Wigner-Smith time-delay matrix for chaotic cavities: new results and a conjecture

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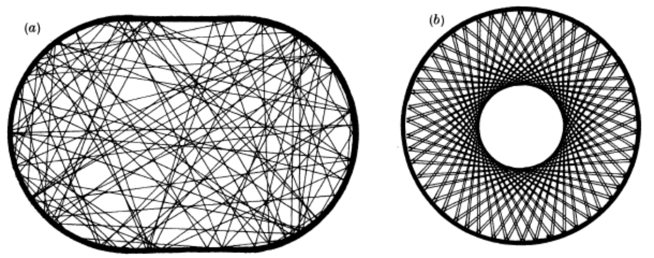
Quantum Chaology

Berry's definition

"Quantum chaology is the study of semiclassical, but non-classical, behaviour characteristic of systems whose classical motion exhibits chaos."



$$\begin{cases} \nabla^2\psi + (2mE/\hbar^2)\psi = 0 & \text{in } D \\ \psi = 0 & \text{on the boundary of } D. \end{cases}$$

One way to see the effect of the classical dynamics is to study local statistics of the energy spectrum, such as the level spacing distribution.

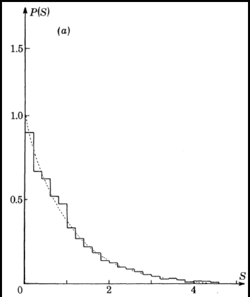
Universality conjectures

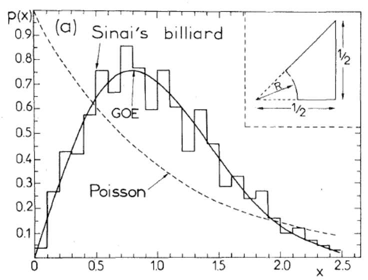
Berry-Tabor conjecture, 1977

If the classical dynamics is completely integrable, then the energy level behaves like a sequence of uncorrelated variables (Poisson statistics).

Bohigas-Giannoni-Schmit conjecture, 1984

The spectral fluctuation measures of a generic classically chaotic system coincide with those of the canonical random matrix ensemble that has the same symmetries.





Random Matrix Approach to Scattering Theory

Scattering in chaotic regime and Dyson's circular ensembles

In systems with ergodic properties, like a chaotic cavity with narrow contacts it is natural to make the following maximum entropy assumption

$S(E)$ is uniformly distributed in a subset of unitary matrices

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 6 NOVEMBER–DECEMBER 1962

The Threefold Way.
Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics

FREEMAN J. DYSON
Institute for Advanced Study, Princeton, New Jersey

$S(E)$ belongs to a 'circular ensemble' determined by physical constraints:

$$\begin{cases} \text{COE}: & U(N)/O(N) & \beta = 1, & \text{TR + SR symmetry;} \\ \text{CUE}: & U(N) & \beta = 2, & \text{No symmetry;} \\ \text{CSE}: & U(N)/Sp(N/2) & \beta = 4, & \text{TR symmetry.} \end{cases}$$

Dyson's Circular ensembles labelled by the Dyson index β.

Time-delay matrix

Time-delay in scattering

PHYSICAL REVIEW VOLUME 98, NUMBER 1 APRIL 1, 1955

Lower Limit for the Energy Derivative of the Scattering Phase Shift

EUGENE P. WIGNER
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

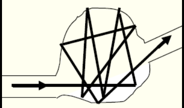
PHYSICAL REVIEW VOLUME 118, NUMBER 1 APRIL 1, 1960

Lifetime Matrix in Collision Theory*

FELIX T. SMITH
Stanford Research Institute, Menlo Park, California

The time-delay matrix Q was invented by Eisenbud (1948), Wigner (1955) and Smith (1960) in order to characterise temporal aspects of a quantum scattering process.

$$\boxed{Q = -i\hbar S^{\dagger}(E)\frac{\mathrm{d}S(E)}{\mathrm{d}E}}$$



Time-delay matrix

$$\boxed{Q = -i\hbar S^\dagger(E)\frac{\mathrm{d}S(E)}{\mathrm{d}E}}$$

$S(E)$: $N \times N$ scattering matrix;
N: number of channels;
Conservation of probability: $S(E)$ unitary; Note that $Q = Q^\dagger$.

Proper delay times

Eigenvalues of the Wigner-Smith matrix Q

$$\tau_1, \tau_2, \ldots, \tau_N$$

Wigner time delay

Time 'spent' by the particle in the scattering region

$$\mathrm{tr}Q = \tau_1 + \tau_2 + \cdots + \tau_N$$

Chaotic cavities

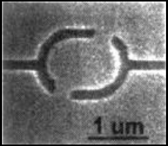
Time-delay matrix in chaotic regime

If $S(E)$ belongs to a circular ensemble then

$$P(\tau_1, \cdots, \tau_N) \propto \prod_k f(\tau_k) \cdot \underbrace{\prod_{i<j} |\tau_i - \tau_j|^\beta}_{\text{repulsion}}$$

Despite the joint distribution for τ_i was explicitly known, the statistical properties of their sum (Wigner time delay) remained unknown for a while. The same for higher moments.

$$\boxed{\mathcal{T}_\kappa = N^{\kappa-1} \mathrm{tr} Q^\kappa}$$



Review of previous results

$$\mathcal{T}_1 = N^0 \mathrm{tr} Q$$
$$\mathcal{T}_2 = N^1 \mathrm{tr} Q^2$$
$$\mathcal{T}_3 = N^2 \mathrm{tr} Q^3$$
$$\vdots$$
$$\mathcal{T}_\kappa = N^{\kappa-1} \mathrm{tr} Q^\kappa$$

- Distribution of $\mathcal{T}_1$ for $N = 1$
  Gopar, Mello and Büttiker (1996);
- Distribution of $\mathcal{T}_1$ for $N = 2$
  Savin, Fyodorov and Sommers (2001);
- Average of $\mathcal{T}_k$ for $N \gg 1$
  Berkolaiko and Kuipers (2010);
- Recursion for cumulants of $\mathcal{T}_1$ for all N
  Mezzadri and Simm (2013);
- Averages and covariances of $\mathcal{T}_1$ and $\mathcal{T}_2$ for $N \gg 1$
  Grabsch and Texier (2014);
- Representation for cumulants of $\mathcal{T}_1$ for $\beta = 2$ for all N
  Novaes (2015);

How? RMT & semiclassical methods. For $\beta = 2$ orthogonal polynomials, representation theory of the symmetric group and connection with Painlevé III.

Large number of scattering channels N

$$\mathcal{T}_\kappa = N^{\kappa-1}\mathrm{tr}Q^\kappa$$

Cumulants

$$\left\langle \mathcal{T}_{\kappa_1}\cdots\mathcal{T}_{\kappa_\nu}\right\rangle_c \sim \left(\frac{1}{\beta N^2}\right)^{\nu-1}\alpha[\kappa_1,\ldots,\kappa_\nu]$$

$\nu = 1$: averages $\left\langle \mathcal{T}_\kappa\right\rangle_c \sim \alpha[\kappa]$;

$\nu = 2$: covariances $\left\langle \mathcal{T}_{\kappa_2}\mathcal{T}_{\kappa_2}\right\rangle_c \sim \dfrac{1}{\beta N^2}\alpha[\kappa_1,\kappa_2]$

Main result: Generating functions

$$F_\nu(z_1,\ldots,z_\nu) = \frac{1}{\beta^{\nu-1}}\sum_{\kappa,\ldots,\kappa_\nu\geq 0}\alpha[\kappa_1,\ldots,\kappa_\nu]z_1^{\kappa_1}\cdots z_\nu^{\kappa_\nu}$$

[FDC, Phys. Rev. E **91***, 060102(R) (2015)]*

[FDC, F. Mezzadri, N. Simm & P. Vivo, Preprint (2015)]

Averages and covariances

$$\boxed{\mathcal{T}_\kappa = N^{\kappa-1} \mathrm{tr}\, Q^\kappa}$$

Averages $\langle \mathcal{T}_\kappa \rangle_c \sim$ 1 1 2 6 22 90 394 1 806 8 558 41 586 $\cdots$

Known in combinatorics as the 'Large Schröder numbers'.

Covariances $\langle \mathcal{T}_{\kappa_1} \mathcal{T}_{\kappa_2} \rangle_c \sim \dfrac{1}{\beta N^2} \begin{bmatrix} 4 & 24 & 132 & 720 & \cdots \\ 24 & 160 & 936 & 5312 & \\ 132 & 936 & 5700 & 33264 & \\ 720 & 5312 & 33264 & 198144 & \\ \vdots & \vdots & & & \ddots \end{bmatrix}$

[FDC, Phys. Rev. E **91**, *060102(R) (2015)]*

Higher order cumulants

Table: A few values of cumulants $\langle \mathcal{T}_{\kappa_1}, \cdots \mathcal{T}_{\kappa_v} \rangle \sim (\beta N^2)^{1-v} \alpha[\kappa_1, \ldots, \kappa_v]$.

(κ_1)	$\alpha[\kappa_1]$	(κ_1, κ_2)	$\alpha[\kappa_1, \kappa_2]$	$(\kappa_1, \kappa_2, \kappa_3)$	$\alpha[\kappa_1, \kappa_2, \kappa_3]$
(1)	1	$(1, 1)$	4	$(1, 1, 1)$	96
(2)	2	$(1, 2)$	24	$(1, 1, 2)$	848
		$(2, 2)$	160	$(1, 2, 2)$	7488
				$(2, 2, 2)$	66112

$(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$	$\alpha[\kappa_1, \kappa_2, \kappa_3, \kappa_4]$	$(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$	$\alpha[\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5]$
$(1, 1, 1, 1)$	5088	$(1, 1, 1, 1, 1)$	437760
$(1, 1, 1, 2)$	54720	$(1, 1, 1, 1, 2)$	5303808
$(1, 1, 2, 2)$	569600	$(1, 1, 1, 2, 2)$	61526016
$(1, 2, 2, 2)$	5792256	$(1, 1, 2, 2, 2)$	690596352
$(2, 2, 2, 2)$	57876480	$(1, 2, 2, 2, 2)$	7553912832
		$(2, 2, 2, 2, 2)$	80925462528

[FDC, F. Mezzadri, N. Simm & P. Vivo, Preprint (2015)]

Further results and a conjecture

Q: Are these cumulants integer-valued? If yes, what do they count?

Affirmative in these cases:

Let $\langle \mathcal{T}_{\kappa_1}, \cdots \mathcal{T}_{\kappa_v} \rangle \sim (\beta N^2)^{1-v} \alpha[\kappa_1, \ldots, \kappa_v]$.

(i) $\alpha[\kappa] \in \mathbb{N}$;

(ii) $\alpha[\kappa_1, \kappa_2] \in \mathbb{N}$;

(iii) $\alpha[\kappa_1, \kappa_2, \kappa_3] \in \mathbb{N}$;

(iv) $\alpha[\underbrace{1, \ldots, 1}_{v \text{ times}}] \in \mathbb{N}$ for all v.

(i)+(ii)+(iii)+(iv) $+$ numerical evidences suggest the following

Conjecture: $\alpha[\kappa_1, \ldots, \kappa_v] \in \mathbb{N}$.

[FDC, F. Mezzadri, N. Simm & P. Vivo, Preprint (2015)]

The conjecture

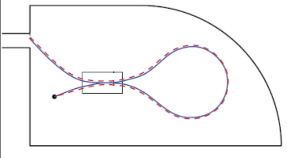
$$F_v(z_1, \ldots, z_v) = \frac{1}{\beta^{v-1}} \sum_{\kappa, \ldots, \kappa_v \geq 0} \underbrace{\alpha[\kappa_1, \ldots, \kappa_v]}_{\text{integers ?}} z_1^{\kappa_1} \cdots z_v^{\kappa_v}$$

Why should we care about this?

- For $v = 1$: $\alpha[\kappa] =$ Large Schröder numbers ;
  What is the underlying counting problem for $v > 1$?
- Semiclassics: sum rules of wave amplitudes over classical trajectories

$$S_{oi} \approx \sum_{\gamma(i \to o)} A_\gamma \exp(iS_\gamma/\hbar).$$

If semiclassics $\equiv$ RMT: heuristic argument supporting the conjecture.



Sum over ‘diagrams’ with ‘encounters’.

Take-home messages

- Connected correlators of Wigner-Smith time delay matrix for chaotic cavities supporting $N \gg 1$ scattering channels: full solution in terms of generating functions;
- Evidences that the statistical behavior of dynamics in chaotic cavities is associated to a counting problem: partial results and a conjecture. Another connection between RMT and quantum chaology.

For more details:

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Thank you and Merry Xmas!