha i} C \psi_{\beta j} \rangle \sim \frac{\Delta}{2} \epsilon_{\alpha\beta I} \epsilon_{ijI}$

We consider the fluctuations of the condensate around the mean field value

$$\Delta(x) = [\Delta_{MF} + \rho(x)] e^{2i\phi(x)}$$

the phase φ is made of a long wavelength background field related to hydrodynamic velocity and a quantum fluctuation.

$$\phi = \bar{\phi} + \varphi$$

$$v_\mu \sim \partial_\mu \bar{\phi}$$

ρ containes long wavelength component associate to vortex configuration and a short wavelegth radial fluctuation

$$\rho = \bar{\rho} + \hat{\rho}$$

$$\bar{\rho} \to -\Delta_{MF} \qquad |x| \to 0$$

The Higgs and phonon modes

The semi-bosonized Lagrangian
with superfluid and Higgs mode

$$\mathscr{L}_\Delta = \bar{\psi}\left(i\gamma^\mu\partial_\mu + \mu\gamma^0 - \gamma^0\partial_0\phi - \gamma^i\partial_i\phi\right)\psi - \frac{1}{2}(\Delta+\rho)\psi^\dagger\psi^* + \frac{1}{2}(\Delta+\rho)\psi^t\psi\,.$$

$$\psi \longrightarrow e^{i\phi}\psi$$

$$\tilde{\mu} = \mu - \partial_0\phi$$

long wavelength fluctuations of the chemical potential

useful for LO Lagrangian for the phonon field

$$A_\mu = (\partial_0\phi, \partial_i\phi)$$

phonon fields appear only with derivative as it should be

Evaluating "microscopic" effective action

After HDET treatment

$$\frac{\mathscr{Z}}{\mathscr{Z}_0} = \frac{\int [d\Delta, d\Delta^*] \exp\left[\frac{1}{g}\int d^4x\, \Delta_{AB} W_{ABCD} \Delta^*_{CD}\right] \det[S^{-1}]^{1/2}}{\det[S_0^{-1}]^{1/2}} \equiv \exp\left[i\mathscr{S}\right]$$

$$S^{-1} \equiv S^{-1}_{MF} + \Gamma \qquad\qquad \mathscr{S} = \mathscr{S}_{\rm mf} + \mathscr{S}_{\rm eff}$$

$$\mathscr{S}_{\rm mf} = -\frac{i}{2}{\rm Tr}\ln[S_0 S^{-1}_{\rm mf}] - \frac{i}{g}\left[\Delta_{AB} W_{ABCD} \Delta^*_{CD}\right]$$

mean field part (gap equation)

$$\mathscr{S}_{\rm eff} = -\frac{i}{g}\int d^4x\; 2\Delta^{\rm mf}_{AB} W_{ABCD}\,\rho(x)_{CD} + \rho_{AB}(x) W_{ABCD}\,\rho_{CD}(x) - {\rm Tr}\ln\left(1 + S_{\rm mf}\Gamma_0\right)$$

effective part with all interaction terms

Low-energy effective action by Son (2002)

The ground state is the **supefluid** CFL

At leading order in α_s $P(\mu) = \frac{N_c N_f}{12\pi^2}\mu^4$

$$\frac{N_c N_f}{12\pi^2}\left[\mu^4 - 4\mu^3\partial_0\varphi + 6\mu^2(\partial_0\varphi)^2 - 2\mu^2(\partial_i\varphi)^2 - 4\mu\partial_0\varphi\partial_\mu\varphi\partial^\mu\varphi + (\partial_\mu\varphi\partial^\mu\varphi)^2\right]$$

costant equilibrium pressure

total derivative negligible if not multi-valued

$$(\partial_0\varphi)^2 - \frac{1}{3}(\nabla\phi)^2$$

kinetic term: 1/√3 is hydrodynamic speed sound in ultrarelativistic gas

interaction terms between Goldstone bosons

$$\partial_\mu\left(n_0\frac{D_\mu\varphi}{\mu_0}\right) = 0 \qquad \mu_0 \equiv (D_\mu\varphi D^\mu\varphi)^{1/2} \qquad n_0 = \left.\frac{dP}{d\mu}\right|_{\mu=\mu_0}$$

$$\partial_\mu\varphi \ll m_N \qquad u^i = \frac{1}{m_N}\partial_i\varphi$$

hydrodynamic interpretation

Effective action for phonon field

The free energy of the system is given by

$$\Omega = \frac{3}{4\pi^2}\tilde{\mu}^4$$

$$\frac{3}{4\pi^2}\mu^4 - \frac{3}{\pi^2}\mu^3\partial_0\phi + \frac{9}{2\pi^2}\mu^2(\partial_0\phi)^2 - \frac{3\mu}{\pi^2}(\partial_0\phi)^3 + \frac{3}{4\pi^2}(\partial_0\phi)^4$$

The spatial derivative for the kinetic equation

$$-\frac{3\mu^2}{2\pi^2}(\partial_i\varphi)^2$$

is obtained evaluating the tadpole contribution to the gauge field self-energy

guess: $\mathscr{L}_\phi = \dfrac{3}{4\pi^2}\left[(\mu - \partial_0\phi)^2 - (\partial_i\phi)^2\right]^2$

Effective action for the radial fluctuation

Kinetic term + mass term of the rho field

$$\mathscr{L}_K(\rho) = \frac{1}{2}\frac{3\mu^2}{4\pi^2}\frac{1}{\Delta^2}\left[(\partial_0\rho)^2 - \frac{1}{3}(\partial_i\rho)^2\right] - \frac{1}{2}\frac{12\mu^2}{\pi^2}\rho^2$$

Note that after wave function renormalization, the mass of the Higgs field depends on Δ as it should be

$$m_\rho^2 = 16\Delta^2$$

In the limit of Δ comparable with T thermal phonon can excite a "vortex", i.e. it can be kinematically possible to have elementary excitations from a quantized vortex line.

Interaction terms

Loop expansion: second order in Γ

$$-\frac{6}{\pi^2}\tilde{\mu}^2\rho(x)^2 = -\frac{6}{\pi^2}(\mu - \partial_0\phi)^2\rho(x)^2$$

Loop expansion: third order in Γ

$$+\frac{2}{\pi^2}(\partial_i\phi)^2\rho^2$$

$$\mathscr{L}(\rho, X^\mu) = \frac{3}{4\pi^2}(\eta^{\mu\nu}X_\mu X_\nu)^2 - \frac{6}{\pi^2}\mathcal{G}^{\mu\nu}X_\mu X_\nu\rho^2$$

$$X^\mu = (\mu - \partial_0\varphi, \partial^i\varphi) \quad \mathcal{G}^{\mu\nu} = \mathrm{diag}(1, -1/3, -1/3, -1/3)$$

Note: Absence of term linear in rho

$$\mathscr{L}_{N.R.}(\rho, Y) = \frac{A}{2}\rho^2 + B\rho Y + \frac{C}{2}Y^2 + DY$$

$$Y = \partial_0\phi + \frac{(\nabla\phi)^2}{2m}$$

m = mass of non relativistic fermion

$$c_s^2 = -\frac{D}{mC}$$

sound speed

$$\mathscr{L}_{N.R.}(\phi) = \frac{B^2 + AC}{2A}Y^2 + DY$$

$$c_s^2 = -\frac{DA}{m(B^2 + AC)}$$

"modified" sound speed

Schakel, 0912.1955

Conclusions and outlook

[V] Study of r-modes can be useful to costraint stellar parameter: V-P-I can play an important role;

[V] a low energy effective action for collective modes of color flavor locked phase of QCD from a **microscopic** HDET approach;
[g] this Lagrangian reproduces the quantum effective action obtained by Son

next steps

[>] vortex solution profile (some terms are missing)
[>] hydrodynamic equations (vortex-phonon interaction)

final aim
[>>] role of “inelastic” V-P-I as dissipative phenomenon

THANKS

High Density Effective Theory

QCD at high density can be conveniently studied through a hierarchy of effective field theories.

The way of obtaining a low energy effective lagrangian is to integrate out high-energy degrees of freedom.

Polchinski (1993) has shown that the physics is particularly simple for energies close to the Fermi energy.

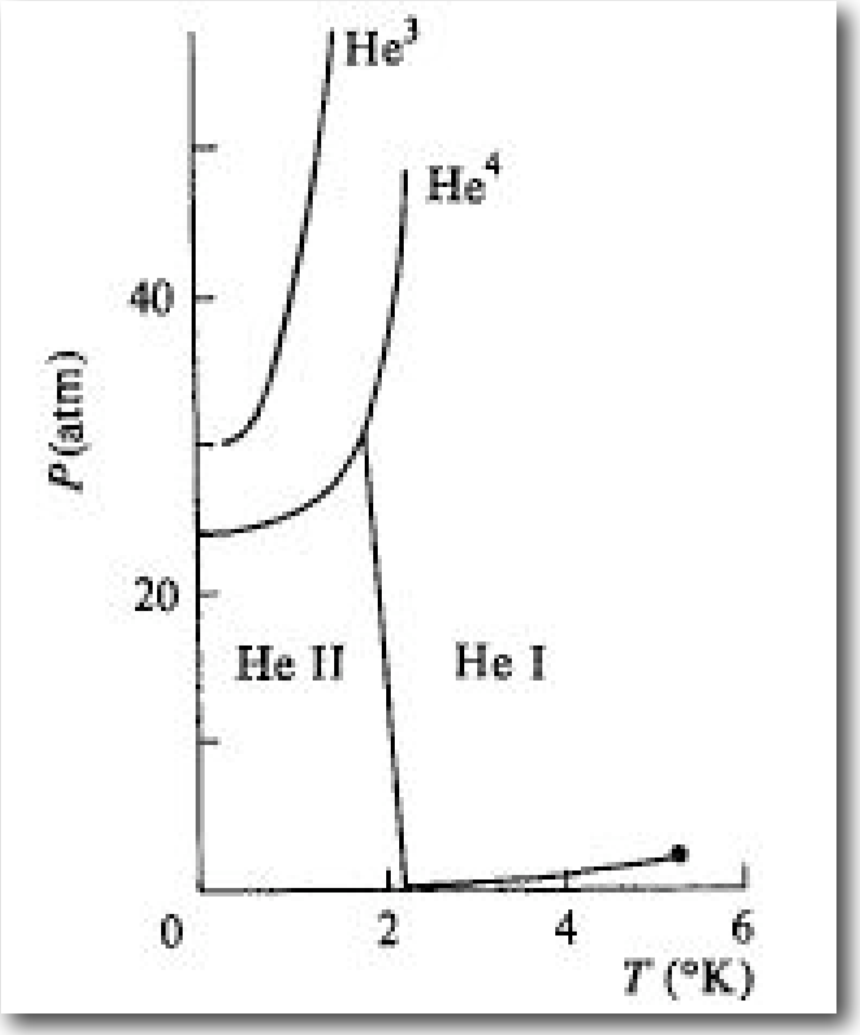
Relevant: four-fermi interaction coupling pair of fermions with opposite momenta: i.e. the interaction giving rise to the BCS condensation.

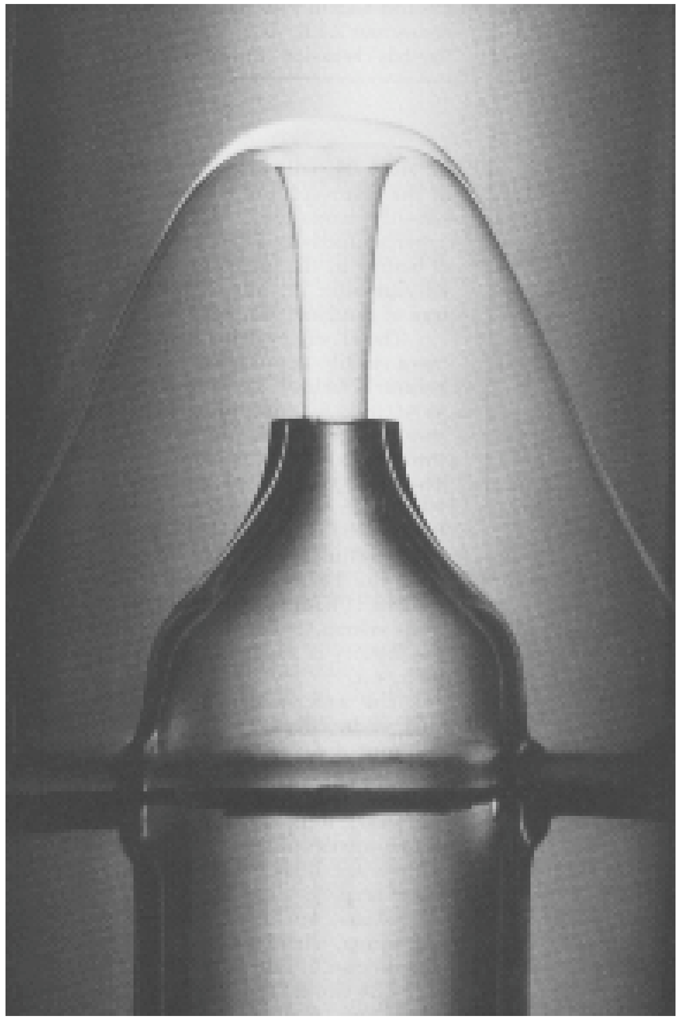
This physics can be described using the High Density Effective Theory (HDET) [1]

[1] Beane et al., (2000); Casalbuoni and Nardulli, (2001); Hong, (2000).

Facts of superfluidity

λ-point @2.18 K: Helium undergoes a second order phase transition. **Below λ**: *Helium II* presents several unusual macroscopic properties [1][2][3]





from *Nature* website

Superfluidity
flowing without friction through narrow capillaries

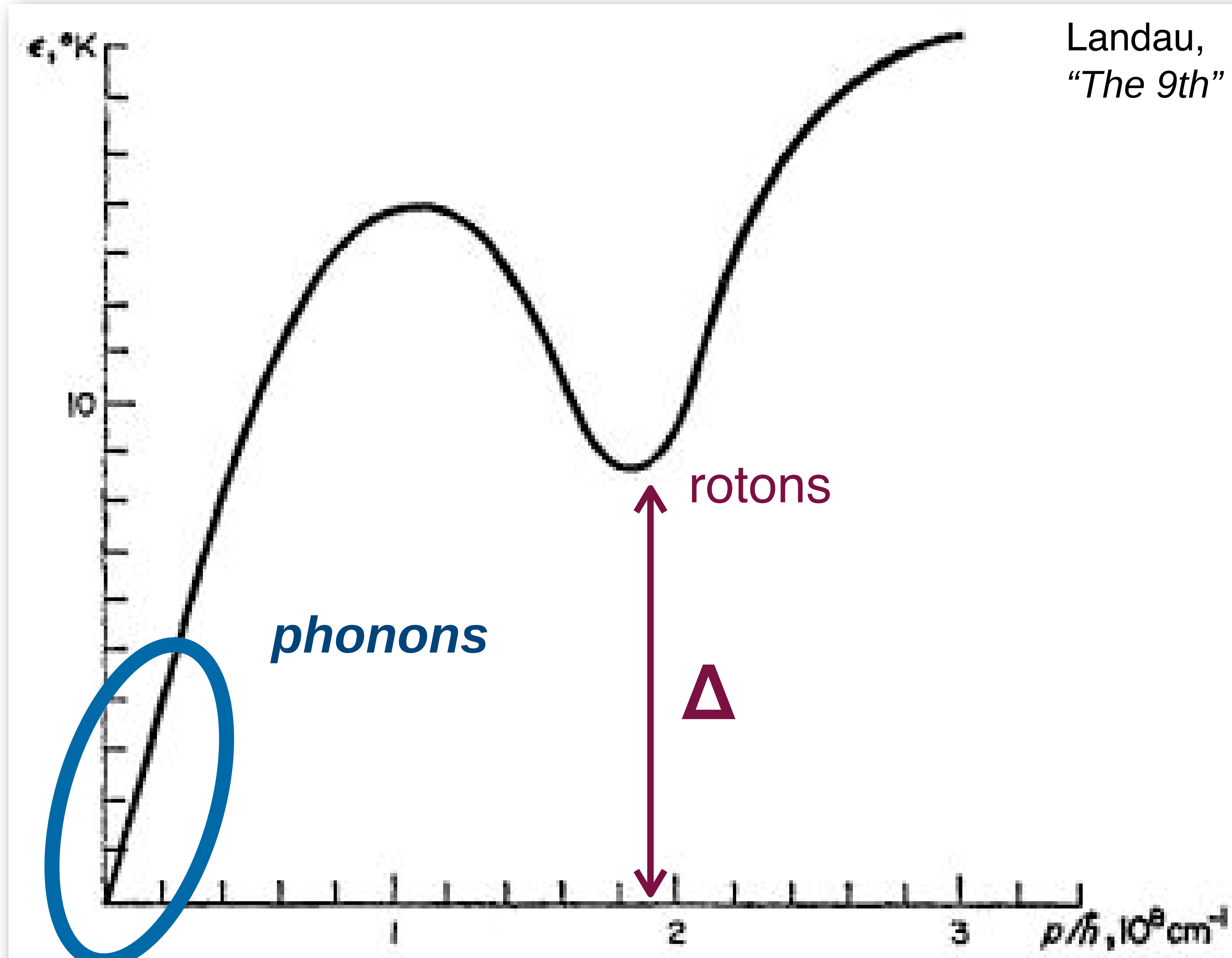
@1-2 K: He II is a *quantum* liquid, i.e. presents quantum properties *before* solidification.

[1] Fetter-Walecka, Quantum theory of Many Particle Systems
[2] A. Khalatnikov, An Introduction to the Theory of Superfluidity
[3] Landau, "The 9th"

The spectrum of Bose quantum liquid

Any Bose quantum liquid (like He II) can be described in terms of elementary excitations that present a spectrum with universal character.

@ **Low energy:** the dynamical properties of the system are ruled by long-wavelength quasiparticles that correspond to ordinary hydrodynamic sound waves, i.e. ***phonons***



If the energy spectrum of the elementary excitations of the particles of the fluid has a phonon-like spectrum, the fluid is a superfluid.

Two fluid Tisza model

The Helium II phase can be considered made of two components, with charateristic velocity fields and densities:

normal fluid		superfluid
ρ_n, v_n	+	ρ_s, v_s
behaves like a ordinary liquid		(1) flows without friction in capillaries (2) has zero entropy

$$\rho = \rho_s + \rho_s$$

$$\rho \boldsymbol{v} = \rho_s \boldsymbol{v}_s + \rho_n \boldsymbol{v}_n$$

$\Xi(t,\mathbf{r}) = \sqrt{n_0(t,\mathbf{r})}e^{-i\phi(t,\mathbf{r})}$ condensate wave function

$\mathbf{j} = (\hbar/m)n_0\nabla\phi$ macroscopic current density of condensate particles

$\mathbf{j} = n_0\,\mathbf{v}_s$ v_s macroscopic velocity of the superfluid motion

$$\mathbf{v}_s = (\hbar/m)\boldsymbol{\nabla}\phi$$

n_0= number of condensate particle per unit volume; m = mass of a liquid particle

This is not a mixing of two elements. In a Bose liquid two different “motions” exist: they have different properties and they show up without momentum transfer from each other.

Vortex filaments

An ordinary liquid contained in cylindrical vessel rotating about its axis, after a certain amount of time, rotates as a whole together with vessel.

The normal component is brought to rotation but

the superfluid part cannot rotate as a whole otherwise

$$\mathrm{curl}\ \mathbf{v}_s \neq 0$$

if $\boldsymbol{\Omega}$ is the angular velocity, the velocity would be $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}$, then $\nabla \times \boldsymbol{v} = 2\boldsymbol{\Omega} \neq 0$

For large rotation rates (> a critical value), a static superfluid condition becomes thermodynamically unfavourable.

Onsager (1949) and Feynman (1955) solved this apparent contraddiction assuming that *the potential flow is lost only at certain lines of singularity*: **vortex filaments**

Characterization of vortex lines

Each vortex filament has a particular value of the velocity circulation along a closed contour round the filament

$$\oint \mathbf{v}_s \cdot d\mathbf{l} = 2\pi\kappa$$

The flux of curl ***v***s must have the same value: this means that a vortex filament either it is *closed* or it *ends at the boundary* of liquid.

For any filament shape, the velocity distribution is

$$\mathbf{v}_s = \frac{\kappa}{2}\int \frac{d\mathbf{z}\times\mathbf{R}}{R^3}$$

"Quantized" vortex lines

Since $\mathbf{v}_s = (\hbar/m)\boldsymbol{\nabla}\phi$ then $\oint \mathbf{v}_s \cdot d\mathbf{l} = (\hbar/m)\Delta\phi$ but the condensate wave function is single valued, i.e. $\Delta\Phi = 2\pi n$

$\kappa = n\,\hbar/m$ only $n = 1$ is thermodynamically stable

The vortex filaments are quantized as $\displaystyle \oint \mathbf{v}_s \cdot d\mathbf{l} = \frac{2\pi\hbar}{m} n$

Above the *critical rotation rate* the superfluid should behave like a rotating rigid body $\boldsymbol{\nabla} \times \langle \mathbf{v}_s \rangle = 2\boldsymbol{\Omega}$

The # of lines per unit area is defined as $\oint \mathbf{v}_s \cdot d\mathbf{l} = N(2\pi\hbar)/m$

The number of vortices increases with Ω: $N = m\Omega/\pi\hbar$

A $T \neq 0$ the elementary excitations from the normal component can scatter with filaments, causing **friction with the superfluid part (!)**

CFL: superfluid but not electric superconductor

The appearance of the Meissner mass means the appearance of the Meissner effect.

In electromagnetism, when the photon acquires a Meissner, the magnetic field can penetrate the superconductor up to a certain penetration lenght

In CFL phase, all gluons acquire a mass but

$$U(1)_{Q'} \subseteq SU(3)_{c+L+R}$$

a linear combination of a gluon and the photon remain massless: and it has been shown [1] that an external magnetic field may penetrate the CFL matter.

The global $U(1)$ symmetry is broken: the corresponding long-wavelength disturbance of the order parameter is the superfluid phonon and it dominates the dynamical properties of the system for low temperature ($< 10^{-2}$ MeV)

[1] M. Alford, J. Berges, and K. Rajagopal, Nucl. Phys. **B571**, 269 (2000).

Relativistic superfluid: quantum effective action

$$Z[J] = \int [d\phi_i] e^{\left[iS + i\int d^4x \; J(x)\Phi(x)\right]}$$

Φ(*x*) is the order parameter of a field theory [1] in which the ground state breaks spontaneaously U(1). *M* is the baryon charge of the order parameter.

$$W[J] = -i \ln Z[J]$$

generates the connected Feynman diagrams constructed from *Φ*

$$\Gamma[\Phi] = W[J] - J\Phi$$

Quantum effective action as Legendre transform of *W*[*J*]. It reproduces at tree level all Green functions constructed from Φ

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{3} A_\mu \bar{q}\gamma^\mu q$$

$$q \longrightarrow e^{i\alpha/3} q \qquad A_\mu \longrightarrow A_\mu + \partial_\mu \alpha$$

$$\Gamma[A_\mu, \Phi] = \Gamma[A_\mu + \partial_\mu\alpha, \Phi e^{iM\alpha}]$$

The quantum effective action respects the same gauge symmetries of the Lagrangian

[1] Son, hep-ph/0204199

$$A_\mu = (\mu, 0)$$

Low energy effective action

The final goal is deriving the quantum effective action for the Goldstone field, i.e the phase of the order parameter (integrating out the amplitude |Φ|)

$$\Phi = |\Phi| e^{iM\varphi} \qquad \varphi \to \varphi + 2\pi/M$$

$$\Gamma[A_\mu, \phi] \equiv \min_{\{|\Phi|\}} \Gamma[A_\mu, |\Phi| e^{iM\varphi}]$$

From symmetries considerations, Son [1] has shown that given an equation of state $P = P(\mu)$, the full low energy effective action is completely determined

$$\Gamma[A_\mu, \varphi] = \int d^4x \; P\left((D_\mu \varphi D^\mu \varphi)^{1/2} \right)$$

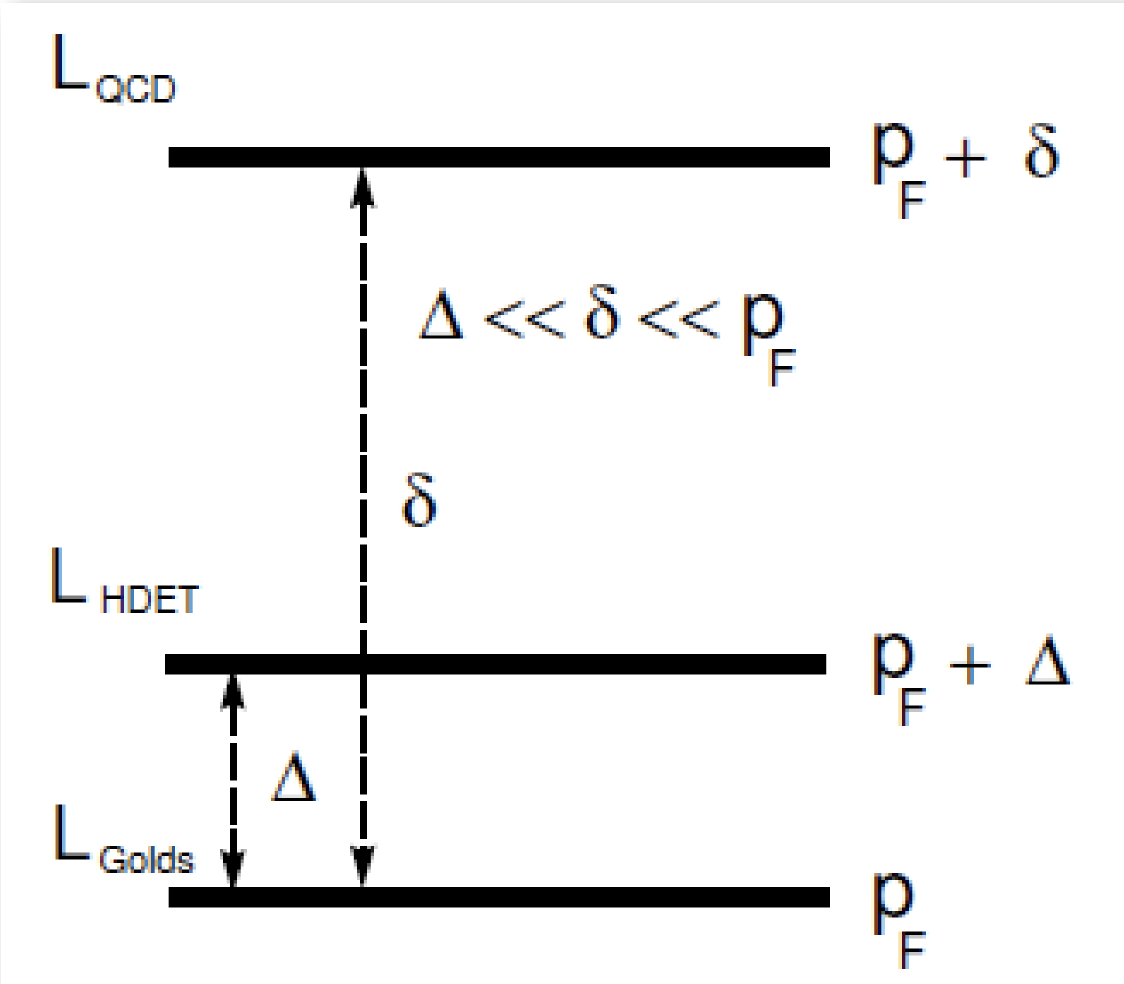
$$D_\mu \equiv \partial_\mu \varphi - A_\mu$$

[1] Son, hep-ph/0204199

$$A_\mu = (\mu, 0)$$

The High Density Effective Theory Approach

High Density Effective Theory - 2



In this theory the condensation effects are described through the introduction of a Majorana mass term.

The d.o.f are quasi-particles (dressed fermions), holes and gauge fields.
This description holds up to a cutoff p_F ($\Delta << \delta << p_F$).

At momenta $p < \Delta$ all the gapped particles decouple and dynamics is ruled by low energy modes (i.e. Goldstone bosons ungapped fermions and massless gauged fields according to the breaking scheme.

Color superconductors

Cooper's Theorem [1] tells us that an attractive interaction, arbitrarily weak, favors to the formation of a condensate of Cooper pairs of quarks, correlated in the momentum space.

At asymptotically high densities, this attractive interaction is provided by single-gluon exchange.

The quark Cooper pair is color-charged: it breaks the color symmetry $SU(3)_C$ spontaneously.

In analogy to electronic superconductors that break $U(1)_{\rm em}$, quark Cooper pairing is a *color*(ed) *superconductivity*.

The pairing patterns depend on many factors: hadronic density, # of flavors, temperature, coupling strength, etc.

At high densities, the favored phase is the *color flavor locked* (CFL) phase.

[1] Cooper, Phys. Rev. **104** 1189 (1956)

Effective chemical potential

$$\mathscr{L}_D = \int \frac{dv}{8\pi} \sum_{A,B=1}^{9} \chi^{\dagger\,A} \begin{pmatrix} i\,V\cdot D\,\delta_{AB} & -\Delta_{AB} \\ -\Delta_{AB} & i\,V\cdot D^*\,\delta_{AB} \end{pmatrix} \chi^B + (L \to R)$$

Note that the phonon appears only with derivative (as gauge fields), as it should be.

The time component of the ‘new’ gauge field, A_0, can be absorbed in an effective chemical potential and the spatial derivative can be threated as insertion

$$\tilde{\mu} \equiv \mu - \partial_0 \varphi$$

Evalutating “microscopic” effective action

$$\frac{\mathscr{Z}}{\mathscr{Z}_0} = \frac{\int [d\Delta, d\Delta^*] \exp\left[\frac{1}{g}\int d^4x\ \Delta_{AB} W_{ABCD} \Delta^*_{CD}\right] \det[S^{-1}]^{1/2}}{\det[S_0^{-1}]^{1/2}} \equiv \exp\left[i\mathscr{S}\right]$$

after introducing fluctuations $\mathscr{S} = \mathscr{S}_{\rm mf} + \mathscr{S}_{\rm eff}$

$$\mathscr{S}_{\rm mf} = -\frac{i}{2}\mathrm{Tr}\ln[S_0 S_{\rm mf}^{-1}] - \frac{i}{g}\left[\Delta_{AB} W_{ABCD} \Delta^*_{CD}\right]$$

mean field part (gap equation)

$$\mathscr{S}_{\rm eff} = -\frac{i}{g}\int d^4x\ 2\Delta^{\rm mf}_{AB} W_{ABCD}\rho(x)_{CD} + \rho_{AB}(x) W_{ABCD}\rho_{CD}(x) - \mathrm{Tr}\ln\left(1 + S_{\rm mf}\Gamma_0\right)$$

effective part with all interaction terms

High Density Effective Theory - 3

$$\int \frac{d^4p}{(2\pi)^4} = \frac{\mu^2}{(2\pi)^4}\int d\Omega \int_{-\delta}^{+\delta} d\ell_{\parallel} \int_{-\infty}^{+\infty} d\ell_0 = \int \frac{d\mathbf{v}}{4\pi}\frac{\mu^2}{\pi}\int \frac{d^2\ell}{(2\pi)^2}$$

the measure

$$\psi(x) = \int \frac{d\mathbf{v}}{4\pi} e^{-i\mu v\cdot x}\left[\psi_+(x) + \psi_-(x)\right]$$

$$\psi_\pm(x) = P_\pm \psi_{\mathbf{v}}(x) = P_\pm \frac{\mu^2}{\pi}\int \frac{d^2\ell}{(2\pi)^2} e^{-i\ell\cdot x}\psi_{\mathbf{v}}(\ell)$$

we introduce the velocity dependent, positive- and negative-energy fields

$$P_\pm = \frac{1 \pm \boldsymbol{\alpha}\cdot\mathbf{v}_F}{2}$$

$$V^\mu = (1,\, \mathbf{v}) \quad \tilde{V}^\mu = (1,\, -\mathbf{v}) \quad \gamma_\parallel^\mu = (\gamma^0,\, (\mathbf{v}\cdot\boldsymbol{\gamma})\,\mathbf{v}) \quad \gamma_\perp^\mu = \gamma^\mu - \gamma_\parallel^\mu$$

$$\begin{aligned}
\bar\psi_+\gamma^\mu\psi_+ &= V^\mu\bar\psi_+\gamma^0\psi_+ \ ,\\
\bar\psi_-\gamma^\mu\psi_- &= \tilde{V}^\mu\bar\psi_-\gamma^0\psi_- \ ,\\
\bar\psi_+\gamma^\mu\psi_- &= \bar\psi_+\gamma_\perp^\mu\psi_- \ ,\\
\bar\psi_-\gamma^\mu\psi_+ &= \bar\psi_-\gamma_\perp^\mu\psi_+ .
\end{aligned}$$

useful identities to recast the Lagrangian terms

High Density Effective Theory - 4

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi}\left[\psi_+^\dagger iV\cdot D\psi_+ + \psi_-^\dagger(2\mu + i\tilde{V}\cdot D)\psi_- + (\bar{\psi}_+ i\not{D}_\perp\psi_- + h.c.)\right].$$

$$\psi_- = \frac{-i\gamma_0\gamma_\perp^\mu D_\mu}{2\mu + i\tilde{V}\cdot D}\psi_+$$

$$iV\cdot D\psi_+ = 0$$

To get rid of the negative energy solutions, we integrate out the Ψ_- in the generating functional. @ tree level, the infinite μ limit shows the decoupling of Ψ_-

$$\mathcal{L}_D = \int \frac{d\mathbf{v}}{4\pi}\left[\psi_+^\dagger iV\cdot D\psi_+ - P^{\mu\nu}\psi_+^\dagger\frac{1}{2\mu + i\tilde{V}\cdot D}D_\mu D_\nu\psi_+\right]$$

with $P^{\mu\nu} = g^{\mu\nu} - \frac{1}{2}\left[V^\mu\tilde{V}^\nu + V^\nu\tilde{V}^\mu\right]$

To make the Lagrangian more symmetric we introduce

$\psi_\pm \equiv \psi_{+,\pm\vec{v}}$ with the new sum over Fermi velocities $\sum_{\vec{v}} = \int \frac{d\vec{v}}{8\pi}$

High Density Effective Theory - 6

Finally $$\mathcal{L}_D = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$$

$$\mathcal{L}_0 = \sum_{\vec{v}} \left[\psi_+^\dagger iV \cdot \partial \psi_+ + \psi_-^\dagger i\tilde{V} \cdot \partial \psi_-\right]$$

Kinetic term for fermions

$$\mathcal{L}_1 = i\, g \sum_{\vec{v}} \left[\psi_+^\dagger iV \cdot A \psi_+ + \psi_-^\dagger i\tilde{V} \cdot A \psi_-\right]$$

Gluon-fermions coupling

$$\mathcal{L}_2 = -\sum_{\vec{v}} P^{\mu\nu} \left[\psi_+^\dagger \frac{1}{2\mu + i\tilde{V} \cdot D} D_\mu D_\nu \psi_+ + \psi_-^\dagger \frac{1}{2\mu + iV \cdot D} D_\mu D_\nu \psi_-\right]$$

Non local Lagrangian, coupling two quarks with any number of gluons, giving contribution to Meissner mass

The CFL condensate

In CFL phase the symmetry breaking is induced by the condensate

$$\langle \psi^{L\,T}_{\alpha i} C \psi^{L}_{\beta j} \rangle = -\langle \psi^{R\,T}_{\alpha i} C \psi^{R}_{\beta j} \rangle = \frac{\Delta}{2}\,\epsilon_{\alpha\beta I}\epsilon_{ijI}$$

After introducing the following notation for a different basis of quark fields

$$\chi = \begin{pmatrix} \psi_+ \\ C\psi_-^* \end{pmatrix} \qquad\qquad \psi_\pm = \frac{1}{\sqrt{2}}\sum_{A=1}^{9} \lambda_A \psi_\pm^A$$

The Lagrangian containing the BCS-like interaction will be

$$\sum_{\vec{v}} \sum_{A,B=1}^{9} \chi^{A\dagger} \begin{pmatrix} iTr[T_A V\cdot D\, T_B] & \Delta_{AB} \\ \Delta_{AB} & iTr[T_A \tilde{V}\cdot D^*\, T_B] \end{pmatrix} \chi^B \;+\; (L \to R)$$

$$\Delta_{AB} = \Delta_A \delta_{AB} \quad \Delta_1 = \cdots = \Delta_8 = \Delta \qquad \Delta_9 = -2\Delta \qquad T_A = \frac{\lambda_A}{\sqrt{2}}$$

Notes

$$\psi(x) = \int \frac{d\mathbf{v}}{4\pi} e^{-i\tilde{\mu} v x} \left(\psi_+(x) + \psi_-(x)\right) \qquad \mathscr{L}_0 = \int \frac{d\mathbf{v}}{4\pi} \psi_+^\dagger (iV^\mu \partial_\mu + v \cdot x V^\mu \partial_\mu \tilde{\mu} - V^i \partial_i \phi/2) \psi_+$$

$$\mathscr{L}_D = \int \frac{dv}{8\pi} \sum_{A,B=1}^{9} \chi^{\dagger\, A} \begin{pmatrix} i\, V \cdot D\, \delta_{AB} & -\Delta_{AB} \\ -\Delta_{AB} & i\, V \cdot D^*\, \delta_{AB} \end{pmatrix} \chi^B + (L \to R)$$

$$\mathscr{L}_{\chi\chi AA} = \int \frac{dv}{8\pi} \sum_{A,B=1}^{9} \chi^{\dagger\, A} \begin{pmatrix} \frac{-P^{\mu\nu}}{2\mu + \tilde{V}\cdot \ell} \delta_{AB} & 0 \\ 0 & \frac{-P^{\mu\nu}}{2\mu + V\cdot \ell} \delta_{AB} \end{pmatrix} A_\mu A_\nu \chi^B$$

$$\mathscr{S}_{\text{eff}}^{(1)} = -i \text{Tr} \int d^4 x\, S_{AB}(x,x) \Gamma_{BC}(x) \qquad \Gamma_{AB}(0) = \text{diag} \left[\begin{pmatrix} \frac{-P^{\mu\nu}}{2\mu + \tilde{V}\cdot \ell} & 0 \\ 0 & \frac{-P^{\mu\nu}}{2\mu + V\cdot \ell} \end{pmatrix}, \ldots, \begin{pmatrix} \frac{-P^{\mu\nu}}{2\mu + \tilde{V}\cdot \ell} & 0 \\ 0 & \frac{-P^{\mu\nu}}{2\mu + V\cdot \ell} \end{pmatrix} \right]$$

$$[S_{\text{MF}}(p)]_{AB} = \text{diag} \left[\underbrace{\frac{1}{D(p)} \begin{pmatrix} \tilde{V} \cdot p & \Delta \\ \Delta^* & V \cdot p \end{pmatrix}, \ldots,}_{8 \text{ times}} \frac{1}{D_9(p)} \begin{pmatrix} \tilde{V} \cdot p & -2\Delta \\ -2\Delta^* & V \cdot p \end{pmatrix} \right]$$

$$\Sigma_{AB}(0) = \text{diag} \left[\begin{pmatrix} -V^\mu \partial_\mu \phi & 0 \\ 0 & \tilde{V}^\mu \partial_\mu \phi \end{pmatrix}, \ldots, \begin{pmatrix} -V^\mu \partial_\mu \phi & 0 \\ 0 & \tilde{V}^\mu \partial_\mu \phi \end{pmatrix} \right]$$

$$\mathscr{S}^{(2)} = + \frac{i}{2} \frac{\mu^2}{2\pi^3} \text{Tr} \int \frac{d\mathbf{v}}{8\pi} \int d^2 \ell\ [S(\ell)\Sigma(0)]^2 = \int \frac{d\mathbf{v}}{8\pi} \tilde{V}^\mu V^\nu \frac{9}{2} \frac{\mu^2}{\pi^2} \partial_\mu \phi \partial_\nu \phi\,.$$

Microscopic effective Lagrangian

$$\frac{N_c N_f}{12\pi^2}\left[\mu^4 - 4\mu^3\partial_0\varphi + 6\mu^2(\partial_0\varphi)^2 - 2\mu^2(\partial_i\varphi)^2 - 4\mu\partial_0\varphi\partial_\mu\varphi\partial^\mu\varphi + (\partial_\mu\varphi\partial^\mu\varphi)^2\right]$$

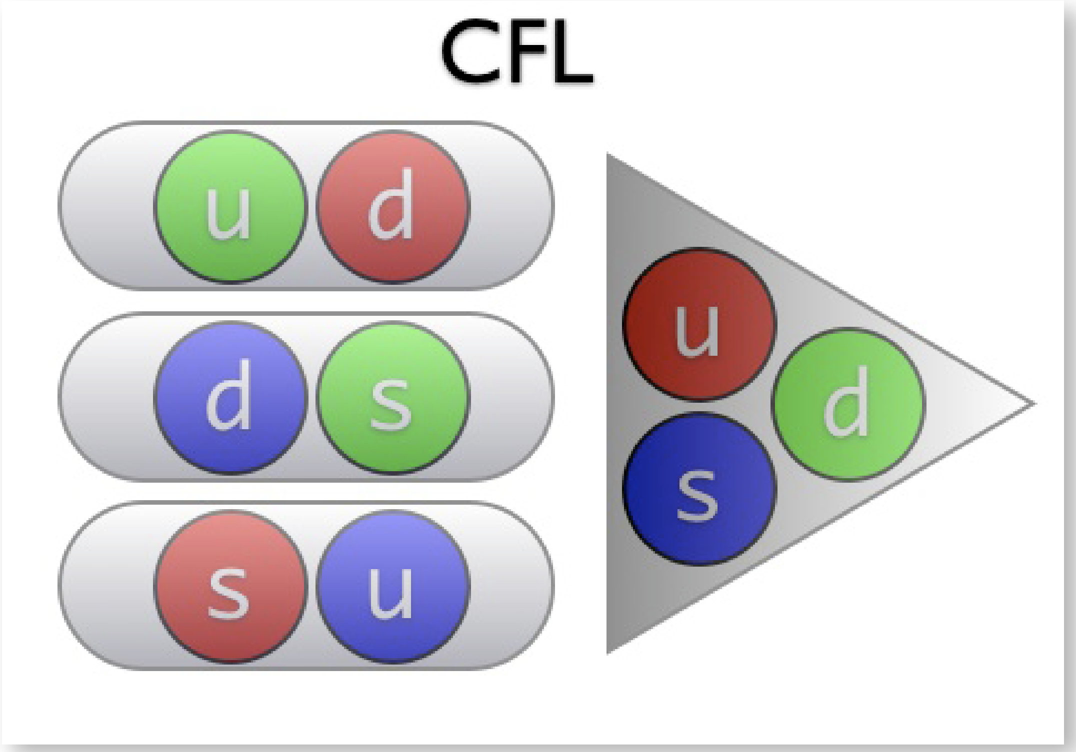
+

$$\frac{3\mu^2}{8\pi^2\Delta^2}\frac{1}{2}\left((\partial_0\rho(x))^2 - \frac{1}{3}(\partial_i\rho(x))^2\right) - \frac{1}{2}\frac{12\mu^2}{\pi^2}\rho(x)^2$$

+

$$-\frac{1}{2}\frac{12}{\pi^2}\left[(\mu - \partial_0\varphi)^2 - (\nabla\varphi)^2\right]\rho^2(x)$$

Symmetry breaking patterns of CFL



$$\langle \psi^T_{\alpha i} C \psi_{\beta j} \rangle = \frac{\Delta}{2} \epsilon_{\alpha\beta I} \epsilon_{ijI}$$

3 flavors participate to the pairing. All nine quarks present a **gap** in the dispersion relation.

$$SU(3)_c \times SU(3)_R \times SU(3)_L \times U(1)_B \rightarrow SU(3)_{c+L+R} \times \mathbb{Z}_2$$

Spontaneous breaking of

chiral simmetry = octet of Goldstone modes

color gauge group = gauge bosons acquire a Meissner m.

$U(1)_B$ = a superfluid Goldstone mode (phonon)

Notes

A finite baryon chemical potential breaks Lorentz invariance of the effective theory. The temporal and spatial decay constants can thereby differ. This difference is encoded in the velocity factor v being different from unity. An explicit calculation shows that $v = 1/\sqrt{3}$ and is common to all Goldstone bosons, including the massless U(1)B Goldstone boson [15].

Notice that if instead of $P \sim \mu^4$ the equation of state was $P \sim \mu^2$, then the superfluid mode would move with the speed of light and is non-interacting. Such a situation occurs, for example, in QCD at finite isospin chemical potential μ_I when $m \ll \mu_I \ll 4\pi f$

A rotating star has two sets of low-frequency modes, the spheroidal g-modes and the toroidal r-modes.