Bethe–Salpeter Equations with Instantaneous Confinement: Establishing Stability of Bound States

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Naïvely implemented confinement allows for unstable states

The most popular 3-dimensional reduction of the Bethe–Salpeter formalism for the description of bound states within quantum field theory is Salpeter's equation, found as the instantaneous limit of the Bethe–Salpeter framework if allowing, in addition, for free propagation of the bound-state constituents. Unfortunately, depending on the chosen Dirac nature of the Bethe–Salpeter kernel which encodes all interactions between the bound-state constituents, supposedly stable results of the Salpeter equation with confining interaction exhibit instabilities (probably related to Klein's paradox). Such instabilities have been reported by numerous—mainly numerical—studies investigating confining interactions of, in configuration space, either solely linear shape [1] or a type interpolating between harmonic-oscillator and linear behaviour [2]. Generally, for kernels being mixtures of Lorentz scalar and time-component Lorentz vector, stability can be assured—for linear confinement—only if the kernel's Dirac structure is predominantly a time-component Lorentz vector.

Building on experience gained in previous analyses focused to the simpler reduced Salpeter equation [3,4] and its improvement [5] by including dressed propagators for all bound-state constituents [6–8], this investigation aims at rigorous proofs [9] of the full-Salpeter solutions' stability for confining (such as harmonic-oscillator) interactions of frequently chosen Lorentz structures, identifying the kernels for which stability may be taken as granted *ab initio*. Bound states can be regarded as stable if, for appropriate interactions, their energy eigenvalues or, in the center-of-momentum system, mass eigenvalues belong to a real, discrete (part of the) spectrum that is bounded from below.

Full Salpeter formalism for fermion–antifermion bound states

Assuming, as usual, the Lorentz structures of the effective couplings of both fermion and antifermion to be represented by identical Dirac matrices Γ and denoting the associated Lorentz-scalar interaction function by $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$, the Salpeter eigenvalue equation for a fermion–antifermion bound state of mass M and the distribution $\Phi(\boldsymbol{p})$ of internal momenta $\boldsymbol{p}$ reads, in the rest frame,

$$\Phi(\boldsymbol{p}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \sum_\Gamma V_\Gamma(\boldsymbol{p},\boldsymbol{q}) \left(\frac{\Lambda^+(\boldsymbol{p})\,\gamma_0\,\Gamma\,\Phi(\boldsymbol{q})\,\Gamma\,\Lambda^-(\boldsymbol{p})\,\gamma_0}{M - 2\,E(p)} - \frac{\Lambda^-(\boldsymbol{p})\,\gamma_0\,\Gamma\,\Phi(\boldsymbol{q})\,\Gamma\,\Lambda^+(\boldsymbol{p})\,\gamma_0}{M + 2\,E(p)} \right),$$

with one-particle kinetic energy $E(p)$ and energy projectors $\Lambda^\pm(\boldsymbol{p})$ given by

$$E(p) \equiv \sqrt{p^2 + m^2}\,, \quad p \equiv |\boldsymbol{p}|\,, \quad \text{and} \quad \Lambda^\pm(\boldsymbol{p}) \equiv \frac{E(p) \pm \gamma_0\,(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m)}{2\,E(p)}\,,$$

where m is the common mass of the bound fermion and its antiparticle. The equation's projector structure constrains $\Phi(\boldsymbol{p})$ to $\Lambda^\pm(\boldsymbol{p})\,\Phi(\boldsymbol{p})\,\Lambda^\pm(-\boldsymbol{p}) = 0$.

General features of eigenvalue spectra of Salpeter's equation

The random-phase-approximation nature of the Salpeter equation or direct inspection allow to establish several characteristics common to all solutions:

- Although the squares of the mass eigenvalues, M^2, are guaranteed to be real [10] the spectrum is in general not necessarily real and even in those cases where it can be shown to be real it is not bounded from below [10].
- In particular, for the maybe most important example of Bethe–Salpeter kernels involving only coupling matrices Γ satisfying $\gamma_0\,\Gamma^\dagger\,\gamma_0 = \pm\Gamma$ and potential functions $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ satisfying $V_\Gamma^*(\boldsymbol{p},\boldsymbol{q}) = V_\Gamma(\boldsymbol{p},\boldsymbol{q}) = V_\Gamma(\boldsymbol{q},\boldsymbol{p})$ the spectrum of mass eigenvalues M consists (in the complex-M plane) of real opposite-sign pairs $(M, -M)$ and imaginary points $M = -M^*$.

Since eigenvalues embedded in a continuous part of the spectrum may cause instabilities the total spectrum is crucial: apart from demanding eigenvalues to be real and bounded from below stability is established if eigenvalues and continuous spectrum are disjoint or if the spectrum is purely discrete at all.

Harmonic-oscillator confinement simplifies Salpeter's *integral* equation to a system of radial eigenvalue *differential* equations

Since the instabilities under consideration are expected to appear first in the pseudoscalar sector [1], the main target of all analyses of the present kind are bound states with a spin-parity-charge conjugation assignment $J^{PC} = 0^{-+}$; their Salpeter amplitude $\Phi(\boldsymbol{p})$ involves two independent components ϕ_1, ϕ_2:

$$\Phi(\boldsymbol{p}) = [\phi_1(\boldsymbol{p})\,\gamma_0\,(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m)/E(p) + \phi_2(\boldsymbol{p})]\,\gamma_5$$

is the unique form of Φ for fermion–antifermion bound states of total spin J, parity $P = (-1)^{J+1}$, and charge-conjugation quantum number $C = (-1)^J$.

Let the interaction kernels be of convolution type $[V_\Gamma(\boldsymbol{p},\boldsymbol{q}) = V_\Gamma(\boldsymbol{p}-\boldsymbol{q})]$, arising from a central potential $V(r)$, $r \equiv |\boldsymbol{x}|$, in configuration space. Using harmonic-oscillator potentials $V(r) = a\,r^2$, where $a \neq 0$ avoids triviality, to exemplify the line of reasoning, the Salpeter equation reduces to a system of (second-order) differential equations involving only the differential operator

$$D \equiv \frac{\mathrm{d}^2}{\mathrm{d}p^2} + \frac{2}{p}\,\frac{\mathrm{d}}{\mathrm{d}p} \qquad \text{(which is just the Laplacian } \Delta \text{ acting on } \ell = 0 \text{ states)}\ .$$

Its study is greatly simplified for massless bound-state constituents: $m = 0$.

The time-component Lorentz-vector ($\Gamma\otimes\Gamma = \gamma^0\otimes\gamma^0$) and Lorentz-scalar ($\Gamma\otimes\Gamma = 1\otimes 1$) interactions may be discussed simultaneously by introducing

$$\sigma = \begin{cases} +1 & \text{for } \Gamma\otimes\Gamma = \gamma^0\otimes\gamma^0 \quad \text{(time-component Lorentz vector)}\ , \\ -1 & \text{for } \Gamma\otimes\Gamma = 1\otimes 1 \quad \text{(Lorentz scalar)}\ . \end{cases}$$

By factorizing off all dependence on angular variables, the Salpeter equation for harmonic-oscillator interactions of both time-component Lorentz-vector and Lorentz-scalar structure is given by a set of radial differential equations:

$$\begin{aligned} (2\,p - a\,\sigma\,D)\,\phi_2(p) &= M\,\phi_1(p)\ , \\ \left[2\,p - a\left(D - 2/p^2\right)\right]\phi_1(p) &= M\,\phi_2(p)\ . \end{aligned}$$

Adding the $\sigma = +1$ and $\sigma = -1$ sets yields their version for Lorentz-scalar/time-component Lorentz-vector mixing $\Gamma\otimes\Gamma = \xi\,\gamma^0\otimes\gamma^0 + \eta\,1\otimes 1$ $(\xi,\eta \in \mathbb{R})$

$$\begin{aligned} [2\,p - a\,(\xi - \eta)\,D]\,\phi_2(p) &= M\,\phi_1(p)\ , \\ \left[2\,p - a\,(\xi + \eta)\left(D - 2/p^2\right)\right]\phi_1(p) &= M\,\phi_2(p)\ . \end{aligned}$$

Clearly, w.l.o.g. the units of the momentum can be chosen such that $|a| = 1$.

Spectral analysis of the simplest case: confining interactions of time-component Lorentz-vector Dirac form ($\Gamma\otimes\Gamma=\gamma^0\otimes\gamma^0$)

For the time-component Lorentz-vector coupling ($\sigma=+1$), the existence of bound states requires $a>0$. In this case a totally analytic proof of the so far only numerically observed stability of the bound states may be constructed. Expressed by the positive self-adjoint operators on the Hilbert space $L^2(\mathbb{R}^3)$

$$A\equiv-\Delta+2\,r=A^\dagger\geq 0\ ,\qquad B\equiv-\Delta+2\,r+\frac{2}{r^2}=B^\dagger\geq 0\ ,\qquad r\equiv|\boldsymbol{x}|\ ,$$

our radial Salpeter equation with harmonic-oscillator confining interactions is just the sector of vanishing angular momentum of the eigenvalue problem

$$\begin{array}{l}A\,f_2=M\,f_1\\ B\,f_1=M\,f_2\end{array}\qquad\Longleftrightarrow\qquad\left(\begin{array}{cc}0&A\\ B&0\end{array}\right)\left(\begin{array}{c}f_1\\ f_2\end{array}\right)=M\left(\begin{array}{c}f_1\\ f_2\end{array}\right).$$

'Squaring' this relation gives an equivalent problem for real eigenvalues M^2:

$$\left(\begin{array}{cc}A\,B&0\\ 0&B\,A\end{array}\right)\left(\begin{array}{c}f_1\\ f_2\end{array}\right)=M^2\left(\begin{array}{c}f_1\\ f_2\end{array}\right)\qquad\Longleftrightarrow\qquad\begin{array}{l}A\,B\,f_1=M^2\,f_1\ ,\\ B\,A\,f_2=M^2\,f_2\ .\end{array}$$

The spectral theorem for self-adjoint operators allows to define the (unique) positive self-adjoint square root $A^{1/2}=(A^{1/2})^\dagger\geq 0$ of A. With this, setting $g\equiv A^{1/2}\,f_2$ converts the relation $B\,A\,f_2=M^2\,f_2$ to an eigenvalue equation $Q\,g=M^2\,g$ of a positive self-adjoint operator: $Q\equiv A^{1/2}\,B\,A^{1/2}=Q^\dagger\geq 0$. Moreover, multiplying the operator inequality $A\leq B$, clearly satisfied by A and B, from left and right by $A^{1/2}$ transforms it to $A^2\leq A^{1/2}\,B\,A^{1/2}\equiv Q$.

A well-known theorem about the spectra of Hamiltonians with potentials increasing without bounds states that a Schrödinger operator $H\equiv-\Delta+V$, defined as sum of quadratic forms, with locally bounded, positive, infinitely rising potential V [$V(\boldsymbol{x})\to+\infty$ for $r\to\infty$] has purely discrete spectrum. Application of this theorem shows that the spectrum of A is purely discrete. By the spectral theorem for self-adjoint operators, A^2 may be represented in terms of the same projection-valued spectral measure as A; accordingly, the spectrum of A^2 is also purely discrete. As positive operators, both A^2 and Q are—trivially—bounded from below. Combining [11] the characterization of all discrete eigenvalues of a self-adjoint operator bounded from below by the famous minimum–maximum principle with the operator inequality $A^2\leq Q$

shows that the spectrum of the real squared mass eigenvalues M^2 of Q must be purely discrete and therefore the spectrum of bound-state masses M too.

Stability of the full-Salpeter bound states as shown for a time-component Lorentz-vector ($\sigma = 1$) harmonic-oscillator interaction with coupling $a > 0$ excludes the possibility that there instabilities of bound states are caused by the associated energy eigenvalues being embedded in a continuous spectrum of the Salpeter operator controlling all bound states, instead of belonging to its discrete spectrum as expected for true confinement. The proof cannot be copied for $a < 0$, since there the counterparts of the operators A and B read

$$\widetilde{A} \equiv \Delta + 2\,r \ , \quad \widetilde{B} \equiv \Delta + 2\,r - \frac{2}{r^2} \ .$$

Positivity is lost for both $\widetilde{A}$ and $\widetilde{B}$, invalidating thus most steps of the proof.

Confining interaction kernels of Lorentz-scalar ($\Gamma \otimes \Gamma = 1 \otimes 1$) or Lorentz-pseudoscalar ($\Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5$) Dirac nature

In the limit of massless bound-state constituents, our Salpeter equations for Lorentz-scalar (i.e., $\sigma = -1$) and Lorentz-pseudoscalar interactions become identical—just as manifestation of chiral symmetry. The operators involved here are $\widetilde{A}$ and B for $a > 0$, and A and $\widetilde{B}$ for $a < 0$. In both of these pairs of operators, one of the latter is not positive, spoiling the above stability proof.

Linear combinations of both time-component Lorentz-vector and Lorentz-scalar confining interaction kernels

From the above, for arbitrary mixtures of scalar and time-component vector Lorentz structure the Salpeter eigenvalue problem is posed by the operators

$$\mathcal{A} \equiv (\alpha - \beta)\,(-\Delta) + 2\,r \ , \quad \mathcal{B} \equiv (\alpha + \beta)\left(-\Delta + \frac{2}{r^2}\right) + 2\,r \ , \quad r \equiv |\boldsymbol{x}| \ ,$$

with $\alpha \equiv a\,\xi \in \mathbb{R}$, $\beta \equiv a\,\eta \in \mathbb{R}$. Positivity of the (symmetric) operators $\mathcal{A}$, $\mathcal{B}$ and presence of both derivatives demands $\alpha - \beta > 0$ and $\alpha + \beta > 0$, which restrains the couplings to the range $\alpha = |\alpha| > |\beta| \geq 0$, or $-1 < \beta/\alpha < +1$ and $\alpha > 0$. A similar proof as for the time-component Lorentz-vector kernel establishes stability irrespective of the relative sign of the two contributions.

Level	$\beta/\alpha = \eta/\xi$								
	+0.99	+0.75	+0.50	+0.25	0.00	−0.25	−0.50	−0.75	−1.00
0	4.53	4.69	4.71	4.67	4.60	4.47	4.28	3.97	2.93
1	6.15	6.84	7.07	7.16	7.15	7.06	6.87	6.49	4.68
2	7.63	8.78	9.14	9.30	9.33	9.24	9.01	8.55	6.14
3	9.01	10.54	11.02	11.23	11.28	11.18	10.92	10.38	7.45
4	10.31	12.18	12.75	13.01	13.07	12.97	12.68	12.06	8.65
5	11.55	13.73	14.38	14.67	14.75	14.65	14.32	13.63	9.77
6	12.74	15.19	15.92	16.25	16.35	16.23	15.87	15.11	10.83
7	13.88	16.59	17.39	17.76	17.86	17.74	17.35	16.52	11.84
8	14.98	17.93	18.81	19.21	19.33	19.19	18.77	17.87	12.81
9	16.04	19.23	20.19	20.64	20.78	20.63	20.16	19.18	13.74

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