Effective Continuum Thresholds for Quark–Hadron Duality in Dispersive Sum Rules

W. Lucha,[1] **D. Melikhov,**[1,2,3] **and S. Simula**[4]

[1] HEPHY, Austrian Academy of Sciences, Vienna, Austria
[2] Faculty of Physics, University of Vienna, Austria
[3] SINP, Moscow State University, Russia
[4] INFN, Sezione di Roma III, Roma, Italy

Incentive: improving QCD sum rules [1–14]

Within the method of QCD sum rules, the concept of quark–hadron duality is implemented by assuming that above a certain 'continuum threshold' the contributions to suitably defined correlators at the level of the QCD degrees of freedom equal those at the level of hadronic bound states. Here we seek to quantify the uncertainty induced by this approximation and to improve the accuracy of predictions by allowing the threshold to depend on the involved momenta and parameters introduced upon applying Borel transformations.

This idea is best tested in a situation where the outcome for all bound-state characteristics is known exactly. So, we study a quantum-mechanical model defined by a nonrelativistic Hamiltonian with harmonic-oscillator potential

$$H=\frac{\mathbf{p}^2}{2\,m}+\frac{m\,\omega^2\,r^2}{2}\qquad r\equiv|\mathbf{x}|$$

exactly solvable: ground-state (g) energy, decay constant, and form factor

$$E_{\rm g}=\frac{3}{2}\,\omega\qquad R_{\rm g}\equiv|\psi_{\rm g}(\mathbf{x}=\mathbf{0})|^2=\left(\frac{m\,\omega}{\pi}\right)^{3/2}\qquad F_{\rm g}(q)=\exp\left(\frac{-q^2}{4\,m\,\omega}\right)$$

Borelized polarization operator/2-point vacuum–vacuum correlator [1–5,8]

$$\Pi(T)\equiv\langle\mathbf{x}_{\rm f}=\mathbf{0}|\exp(-H\,T)|\mathbf{x}_{\rm i}=\mathbf{0}\rangle\stackrel{\rm HO}{=}\left[\frac{m\,\omega}{2\,\pi\sinh(\omega\,T)}\right]^{3/2}$$

double-Borelized 3-current correlator involving current operator $J(\mathbf{q})$ [6,9]

$$
\begin{aligned}
\Gamma(T_2, T_1, q) &\equiv \langle \mathbf{x}_\mathrm{f} = \mathbf{0} | \exp(-H\,T_2)\, J(\mathbf{q}) \exp(-H\,T_1) | \mathbf{x}_\mathrm{i} = \mathbf{0} \rangle \\
&\xrightarrow{T_{1,2}\to\infty} R_\mathrm{g} \exp[-E_\mathrm{g}\,(T_1 + T_2)]\, F_\mathrm{g}(q)
\end{aligned}
$$

equal Borel "times" $T_1 = T_2 = \dfrac{T}{2}$:

$$
\begin{aligned}
\Gamma(T, q) &\overset{\mathrm{HO}}{=} \Pi(T) \exp\left[-\frac{q^2}{4\,m\,\omega} \tanh\left(\frac{\omega\,T}{2}\right)\right] \\
&\xrightarrow{T\to\infty} R_\mathrm{g} \exp(-E_\mathrm{g}\,T)\, F_\mathrm{g}(q) \equiv \Gamma_\mathrm{g}(T, q)
\end{aligned}
$$

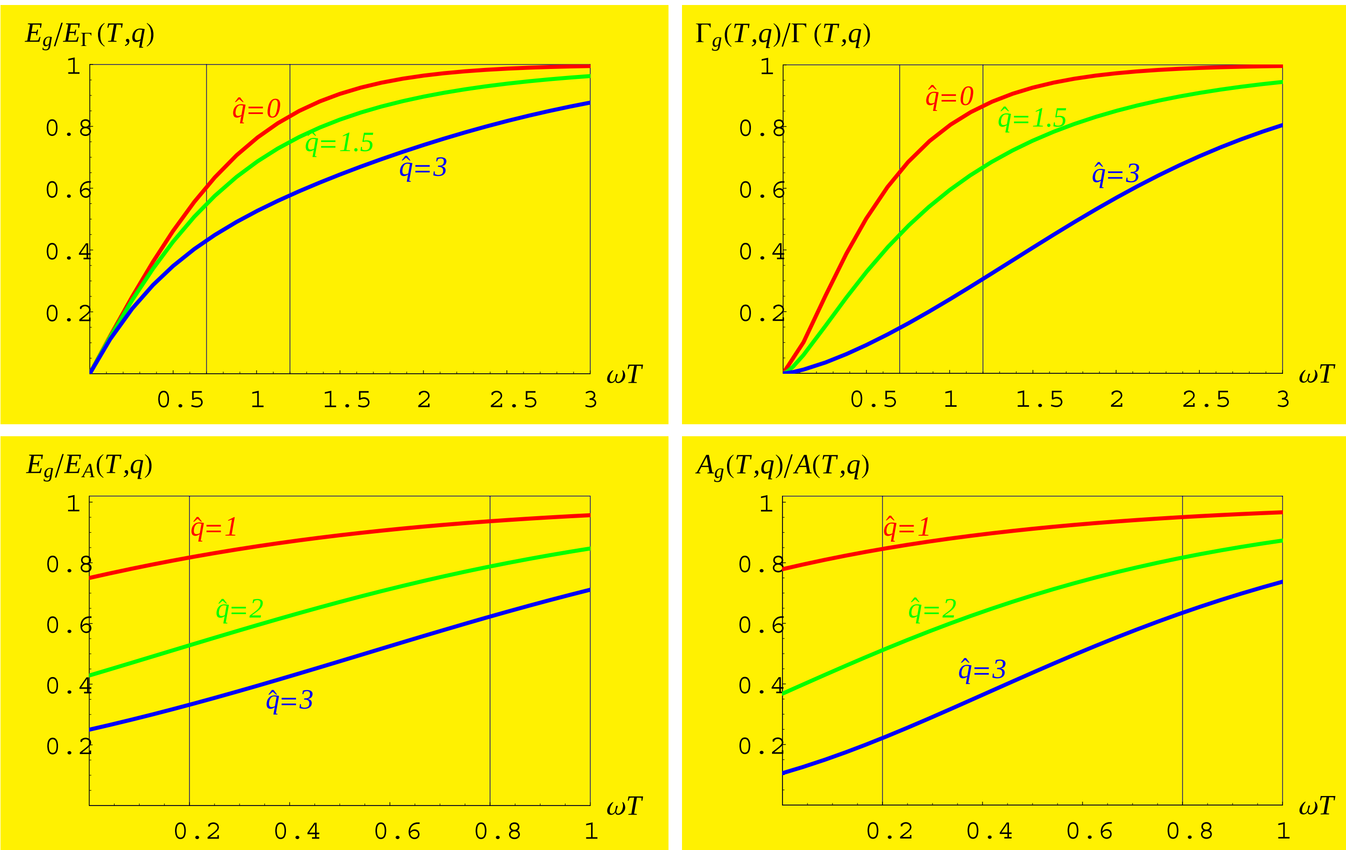
Borelized vacuum–hadron amplitude of 2-quark-currents T-product [7,12]

$$
\begin{aligned}
A(T, q) &\equiv \langle \mathbf{x} = \mathbf{0} | \exp(-H\,T)\, J(\mathbf{q}) | \psi_\mathrm{g} \rangle \\
&\xrightarrow{T\to\infty} \sqrt{R_\mathrm{g}} \exp(-E_\mathrm{g}\,T)\, F_\mathrm{g}(q) \equiv A_\mathrm{g}(T, q)
\end{aligned}
$$

define average energies similarly for $A(T, q)$

$$
E_\Gamma(T, q) \equiv -\frac{\partial}{\partial T} \log \Gamma(T, q) = \frac{3}{2}\,\omega \coth(\omega\,T) + \frac{q^2}{4\,m\,[1 + \cosh(\omega\,T)]}
$$

Relative ground-state contribution to $E_\Gamma(T, q)$ and $\Gamma(T, q)$ (first row) resp. $E_A(T, q)$ and $A(T, q)$ (second row), for several dimensionless $\hat{q} \equiv q/\sqrt{m\,\omega}$



Duality $\Longleftrightarrow$ effective continuum threshold

- perform operator product expansion (OPE) $\widehat{=}$ power series in T [1–3,8]
- represent perturbative part of correlator in form of a dispersion integral
- invoke quark–hadron duality: skip any contribution to correlator above its T- (maybe q-) dependent effective continuum threshold $z_{\rm eff}^{\Pi,\Gamma,A}(T,q)$

$$\Pi_{\rm g}(T)\equiv R_{\rm g}\exp(-E_{\rm g}\,T)$$

$$\stackrel{\rm SR}{=}\Pi_{\rm dual}(T,z_{\rm eff}^{\Pi}(T))\equiv\int_0^{z_{\rm eff}^{\Pi}(T)}{\rm d}z\exp(-z\,T)\,\rho_0(z)+\Pi_{\rm power}(T)$$

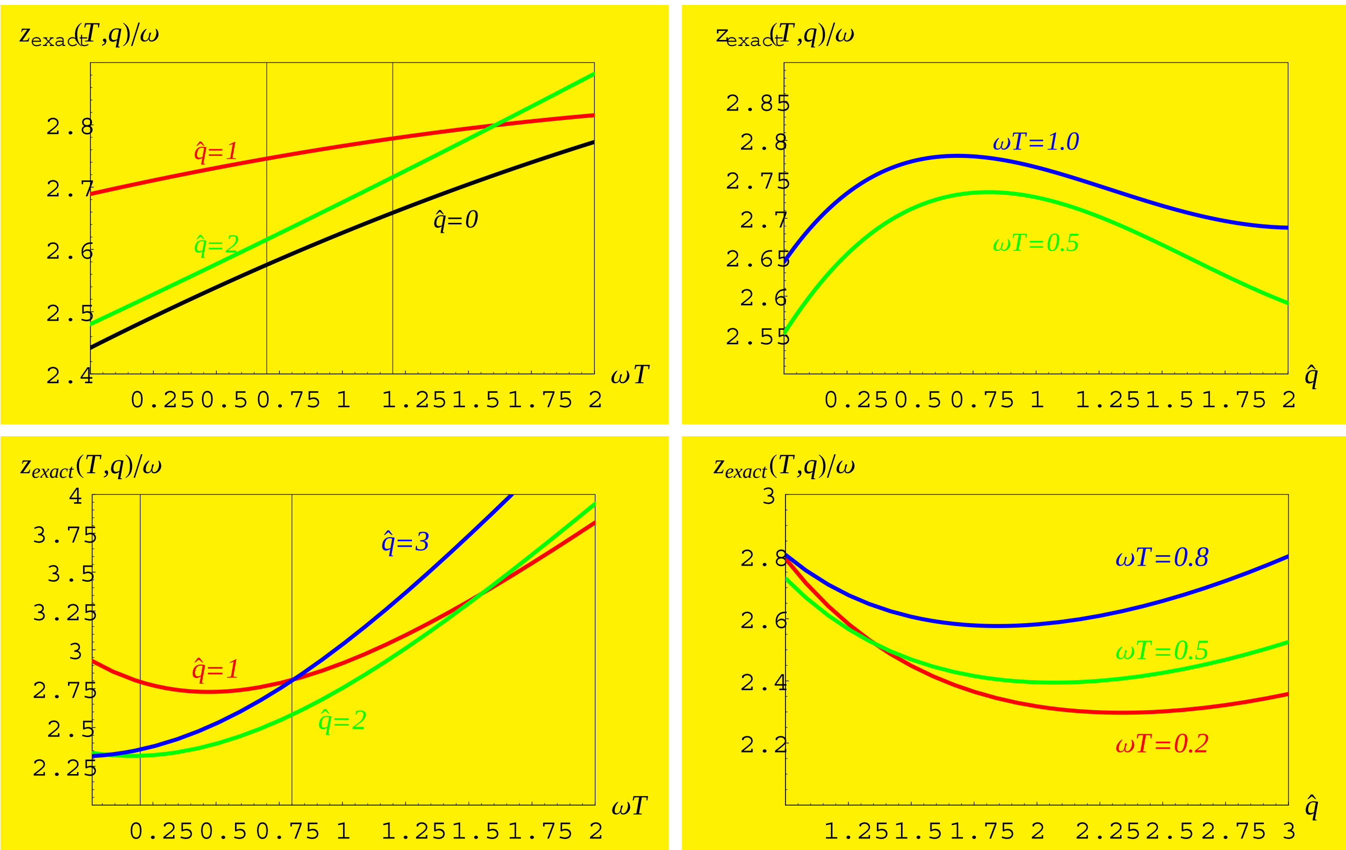
$$\Gamma_{\rm g}(T,q)\equiv R_{\rm g}\exp(-E_{\rm g}\,T)\,F_{\rm g}(q)\stackrel{\rm SR}{=}\Gamma_{\rm dual}(T,q,z_{\rm eff}^{\Gamma}(T,q))$$

$$\equiv\int_0^{z_{\rm eff}^{\Gamma}(T,q)}{\rm d}z_1\int_0^{z_{\rm eff}^{\Gamma}(T,q)}{\rm d}z_2\exp\left(-\frac{z_1+z_2}{2}\,T\right)\Delta_0(z_1,z_2,q)+\Gamma_{\rm power}(T,q)$$

$$A_{\rm g}(T,q)\equiv\sqrt{R_{\rm g}}\exp(-E_{\rm g}\,T)\,F_{\rm g}(q)$$

$$\stackrel{\rm SR}{=}A_{\rm dual}(T,q,z_{\rm eff}^{A}(T,q))\equiv\int_0^{z_{\rm eff}^{A}(T,q)}{\rm d}z\exp(-z\,T)\,\rho_A(z,T,q)$$

Exact effective continuum thresholds $z_{\rm exact}^{\Gamma}(T,q)$ (first row) and $z_{\rm exact}^{A}(T,q)$ (second row): $\Gamma(T_2,T_1,0)=\Pi(T_1+T_2)$ implies $z_{\rm exact}^{\Gamma}(T,q=0)=z_{\rm exact}^{\Pi}(T)$



effective continuum threshold: ansatz [10–12] $z_{\rm eff}^{(n)}(T,q)=\sum\limits_{j=0}^{n} z_j^{(n)}(q)\,(\omega\,T)^j$

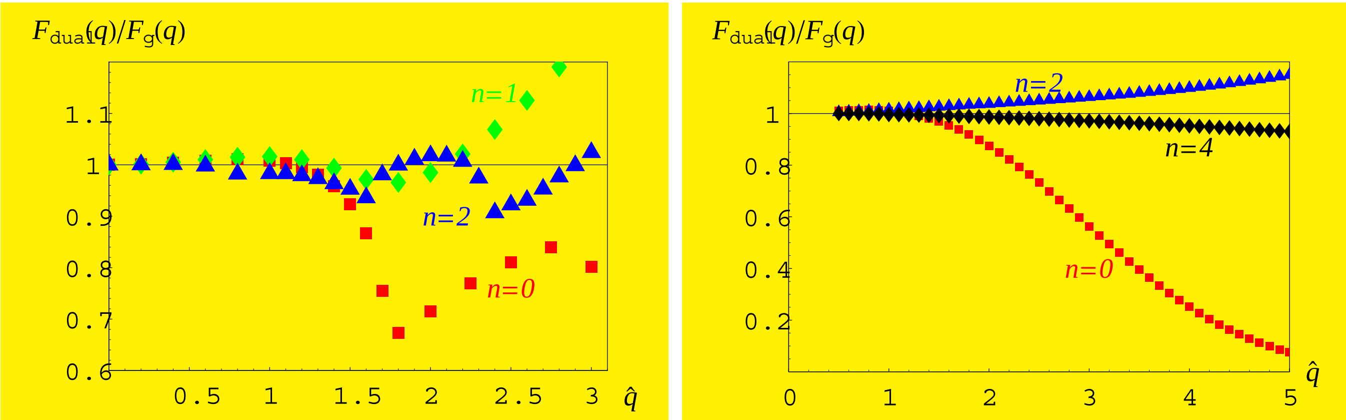
define dual energies

$$E_{\rm dual}^{\Gamma}(T,q)\equiv-\frac{\rm d}{{\rm d}T}\log\Gamma_{\rm dual}(T,q,z_{\rm eff}^{\Gamma}(T,q))$$

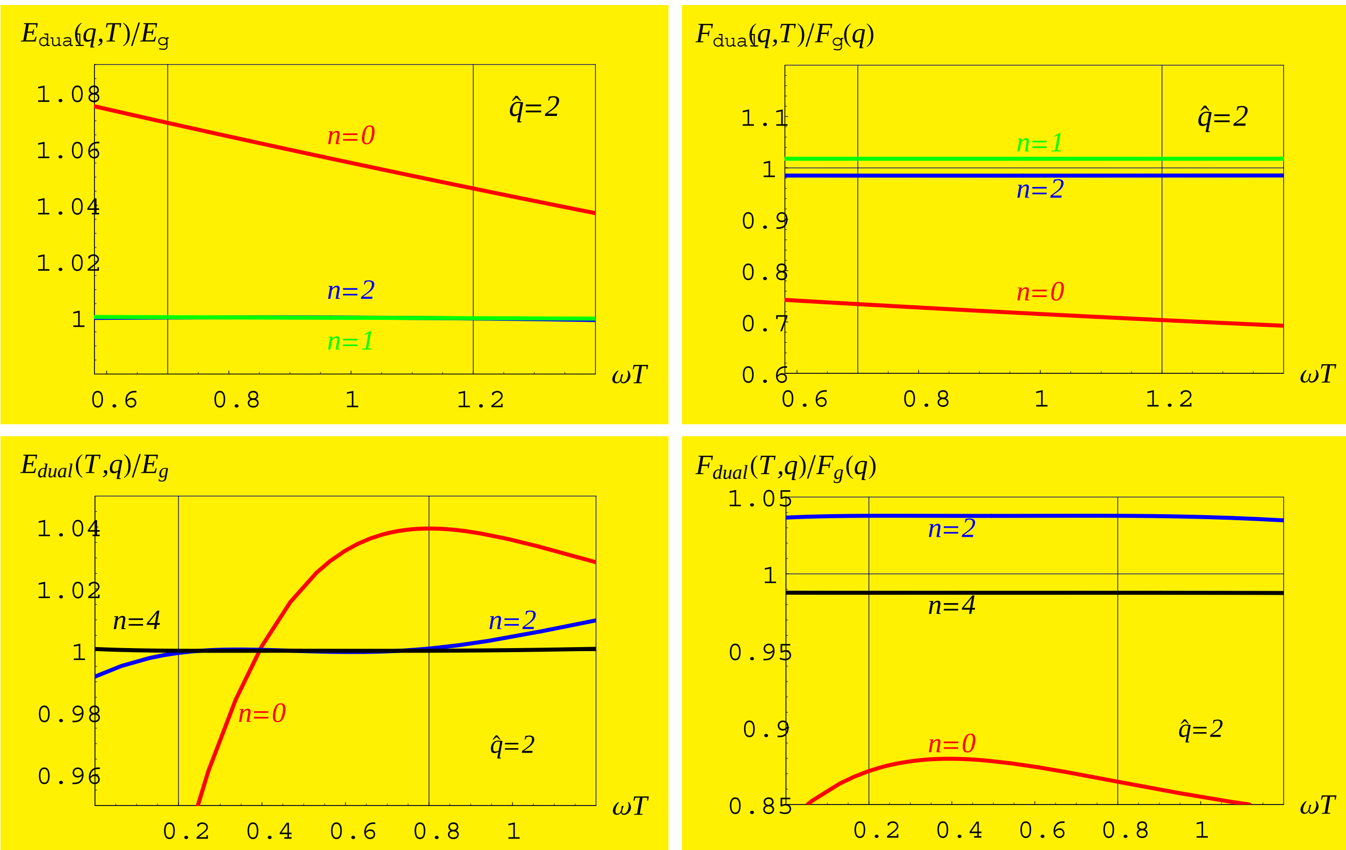
$$E_{\rm dual}^{A}(T,q)\equiv-\frac{\rm d}{{\rm d}T}\log A_{\rm dual}(T,q,z_{\rm eff}^{A}(T,q))$$

minimization ($T_{i\le N}\in$ Borel window) $\quad\chi^2\equiv\frac{1}{N}\sum\limits_{i=1}^{N}\left[E_{\rm dual}^{\Gamma,A}(T_i,q)-E_{\rm g}\right]^2$

Dual form factor $F_{\rm dual}(q)$ from correlators $\Gamma(T,q)$ (left) and $A(T,q)$ (right)



Fitted dual energy $E_{\rm dual}(T,q)$ (left) and form factor $F_{\rm dual}(T,q)$ (right) for correlator $\Gamma(T,q)$ (upper row) and $A(T,q)$ (lower row) @ $\hat q\equiv q/\sqrt{m\,\omega}=2$



Bridging the gap: QCD $\approx$ potential models

QCD [13]

study Borelized correlator of 2 pseudoscalar currents $j_5 \equiv (m_b+m_u)\,\bar{u}\,\mathrm{i}\,\gamma_5\,b$

$$\Pi(\tau) = \int_{(m_b+m_u)^2}^{\infty} \mathrm{d}s\,\exp(-s\,\tau)\,\rho_{\mathrm{pert}}(s,\mu) + \Pi_{\mathrm{power}}(\tau,\mu)$$

$$\stackrel{\mathrm{SR}}{=} f_B^2\,M_B^4\,\exp(-M_B^2\,\tau) + \cdots \xrightarrow{\tau\to\infty} f_B^2\,M_B^4\,\exp(-M_B^2\,\tau) \equiv \Pi_{\mathrm{g}}(\tau)$$

perturbative spectral density: strong coupling α_{s} @ renormalization scale μ

$$\rho_{\mathrm{pert}}(s,\mu) = \rho_0(s,\mu) + \frac{\alpha_{\mathrm{s}}(\mu)}{\pi}\,\rho_1(s,\mu) + \frac{\alpha_{\mathrm{s}}^2(\mu)}{\pi^2}\,\rho_2(s,\mu) + \cdots$$

decay constant f_B of B meson defined by $\qquad (m_b+m_u)\,\langle 0|j_5|B\rangle = f_B\,M_B^2$

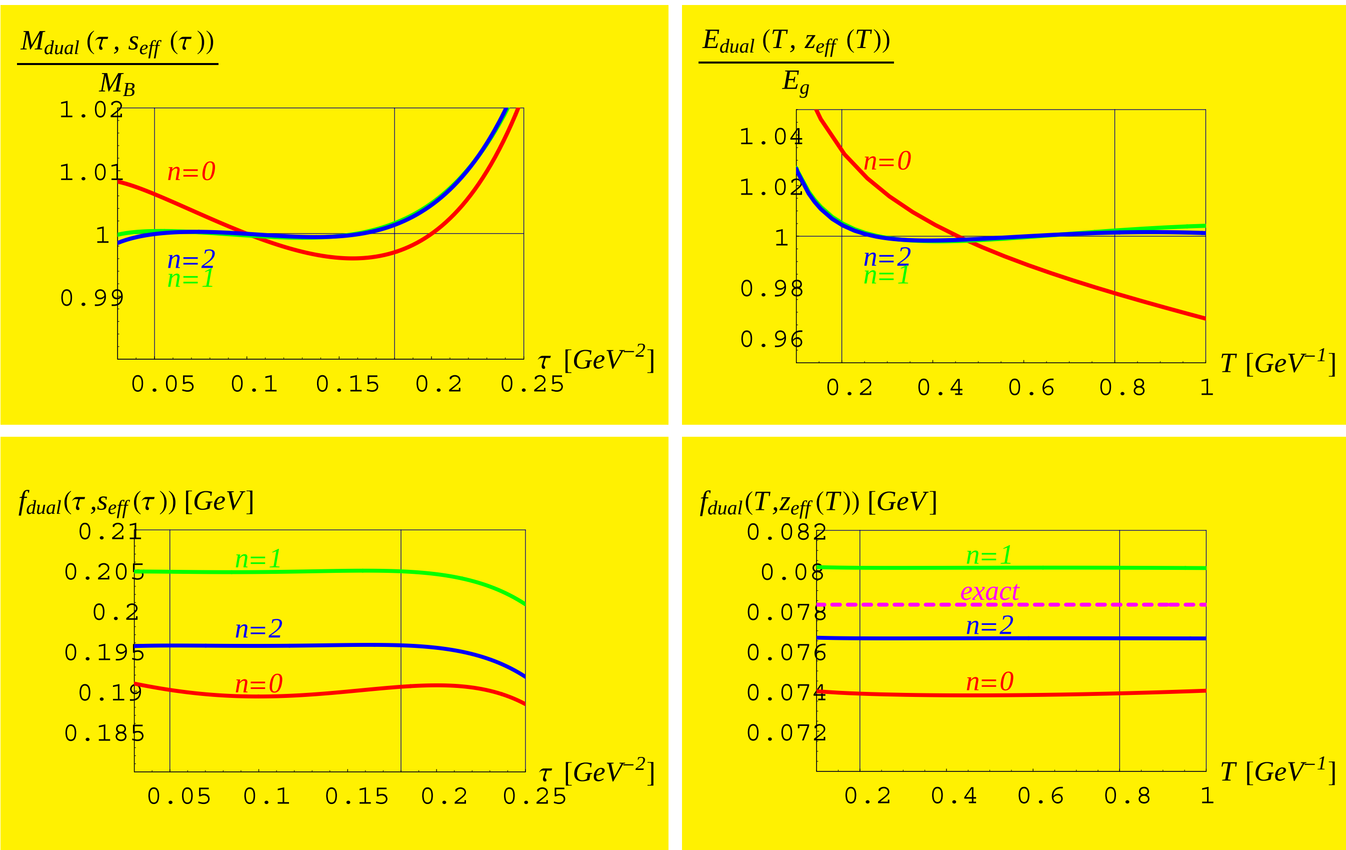
duality $\qquad \Pi_{\mathrm{dual}}(\tau, s_{\mathrm{eff}}(\tau)) \equiv \int_{(m_b+m_u)^2}^{s_{\mathrm{eff}}(\tau)} \mathrm{d}s\,\exp(-s\,\tau)\,\rho_{\mathrm{pert}}(s,\mu) + \Pi_{\mathrm{power}}(\tau,\mu)$

$$\stackrel{\mathrm{SR}}{=} \Pi_{\mathrm{g}}(\tau) \equiv f_B^2\,M_B^4\,\exp(-M_B^2\,\tau)$$

introduce dual mass $\qquad M_{\mathrm{dual}}^2(\tau) \equiv -\frac{\mathrm{d}}{\mathrm{d}\tau}\log\Pi_{\mathrm{dual}}(\tau, s_{\mathrm{eff}}(\tau))$

& decay constant $\qquad f_{\mathrm{dual}}^2(\tau) \equiv M_B^{-4}\,\exp(M_B^2\,\tau)\,\Pi_{\mathrm{dual}}(\tau, s_{\mathrm{eff}}(\tau))$

Dual masses and decay constants in QCD (left) and potential model (right)



power-law ansatz for effective continuum threshold [13] $s_{\rm eff}^{(n)}(\tau)=\sum_{j=0}^{n} s_j^{(n)}\,\tau^j$

minimization ($\tau_{i\le N}\in$ Borel window) $\qquad \chi^2\equiv\frac{1}{N}\sum_{i=1}^{N}\left[M_{\rm dual}^2(\tau_i)-M_B^2\right]^2$

Comparison with quantum-mechanical potential models [14]

confining harmonic-oscillator interaction + attractive Coulomb interaction

$$H=\frac{\mathbf{p}^2}{2\,m}+\frac{m\,\omega^2\,r^2}{2}-\frac{\alpha}{r}\qquad r\equiv|\mathbf{x}|$$

Observations, results, findings, conclusions

- exact effective continuum thresholds depend on Borel parameter T and on relevant momenta q, and are not universal (i.e. vary with correlator)
- proposed algorithm both improves dramatically accuracy of traditional sum-rule predictions and yields firm estimate of intrinsic uncertainties
- striking similarity, even quantitatively, of hadron-parameter extraction procedures in potential models and in QCD calls for QCD applications

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