Multiloop Corrections to Rare B Decays

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1. Introduction
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5. Summary

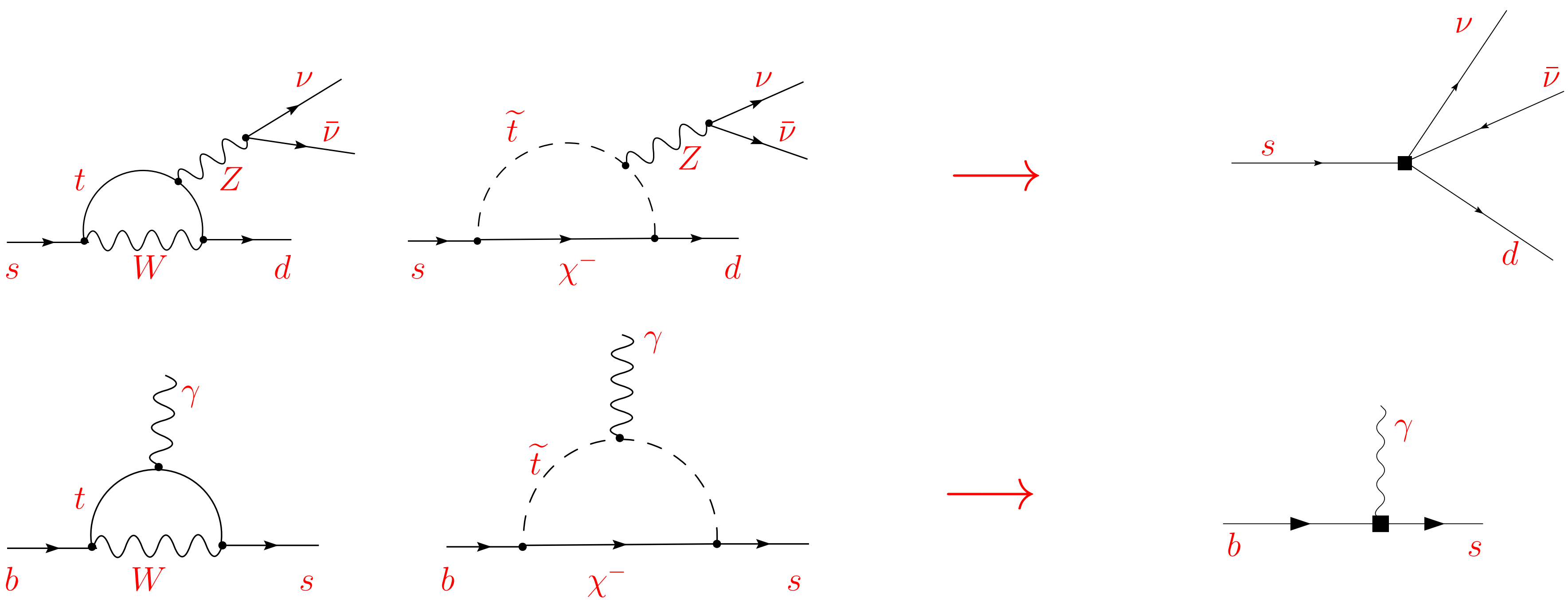
Overnight in Naples on the way to FPCapri2016:

Rare B and K decays are flavour-changing processes that occur at low energies, at scales $\mu \ll M_W$. It is convenient to pass from the full theory of electroweak interactions to an effective theory by removing the high-energy degrees of freedom, i.e. integrating out the W-boson and all the other particles with $m \sim M_W$.

$$\mathcal{L}_{\text{(full EW}\times\text{QCD)}} \longrightarrow \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QED}\times\text{QCD}}\left(\begin{smallmatrix}\text{quarks} \neq t \\ \text{\& leptons}\end{smallmatrix}\right) + \sum_n C_n(\mu)\, Q_n$$

Q_n – **local interaction terms (operators),** C_n – **coupling constants (Wilson coefficients)**

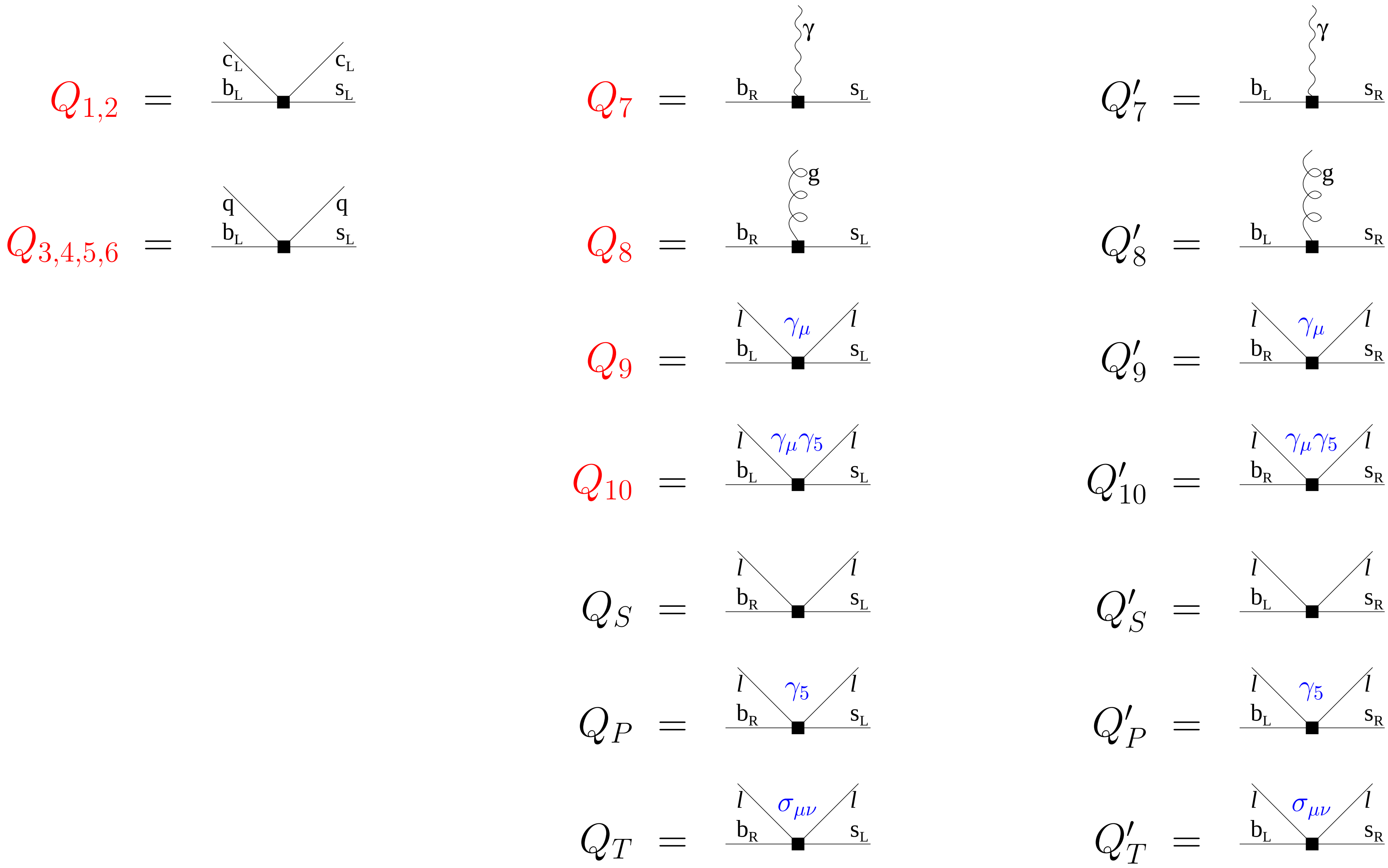
Information on the electroweak-scale physics is encoded in the values of $C_i(\mu)$ **, e.g.,**



Advantages: **Resummation of** $\left(\alpha_s \ln \frac{M_W^2}{\mu^2}\right)^n$ **using renormalization group, easier account for symmetries.**

The following vertices Q_i matter for $b \to s\gamma$ and and $b \to s l^+ l^-$:

(SM – only the red ones)



Assumption: no relevant NP effects in the 4-quark operators.

Our ability to observe or constrain new physics depends on the accuracy of determining the SM "background". Thus, precise evaluation of $C_i(\mu)$ in the SM is particularly important.

Two steps of the Wilson coefficient calculation:

Matching: Evaluating $C_i(\mu_0)$ at $\mu_0 \sim M_W$ by requiring equality of the SM and the effective theory Green's functions.

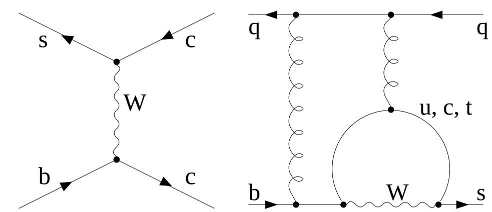
Mixing: Deriving the effective theory Renormalization Group Equations (RGE) from the renormalization constant matrices (the operators mix under renormalization).
Next, using the RGE to evolve C_i from μ_0 to μ ~(external momenta).

Operator bases can chosen in a convention-dependent manner.
For example, two possible conventions for the $|\Delta B| = |\Delta S| = 1$ four-quark operators in the SM read:

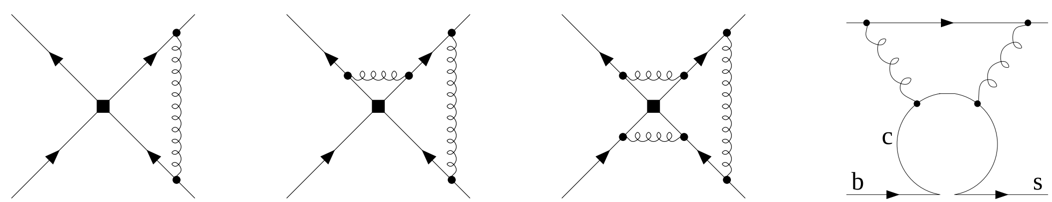
$$
\begin{aligned}
Q_1 &= (\bar{s}^\alpha_L \gamma_\mu c^\beta_L)(\bar{c}^\beta_L \gamma^\mu b^\alpha_L) & P_1 &= (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L) \\
Q_2 &= (\bar{s}^\alpha_L \gamma_\mu c^\alpha_L)(\bar{c}^\beta_L \gamma^\mu b^\beta_L) & P_2 &= (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L) \\
Q_3 &= (\bar{s}^\alpha_L \gamma_\mu b^\alpha_L) \textstyle\sum_q (\bar{q}^\beta_L \gamma^\mu q^\beta_L) & P_3 &= (\bar{s}_L \gamma_\mu b_L) \textstyle\sum_q (\bar{q} \gamma^\mu q) \\
Q_4 &= (\bar{s}^\alpha_L \gamma_\mu b^\beta_L) \textstyle\sum_q (\bar{q}^\beta_L \gamma^\mu q^\alpha_L) & P_4 &= (\bar{s}_L \gamma_\mu T^a b_L) \textstyle\sum_q (\bar{q} \gamma^\mu T^a q) \\
Q_5 &= (\bar{s}^\alpha_L \gamma_\mu b^\alpha_L) \textstyle\sum_q (\bar{q}^\beta_R \gamma^\mu q^\beta_R) & P_5 &= (\bar{s}_L \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} b_L) \textstyle\sum_q (\bar{q} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} q) \\
Q_6 &= (\bar{s}^\alpha_L \gamma_\mu b^\beta_L) \textstyle\sum_q (\bar{q}^\beta_R \gamma^\mu q^\alpha_R) & P_6 &= (\bar{s}_L \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} T^a b_L) \textstyle\sum_q (\bar{q} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} T^a q)
\end{aligned}
$$

Gilman, Wise, 1979 — Chetyrkin, Münz, MM, 1996

Matching:



Operator mixing:



Expansion in external momenta ⇒ spurious IR divergences arise.

Renormalization constant calculation using masses as IR regulators

Münz, MM, 1995	2-loop	dipole operator mixing
van Ritbergen, Vermaseren, Larin, 1997	4-loop	$\beta_{\rm QCD}$
Chetyrkin, Münz, MM, 1997	3-loop	(4-quark) $\to$ dipole
(...)		
Gambino, Gorbahn, Haisch, 2003	3-loop	(4-quark) $\to$ (quark-lepton)
Gorbahn, Haisch, 2004	3-loop	four-quark operator mixing
Czakon, 2004	4-loop	$\beta_{\rm QCD}$
Gorbahn, Haisch, MM, 2005	3-loop	dipole operator mixing
Czakon, Haisch, MM, 2006	4-loop	(4-quark) $\to$ dipole

Exact decomposition of a propagator denominator:

$$\underbrace{\frac{1}{(q+p)^2-M^2}}_{\Delta D=-2} = \underbrace{\frac{1}{q^2-m^2}}_{\Delta D=-2} + \underbrace{\frac{M^2-p^2-2qp-m^2}{q^2-m^2}\,\frac{1}{(q+p)^2-M^2}}_{\Delta D=-3}$$

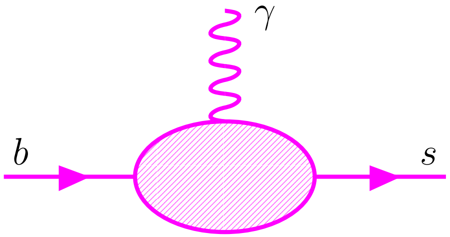
q– linear combination of loop momenta, p– linear combination of external momenta,
M– mass of the considered particle, m– IR regulator mass (arbitrary)

After applying this identity sufficiently many times, the last term can be dropped in each propagator. The only Feynman integrals to perform then are single-scale massive tadpoles.

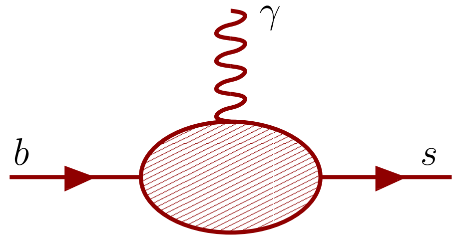
Up to three loops, explicit expressions for pole parts of all the single-scale massive tadpoles are available in terms of solved recurrences [Chetyrkin, Münz, MM, 1997] ($\leftrightarrow$ Ringberg workshop 1994).

At four loops, IBP are used for reduction to less than 20 master integrals [van Ritbergen, 1997; Schröder, 2002; Czakon, 2004] ($\leftrightarrow$ RADCOR 2002).

The matching conditions are most easily found by requiring equality of the full SM and the effective theory 1PI off-shell Green's functions that are expanded in external momenta and light masses prior to loop-momentum integration.



$=$



$+\ \mathcal{O}\left(\frac{1}{M_W^4}\right)$

Full EW theory
UV counterterms included

Spurious IR $\frac{1}{\epsilon^n}$ **remain**

Effective Theory
Loop diagrams vanish

UV $\frac{1}{\epsilon^n}$ **remain**

The $\frac{1}{\epsilon^n}$ poles cancel in the matching equation.

The only Feynman integrals to calculate: partly-massive tadpoles.

Algorithms for calculating 3-loop single-scale partly-massive tadpoles were developed in 1994-2000 [Chetyrkin, Kühn, Steinhauser; Avdeev, Fleischer, Mikhailov, Tarasov, Kalmykov; Broadhurst]. Full automatization in the code `MATAD` by M. Steinhauser (2000).

Differences among the simultaneously decoupled heavy particle masses can be taken into account by Taylor expanding around the equal-mass point. Alternatively, for large mass ratios, either asymptotic expansions or a sequence of effective theories can be applied.

Current status of the Wilson coefficient evaluation in the SM:

$\left(\bar{q}_1\,\sigma^{\alpha\beta}P_{L,R}\,q_2\right)F_{\alpha\beta}, \quad \left(\bar{q}_1\,\sigma^{\alpha\beta}P_{L,R}\,T^a q_2\right)G^a_{\alpha\beta}$ – **dipole-type, the only dim-5 ones, chirality-suppressed**

$\mathcal{O}(\alpha_s^2, \alpha_{\rm em}/\alpha_s, \alpha_{\rm em}/s_w^2)$ **known**

$(\bar{q}_1\,\gamma^\alpha P_L\,q_2)\left(\bar{l}\,\gamma_\alpha \nu\right)$ – **charged-current quark-lepton**

$\mathcal{O}(\alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^n) = 0$

$(\bar{q}_1\,\gamma^\alpha P_L\,q_2)\left(\bar{l}\,\gamma_\alpha\gamma_5\,l\right), \quad (\bar{q}_1\,\gamma^\alpha P_L\,q_2)\,(\bar{\nu}\,\gamma_\alpha\nu)$ – **neutral-current quark-lepton (Z-penguins and W-boxes)**

$\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **known**

$(\bar{q}_1\,\gamma^\alpha P_L\,q_2)\left(\bar{l}\,\gamma_\alpha\,l\right)$ – **neutral-current quark-lepton (photonic penguin)**

$\mathcal{O}(\alpha_s, \alpha_{\rm em}/\alpha_s, \alpha_{\rm em}/s_w^2)$ **known,** $\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **would be possible**

$(\bar{q}_1\,\gamma^\alpha P_L\,q_2)\,(\bar{q}_3\,\gamma^\alpha P_L\,q_4), \quad (\bar{q}_1\,\gamma^\alpha P_L\,T^a q_2)\,(\bar{q}_3\,\gamma^\alpha P_L\,T^a q_4)$ – **charged-current four-quark**

$\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^3)$ **would be possible**

$(\bar{q}_1\,\gamma^\alpha P_L\,q_2)\,(\bar{q}\,\gamma^\alpha P_{L,R}\,q), \quad (\bar{q}_1\,\gamma^\alpha P_L\,T^a q_2)\,(\bar{q}\,\gamma^\alpha P_{L,R}\,T^a q)$ – **neutral-current four-quark** $\Delta F = 1$

$\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^3)$ **would be possible**

$(\bar{q}_1\,\gamma^\alpha P_L\,q_2)\,(\bar{q}_1\,\gamma^\alpha P_L\,q_2)$ – **neutral-current four-quark** $\Delta F = 2$

$\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ **known,** $\mathcal{O}(\alpha_s^3)$ **would be possible**

$B^0_{s,d} \to \mu^+\mu^-$

- They are strongly suppressed, loop-generated process in the SM. Their average time-integrated branching ratios (with the final-state photon bremsstrahlung included) read:

$$\overline{\mathcal{B}}^{\rm SM}_{s\mu} = (3.65 \pm 0.23) \times 10^{-9}$$
$$\overline{\mathcal{B}}^{\rm SM}_{d\mu} = (1.06 \pm 0.09) \times 10^{-10}$$

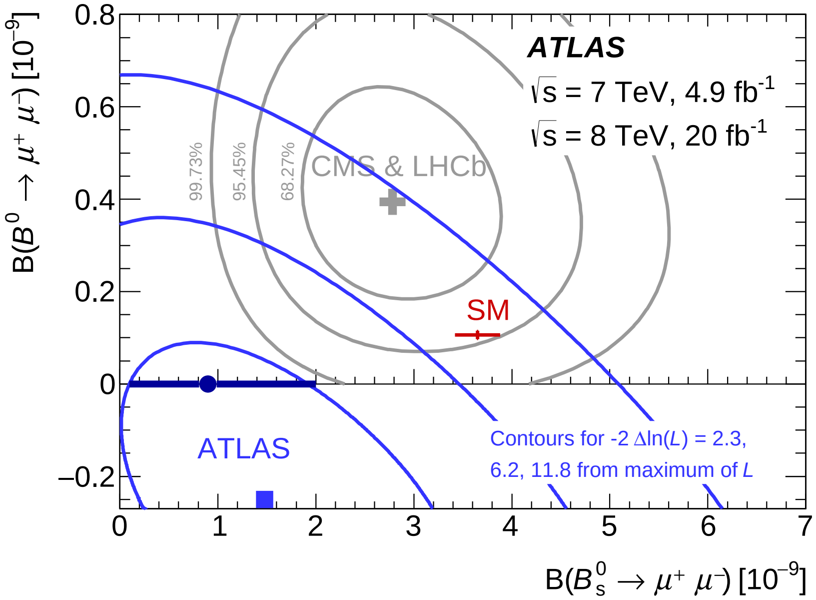
[C. Bobeth, M. Gorbahn, T. Hermann, MM, E. Stamou and M. Steinhauser, Phys. Rev. Lett. 112 (2014) 101801]

- It is very sensitive to new physics even in models with Minimal Flavour Violation (MFV). Enhancements by orders of magnitude are possible even when constraints from all the other measurements are taken into account.

- The measured branching ratios [CMS and LHCb, Nature 522 (2015) 68]

$$\overline{\mathcal{B}}^{\rm exp}_{s} = (2.8^{+0.7}_{-0.6}) \times 10^{-9}$$
$$\overline{\mathcal{B}}^{\rm exp}_{d} = (3.9^{+1.6}_{-1.4}) \times 10^{-10}$$

(ATLAS in arXiv:1604.04263 gives 95% C.L. bounds $\overline{\mathcal{B}}_s < 3.0 \times 10^{-9}$ and $\overline{\mathcal{B}}_d < 4.2 \times 10^{-10}$)



Operators (dim 6) that matter for $B_s^0 \to \mu^+\mu^-$ read

$$Q_A = (\bar{b}\gamma^\alpha\gamma_5 s)\,(\bar{\mu}\gamma_\alpha\gamma_5\mu)$$ – **the only relevant one in the SM**

$$Q_{S(P)} = (\bar{b}\gamma_5 s)\,(\bar{\mu}(\gamma_5)\mu) \;=\; \frac{i(\bar{b}\gamma^\alpha\gamma_5 s)\partial_\alpha(\bar{\mu}(\gamma_5)\mu)}{m_b+m_s} + \boxed{E} + \boxed{T}$$

$\boxed{E}$: vanishing by EOM

$\boxed{T}$: total derivative

Operators (dim 6) that matter for $B_s^0 \to \mu^+\mu^-$ read

$$Q_A = (\bar{b}\gamma^\alpha\gamma_5 s)(\bar{\mu}\gamma_\alpha\gamma_5\mu)$$ – the only relevant one in the SM

$$Q_{S(P)} = (\bar{b}\gamma_5 s)(\bar{\mu}(\gamma_5)\mu) = \frac{i(\bar{b}\gamma^\alpha\gamma_5 s)\partial_\alpha(\bar{\mu}(\gamma_5)\mu)}{m_b+m_s} + \boxed{E} + \boxed{T}$$

$\boxed{E}$: vanishing by EOM; $\boxed{T}$: total derivative

Necessary non-perturbative input: $\langle 0|\bar{b}\gamma^\alpha\gamma_5 s|B_s^0(p)\rangle = ip^\alpha f_{B_s^0}$

Recent lattice determinations of the B_s-meson decay constant:

$f_{B_s} =$

225.0(4.0) MeV,	HPQCD (r),	arXiv:1110.4510
224.0(5.0) MeV,	HPQCD (nr),	arXiv:1302.2644
234.0(6.0) MeV,	ROME,	arXiv:1212.0301
242.0(9.5) MeV,	FNAL/MILC,	arXiv:1112.3051
232.0(10) MeV,	ETM,	arXiv:1107.1441
219.0(12) MeV,	ALPHA,	arXiv:1210.6524
235.4(12) MeV,	RBC/UKQCD,	arXiv:1404.4670
224.0(14) MeV,	ALPHA,	arXiv:1404.3590

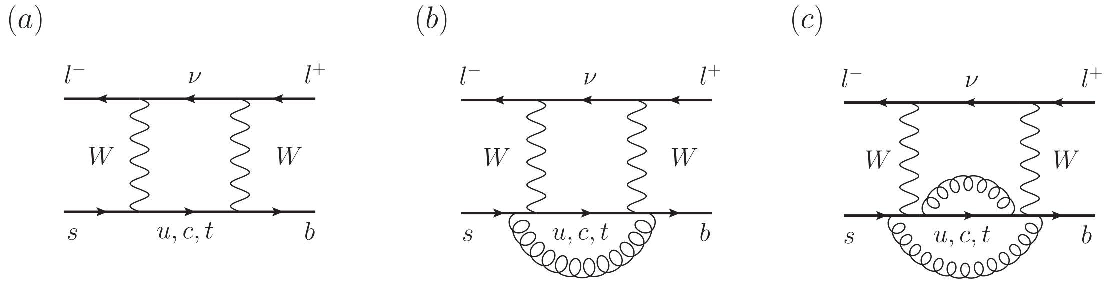
Flavour Lattice Averaging Group (FLAG), arXiv:1310.8555 gives

$$f_{B_s} = 227.7(4.5)\ \text{MeV}.$$

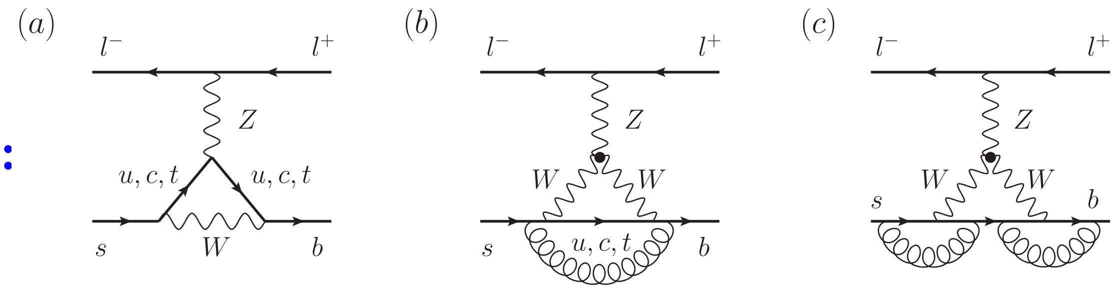
Evaluation of the NNLO QCD matching corrections in the SM

[T. Hermann, MM, M. Steinhauser, JHEP 1312 (2013) 097]

W-boxes:
(1LPI)



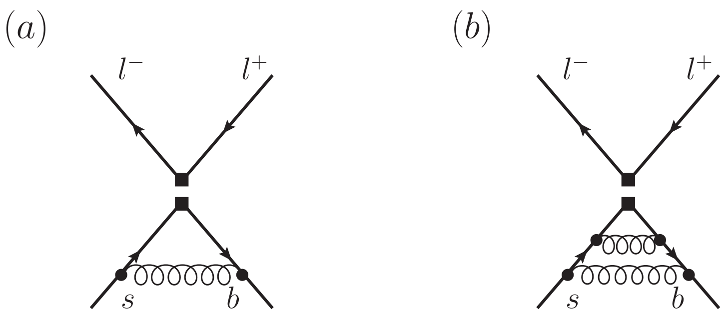
Z-penguins:
(1LPI)



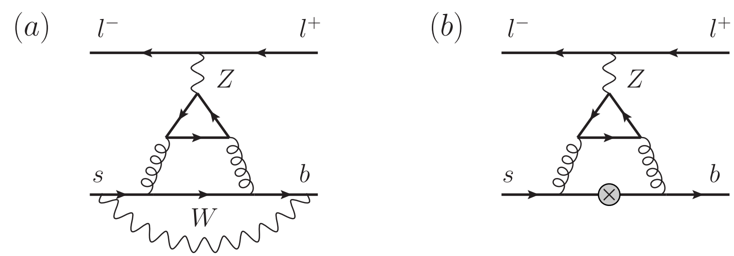
Subtleties: (i) counterterms with finite parts $\sim \bar{b}_L \not{D} s_L$

(ii) evanescent operators: $E_B = (\bar{b}\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 s)(\bar{\mu}\gamma^\sigma\gamma^\rho\gamma^\nu\gamma_5\mu) - 4(\bar{b}\gamma_\alpha\gamma_5 s)(\bar{\mu}\gamma^\alpha\gamma_5\mu)$

$E_T = \mathrm{Tr}\,(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma_5)(\bar{b}\gamma_\nu\gamma_\rho\gamma_\sigma s)(\bar{\mu}\gamma_\alpha\gamma_5\mu) + 24(\bar{b}\gamma_\alpha\gamma_5 s)(\bar{\mu}\gamma^\alpha\gamma_5\mu)$



Renormalization of E_B

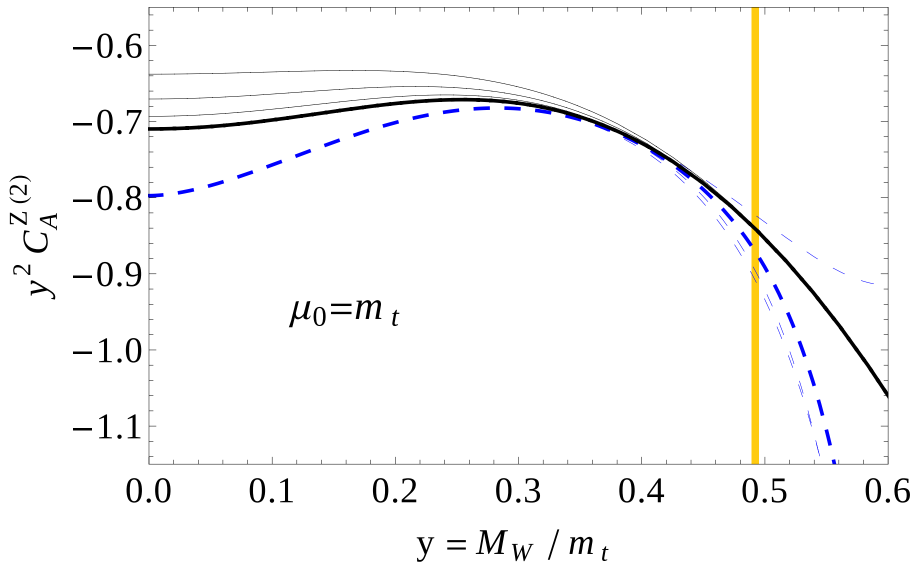


Diagrams generating E_T

Perturbative series for the Wilson coefficient at $\mu = \mu_0 \sim m_t, M_W$:

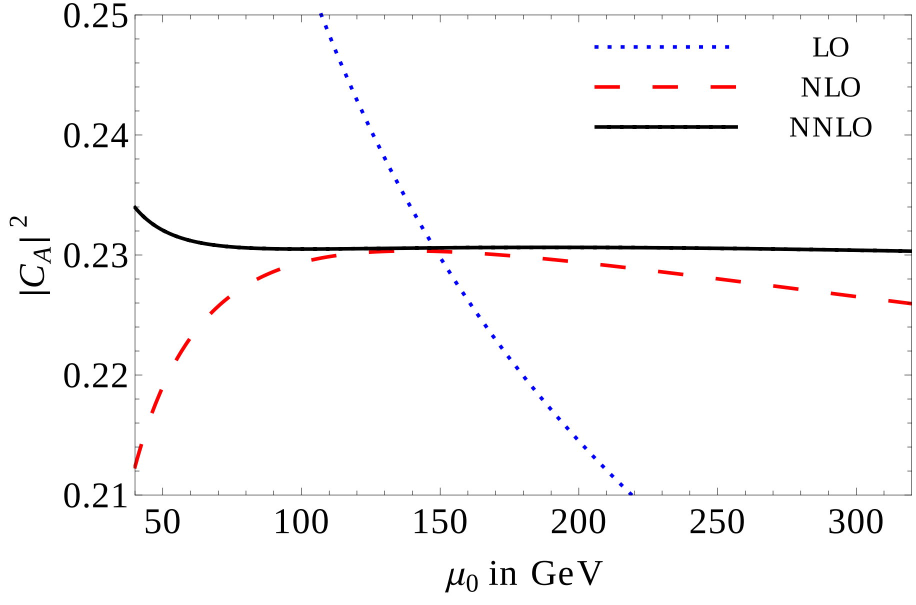
$$C_A(\mu_0) = C_A^{(0)}(\mu_0) + \frac{\alpha_s}{4\pi} C_A^{(1)}(\mu_0) + \left(\frac{\alpha_s}{4\pi}\right)^2 C_A^{(2)}(\mu_0) + \frac{\alpha_{em}}{4\pi} \Delta_{\rm EW} C_A(\mu_0) + \ldots$$

The top quark mass is $\overline{\rm MS}$-renormalized at μ_0 with respect to QCD, and on shell with respect to the EW interactions. Both α_s and α_{em} are $\overline{\rm MS}$-renormalized at μ_0 in the effective theory.



$$C_A^{(n)} = C_A^{W,(n)} + C_A^{Z,(n)}$$

To deal with single-scale tadpole integrals, we expand around $y = 1$ (solid lines) and around $y = 0$ (dashed lines), where $y = M_W/m_t$. The expansions reach $(1 - y^2)^{16}$ and y^{12}, respectively. The blue band indicates the physical region.



Matching scale dependence of $|C_A|^2$ gets significantly reduced. The plot corresponds to $\Delta_{\rm EW} C_A(\mu_0) = 0$. However, with our conventions for m_t and the global normalization, μ_0-dependence is due to QCD only.

NNLO fit (with $\Delta_{\rm EW} C_A(\mu_0) = 0$):

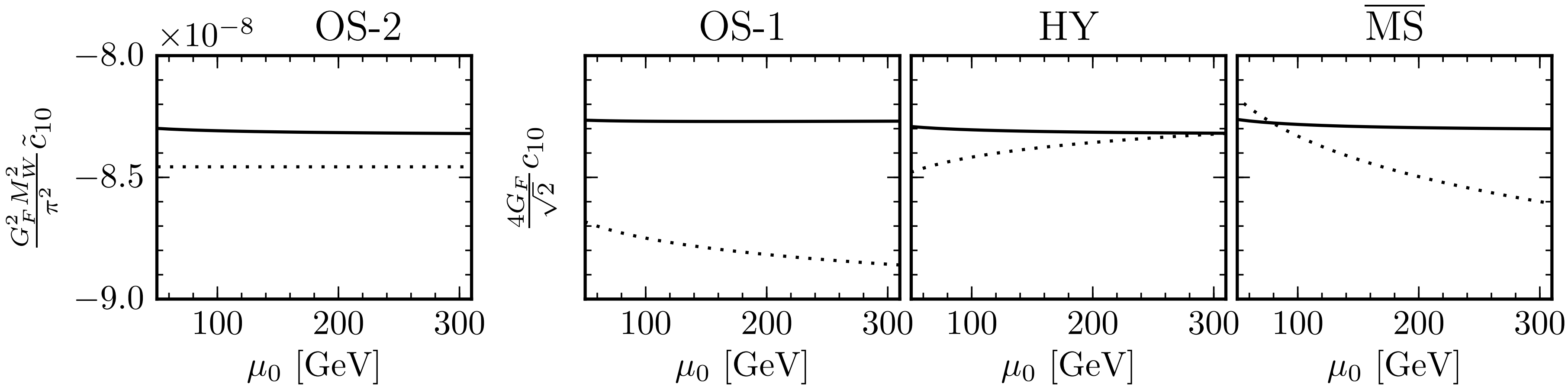
$$C_A = 0.4802 \left(\frac{M_t}{173.1}\right)^{1.52} \left(\frac{\alpha_s(M_Z)}{0.1184}\right)^{-0.09} + \mathcal{O}(\alpha_{em})$$

Evaluation of the NLO EW matching corrections in the SM

[C. Bobeth, M. Gorbahn, E. Stamou, Phys. Rev. D 89 (2014) 034023]

Method: similar to the NNLO QCD case. Two-loop integrals with three mass scales are present.

Dependence of the final result on μ_0 in various renormalization schemes (dotted – LO, solid – NLO):



In all the four plots: no QCD corrections to C_A included, $m_t(m_t)$ w.r.t. QCD used.

OS-2 scheme: Global normalization factor in $\mathcal{L}_{\text{eff}}$ set to $N = V_{tb}^* V_{ts}\, G_F^2 M_W^2/\pi^2$

Masses at the LO renormalized on-shell w.r.t. EW interactions (including M_W in N)

Plotted quantity: $-2C_A\, G_F^2 M_W^2/\pi^2$ in GeV^{-2}

NLO EW matching correction to the BR: -3.7%

other schemes: Global normalization factor in $\mathcal{L}_{\text{eff}}$ set to $4V_{tb}^* V_{ts}\, G_F/\sqrt{2}$

At the LO, $\alpha_{em}(\mu_0)$ used

$\overline{\text{MS}}$: Masses and $\sin^2\theta_W$ renormalized at μ_0

OS-1: Masses as in OS-2, $\sin^2\theta_W$ on-shell

HY (hybrid): Masses as in OS-2, $\sin^2\theta_W$ as in $\overline{\text{MS}}$.

SM predictions for all the branching ratios $\overline{\mathcal{B}}_{q\ell} \equiv \overline{\mathcal{B}}(B_q^0 \to \ell^+\ell^-)$

[C. Bobeth, M. Gorbahn, T. Hermann, MM, E. Stamou, M. Steinhauser, PRL 112 (2014) 101801]

$$\begin{aligned}
\overline{\mathcal{B}}_{se} \times 10^{14} &= (8.54 \pm 0.13)\, R_{t\alpha}\, R_s = 8.54 \pm 0.55, \\
\overline{\mathcal{B}}_{s\mu} \times 10^{9} &= (3.65 \pm 0.06)\, R_{t\alpha}\, R_s = 3.65 \pm 0.23, && (\text{LHCb \& CMS}: \ 2.8^{+0.7}_{-0.6}) \\
\overline{\mathcal{B}}_{s\tau} \times 10^{7} &= (7.73 \pm 0.12)\, R_{t\alpha}\, R_s = 7.73 \pm 0.49, \\
\overline{\mathcal{B}}_{de} \times 10^{15} &= (2.48 \pm 0.04)\, R_{t\alpha}\, R_d = 2.48 \pm 0.21, \\
\overline{\mathcal{B}}_{d\mu} \times 10^{10} &= (1.06 \pm 0.02)\, R_{t\alpha}\, R_d = 1.06 \pm 0.09, && (\text{LHCb \& CMS}: \ 3.9^{+1.6}_{-1.4}) \\
\overline{\mathcal{B}}_{d\tau} \times 10^{8} &= (2.22 \pm 0.04)\, R_{t\alpha}\, R_d = 2.22 \pm 0.19,
\end{aligned}$$

where

$$R_{t\alpha} = \left(\frac{M_t}{173.1\ \text{GeV}}\right)^{3.06} \left(\frac{\alpha_s(M_Z)}{0.1184}\right)^{-0.18},$$

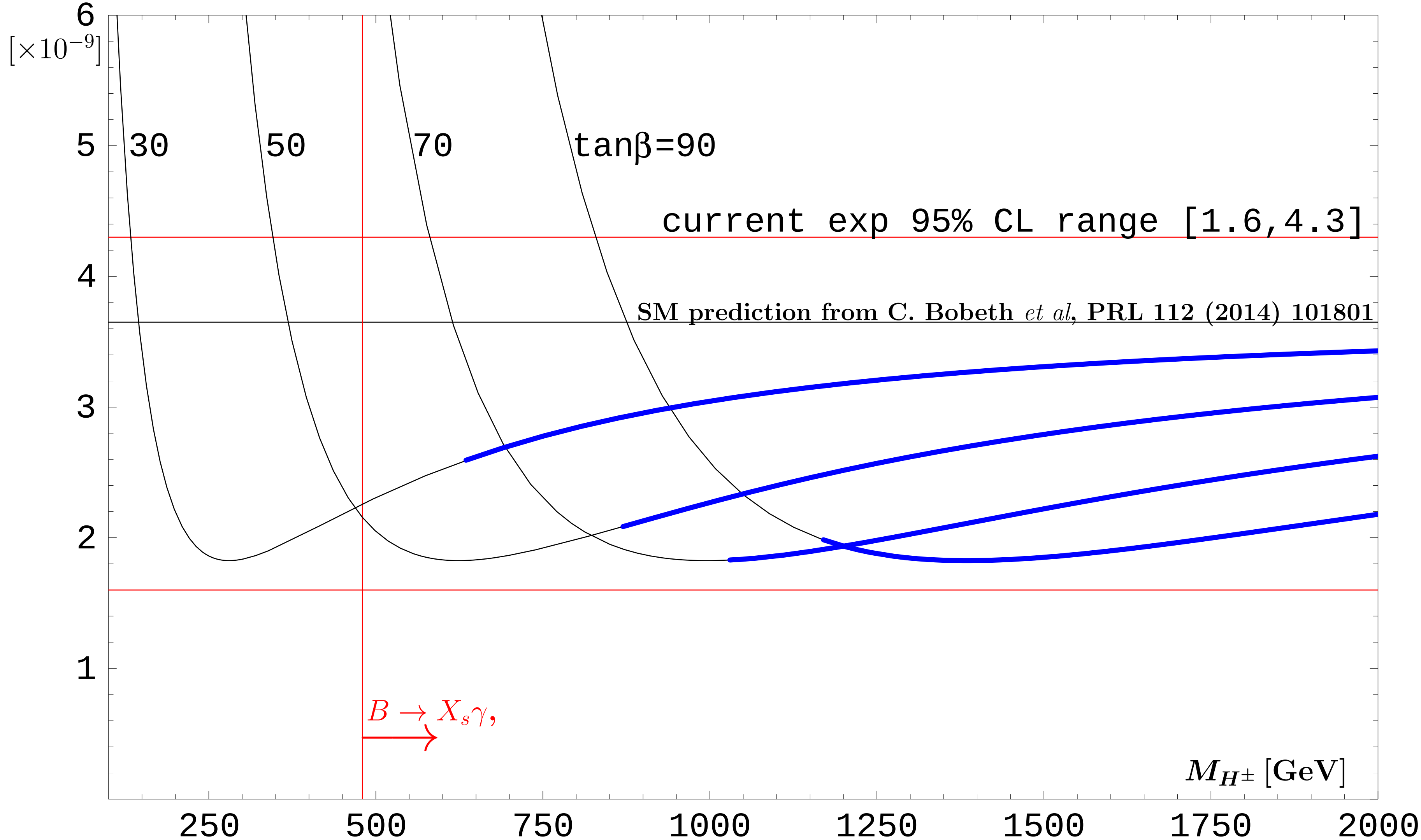
$$R_s = \left(\frac{f_{B_s}[\text{MeV}]}{227.7}\right)^2 \left(\frac{|V_{cb}|}{0.0424}\right)^2 \left(\frac{|V_{tb}^\star V_{ts}/V_{cb}|}{0.980}\right)^2 \frac{\tau_H^s\ [\text{ps}]}{1.615},$$

$$R_d = \left(\frac{f_{B_d}[\text{MeV}]}{190.5}\right)^2 \left(\frac{|V_{tb}^\star V_{td}|}{0.0088}\right)^2 \frac{\tau_d^{\text{av}}\ [\text{ps}]}{1.519}.$$

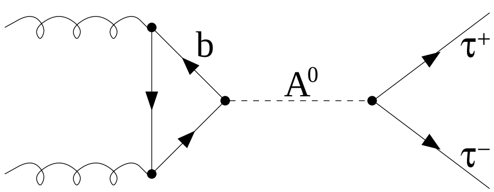
Sources of uncertainties	f_{B_q}	CKM	τ_H^q	M_t	α_s	other parametric	non-parametric	$\sum$	
$\overline{\mathcal{B}}_{s\ell}$	4.0%	4.3%	1.3%	1.6%	0.1%	$< 0.1\%$	1.5%	6.4%	$\longrightarrow$ 4.7% (?)
$\overline{\mathcal{B}}_{d\ell}$	4.5%	6.9%	0.5%	1.6%	0.1%	$< 0.1\%$	1.5%	8.5%	

In the case of $\overline{\mathcal{B}}_{s\ell}$, the main uncertainty (4.2%) originates from $|V_{cb}| = 0.0424(9)$ that comes from a fit to the inclusive semileptonic data [P. Gambino and C. Schwanda, arXiv:1307.4551].

$\mathcal{B}(B_s \to \mu^+\mu^-)$ in the Two-Higgs-Doublet Model II



Blue lines — still allowed for $M_{H^\pm} = \sqrt{M_A^2 + M_W^2}$ after taking into account the LHC searches for $pp \to A^0 \to \tau^+\tau^-$ [CMS arXiv:1408.3316, ATLAS arXiv:1409.6064].



NNLO QCD corrections to $\bar{B} \to X_s \gamma$

The relevant perturbative quantity $P(E_0)$:

$$\frac{\Gamma[b \to X_s \gamma]_{E_\gamma > E_0}}{\Gamma[b \to X_u e \bar{\nu}]} = \left| \frac{V_{ts}^* V_{tb}}{V_{ub}} \right|^2 \frac{6\alpha_{\rm em}}{\pi} \underbrace{\sum_{i,j} C_i(\mu_b)\, C_j(\mu_b)\, K_{ij}}_{P(E_0)}$$

Expansions of the Wilson coefficients and K_{ij} in $\widetilde{\alpha}_s \equiv \frac{\alpha_s(\mu_b)}{4\pi}$:

$$C_i(\mu_b) = C_i^{(0)} + \widetilde{\alpha}_s\, C_i^{(1)} + \widetilde{\alpha}_s^2\, C_i^{(2)} + \ldots$$

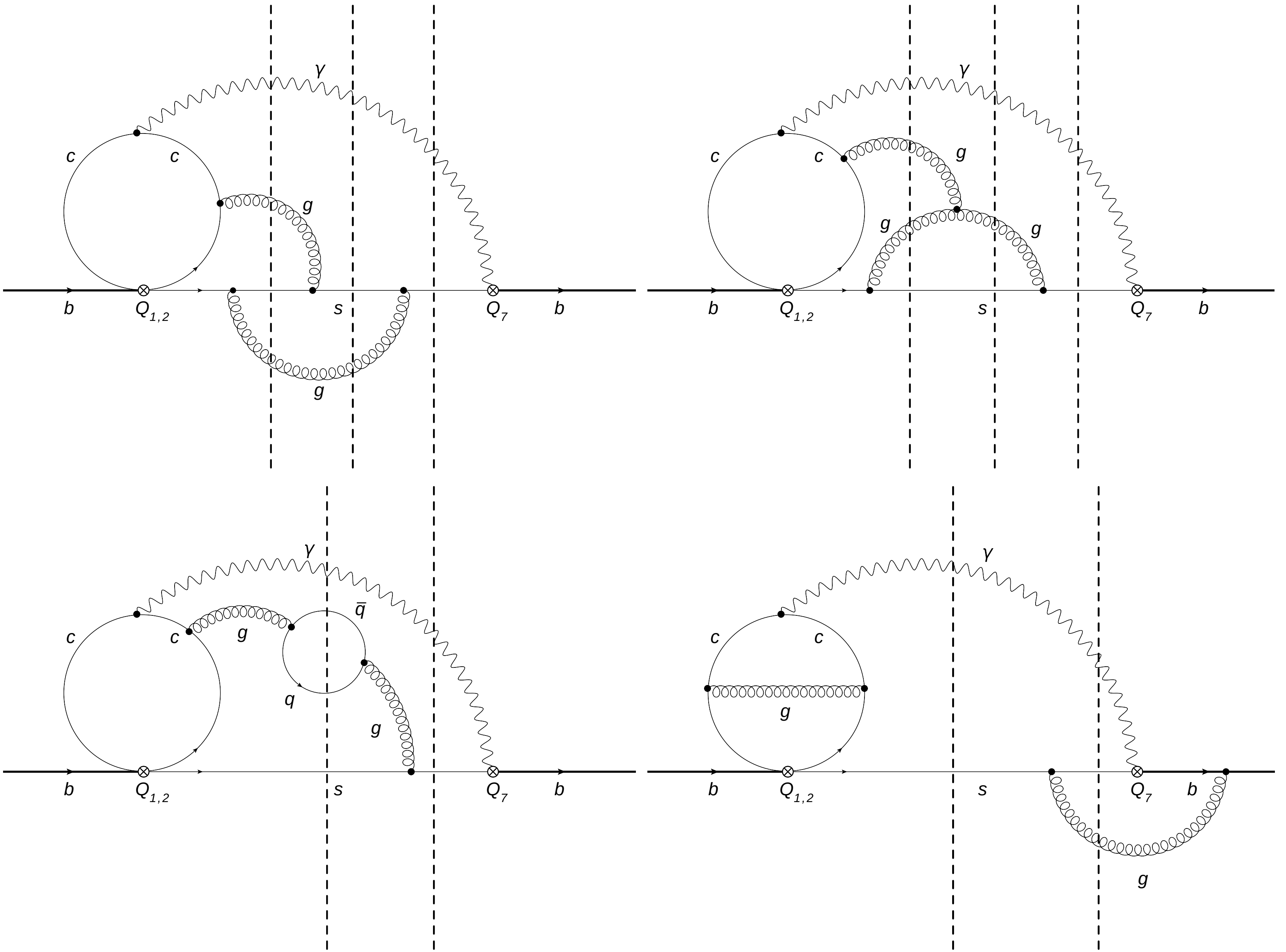
$$K_{ij} = K_{ij}^{(0)} + \widetilde{\alpha}_s\, K_{ij}^{(1)} + \widetilde{\alpha}_s^2\, K_{ij}^{(2)} + \ldots$$

Most important at the NNLO: $K_{77}^{(2)}$, $K_{27}^{(2)}$ and $K_{17}^{(2)}$.

They depend on $\frac{\mu_b}{m_b}$, $\delta = 1 - \frac{2E_0}{m_b}$ and $z = \frac{m_c^2}{m_b^2}$.

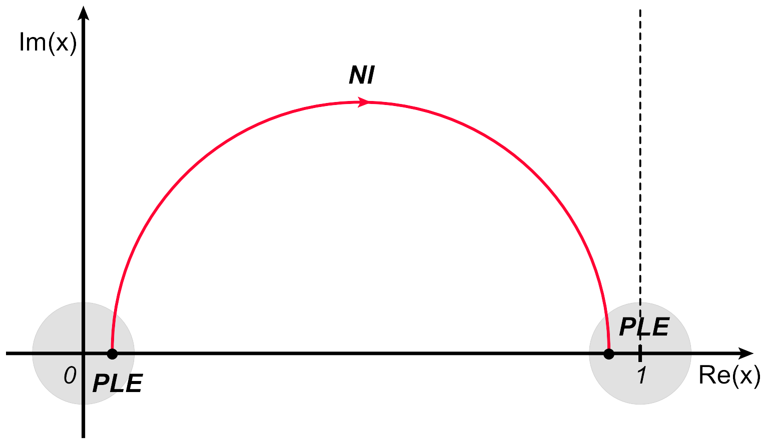
Evaluation of $K_{27}^{(2)}$ and $K_{17}^{(2)}$ for $m_c = 0$ and $\delta = 1$:

[M. Czakon, P. Fiedler, T. Huber, MM, T. Schutzmeier, M. Steinhauser, JHEP 1504 (2015) 168]



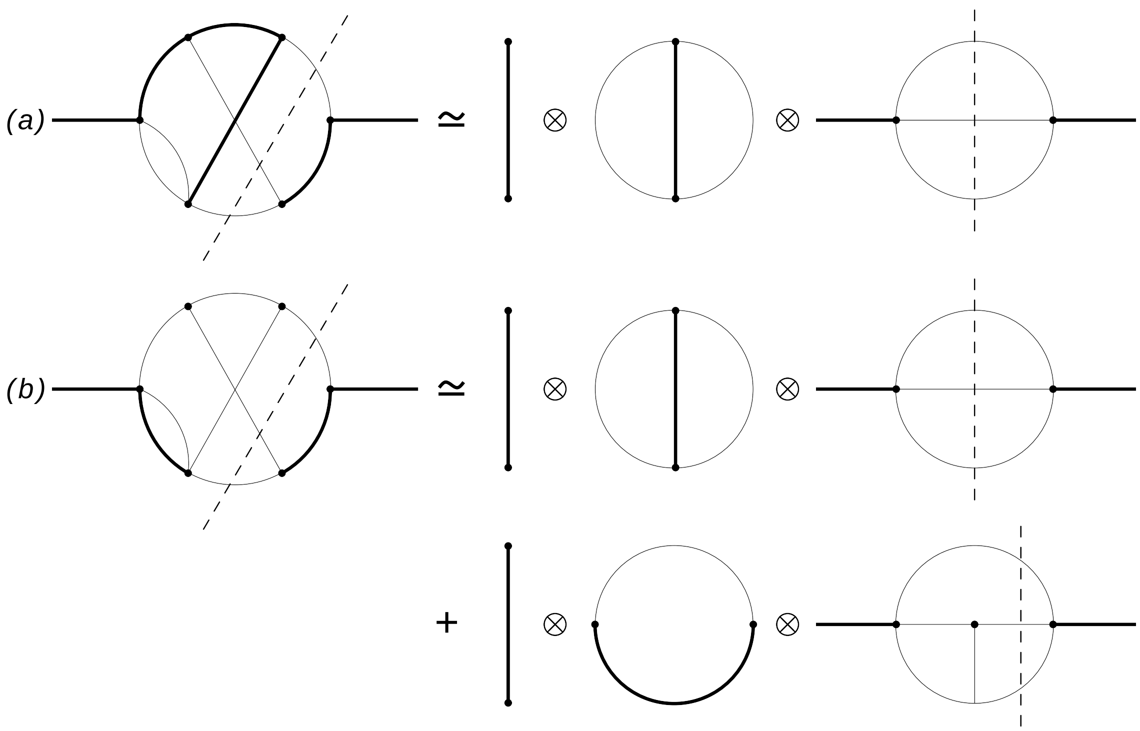
Master integrals and differential equations:

	n_D	n_{OS}	n_{eff}	$n_{massless}$
2-particle cuts	**292**	**92**	**143**	**9**
3-particle cuts	**267**	**54**	**110**	**11**
4-particle cuts	**292**	**17**	**37**	**7**
total	**851**	**163**	**290**	**27**



$$\frac{d}{dx} I_i(x) = \sum_j R_{ij}(x) I_j(x), \qquad x = \frac{p^2}{m_b^2}.$$

Boundary conditions in the vicinity of $x = 0$:



Results for the NNLO corrections:

$$K_{27}^{(2)}(z,\delta) = A_2 + F_2(z,\delta) \underbrace{- \tfrac{27}{2} f_q(z,\delta) + f_b(z) + f_c(z) + \tfrac{4}{3}\phi_{27}^{(1)}(z,\delta)\ln z}_{\text{quark loops on the gluon lines \& BLM approximation}}$$

$$+ \left[\text{terms} \sim \left(\ln\frac{\mu_b}{m_b},\ \ln^2\frac{\mu_b}{m_b},\ \ln\frac{\mu_c}{m_c}\right) \text{ or vanishing when } m_b \to m_b^{\rm pole}\right],$$

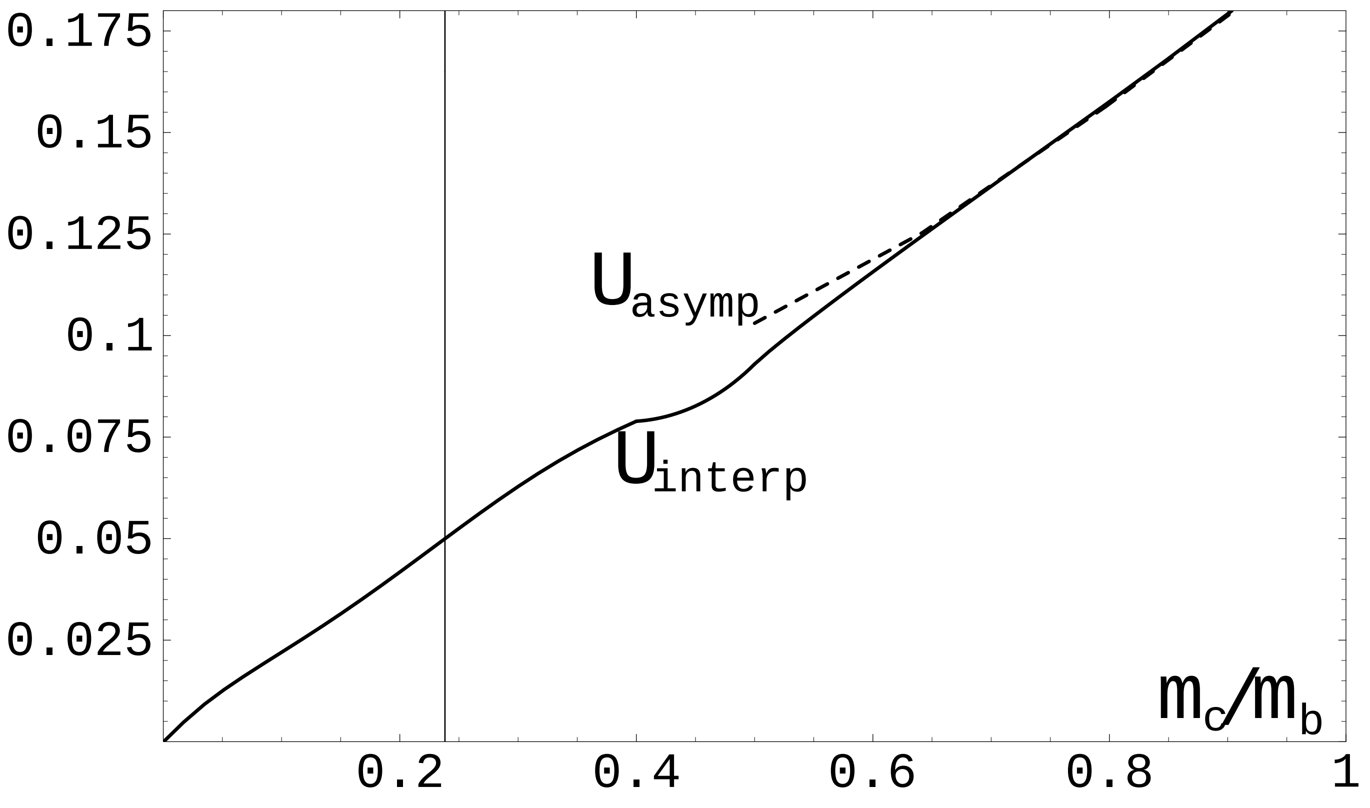
$$K_{17}^{(2)}(z,\delta) = -\tfrac{1}{6}K_{27}^{(2)}(z,\delta) + A_1 + F_1(z,\delta) + \left[\text{terms} \sim \left(\ln\frac{\mu_b}{m_b},\ \ln^2\frac{\mu_b}{m_b}\right)\right].$$

$F_i(0,1) \equiv 0$, $A_1 \simeq 22.605$, $A_2 \simeq 75.603$ from the present calculation.

Next, we interpolate in $z = m_c^2/m_b^2$ by assuming that $F_i(z,1)$ are linear combinations of $f_q(z,1)$, $K_{27}^{(1)}(z,1)$, $z\frac{d}{dz}K_{27}^{(1)}(z,1)$ and a constant term. The known large-z behaviour of F_i [hep-ph/0609241] and the condition $F_i(0,1) \equiv 0$ fix these linear combinations in a unique manner.

Effect of the interpolated contribution on the branching ratio

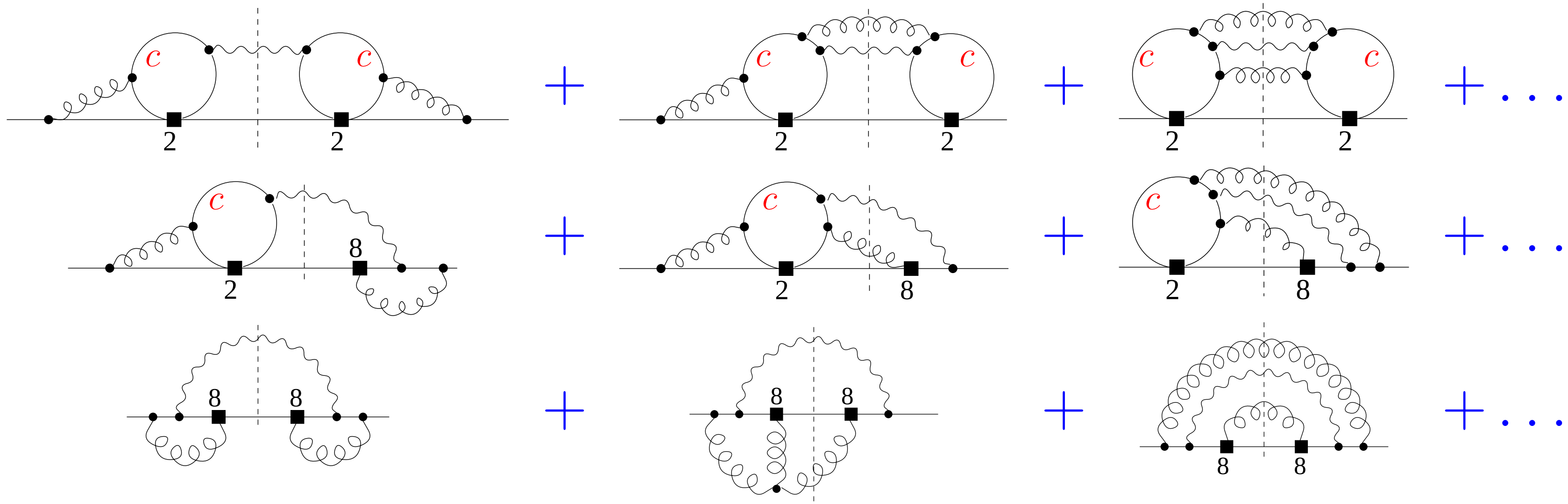
$$\frac{\Delta\mathcal{B}_{s\gamma}}{\mathcal{B}_{s\gamma}} \simeq U(z,\delta) \equiv \frac{\alpha_s^2(\mu_b)}{8\pi^2} \frac{C_1^{(0)}(\mu_b)F_1(z,\delta)+\left(C_2^{(0)}(\mu_b)-\frac{1}{6}C_1^{(0)}(\mu_b)\right)F_2(z,\delta)}{C_7^{(0)\mathrm{eff}}(\mu_b)}$$



K_{22}**:**
(and analogous K_{11} & K_{12})

K_{28}**:**
(and analogous K_{18})

K_{88}**:**



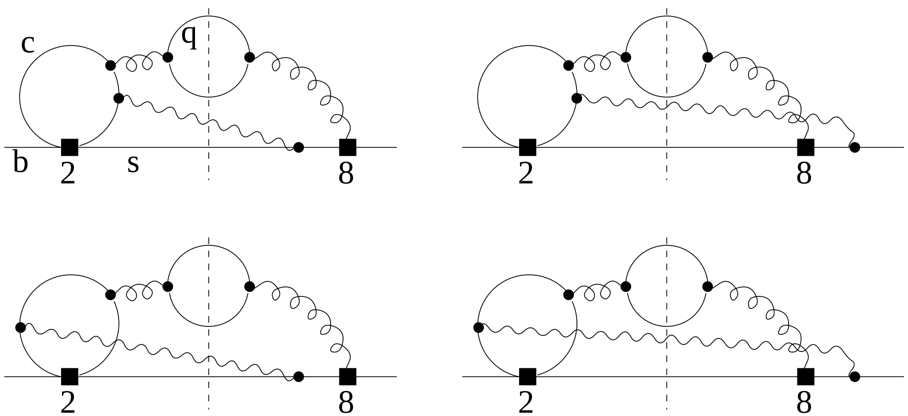
Two-particle cuts are known (just $|\mathrm{NLO}|^2$).

Three- and four-particle cuts are known in the BLM approximation only: [Ligeti, Luke, Manohar, Wise, 1999], [Ferroglia, Haisch, arXiv:1009.2144], [Poradziński, MM, arXiv:1009.5685].

NLO+(NNLO BLM) corrections are not big (+3.8%).

Example:

Evaluation of the $(n > 2)$-particle cut contributions to K_{28} in the Brodsky-Lepage-Mackienzie (BLM) approximation ("naive nonabelianization", large-β_0 approximation) [Poradziński, MM, arXiv:1009.5685]:



q **– massless quark,**

N_q **– number of massless flavours (equals to 3 in practice because masses of u, d, s are neglected).**

Replacement in the final result:

$$-\tfrac{2}{3}N_q \quad \longrightarrow \quad \beta_0 = 11 - \tfrac{2}{3}(N_q + 2).$$

The diagrams have been evaluated using the method of Smith and Voloshin [hep-ph/9405204].

Non-BLM contributions to K_{ij} from quark loops on the gluon lines are quasi-completely known.
[Boughezal, Czakon, Schutzmeier, 2007], [Asatrian, Ewerth, Gabrielyan, Greub, 2007], [Ewerth, 2008].

Incorporating other perturbative contributions evaluated after the previous phenomenological analysis in hep-ph/0609232:

1. Four-loop mixing (current-current) $\to$ (gluonic dipole)
   M. Czakon, U. Haisch, MM, JHEP 0703 (2007) 008 [hep-ph/0612329]
2. Diagrams with massive quark loops on the gluon lines
   R. Boughezal, M. Czakon and T. Schutzmeier, JHEP 0709 (2007) 072 [arXiv:0707.3090]
   H. M. Asatrian, T. Ewerth, H. Gabrielyan and C. Greub, Phys. Lett. B 647 (2007) 173 [hep-ph/0611123]
   T. Ewerth, Phys. Lett. B 669 (2008) 167 [arXiv:0805.3911]
3. Complete interference (photonic dipole)–(gluonic dipole)
   H. M. Asatrian, T. Ewerth, A. Ferroglia, C. Greub and G. Ossola,
   Phys. Rev. D 82 (2010) 074006 [arXiv:1005.5587]
4. New BLM corrections to contributions from 3-body and 4-body final states for interferences not involving the photonic dipole
   A. Ferroglia and U. Haisch, Phys. Rev. D 82 (2010) 094012 [arXiv:1009.2144]
   MM and M. Poradziński, Phys. Rev. D 83 (2011) 014024 [arXiv:1009.5685]
5. LO contributions from $b \to s\gamma q\bar{q}$, $(q = u, d, s)$ from 4-quark operators ("penguin" or CKM-suppressed)
   M. Kamiński, MM and M. Poradziński, Phys. Rev. D 86 (2012) 094004 [arXiv:1209.0965]
6. NLO contributions from $b \to s\gamma q\bar{q}$, $(q = u, d, s)$ from interferences of the above operators with $Q_{1,2,7,8}$
   T. Huber, M. Poradziński, J. Virto, JHEP 1501 (2015) 115 [arXiv:1411.7677]

Taking into account new non-perturbative analyses:

M. Benzke, S. J. Lee, M. Neubert and G. Paz, JHEP 1008 (2010) 099 [arXiv:1003.5012]
T. Ewerth, P. Gambino and S. Nandi, Nucl. Phys. B 830 (2010) 278 [arXiv:0911.2175]

Updating the parameters (Parametric uncertainties go down to 2.0%)

P. Gambino, C. Schwanda, Phys. Rev. D 89 (2014) 014022
A. Alberti, P. Gambino, K. J. Healey, S. Nandi, Phys. Rev. Lett. 114 (2015) 061802

Updated SM estimate for the CP- and isospin-averaged branching ratio of $\bar{B} \to X_s\gamma$ [arXiv:1503.01789, arXiv:1503.01791]:

$$\mathcal{B}_{s\gamma}^{\rm SM} = (3.36 \underbrace{\pm 0.23}_{\pm 6.9\%}) \times 10^{-4}$$

Contributions to the total TH uncertainty (summed in quadrature):

5% non-perturbative, **3%** from the interpolation in m_c

3% higher order $\mathcal{O}(\alpha_s^3)$, **2%** parametric

It is very close the the experimental world average(s):

(a) $\mathcal{B}_{s\gamma}^{\rm exp} = (3.43 \underbrace{\pm 0.21 \pm 0.07}_{\pm 6.5\%}) \times 10^{-4}$ [HFAG, arXiv:1412.7515]

(b) $\mathcal{B}_{s\gamma}^{\rm exp} = (3.41 \underbrace{\pm 0.15 \pm 0.04}_{\pm 4.6\%}) \times 10^{-4}$ [Karim Trabelsi, talk at EPS 2015]

Experiment agrees with the SM to much better than $\sim 1\sigma$ level.

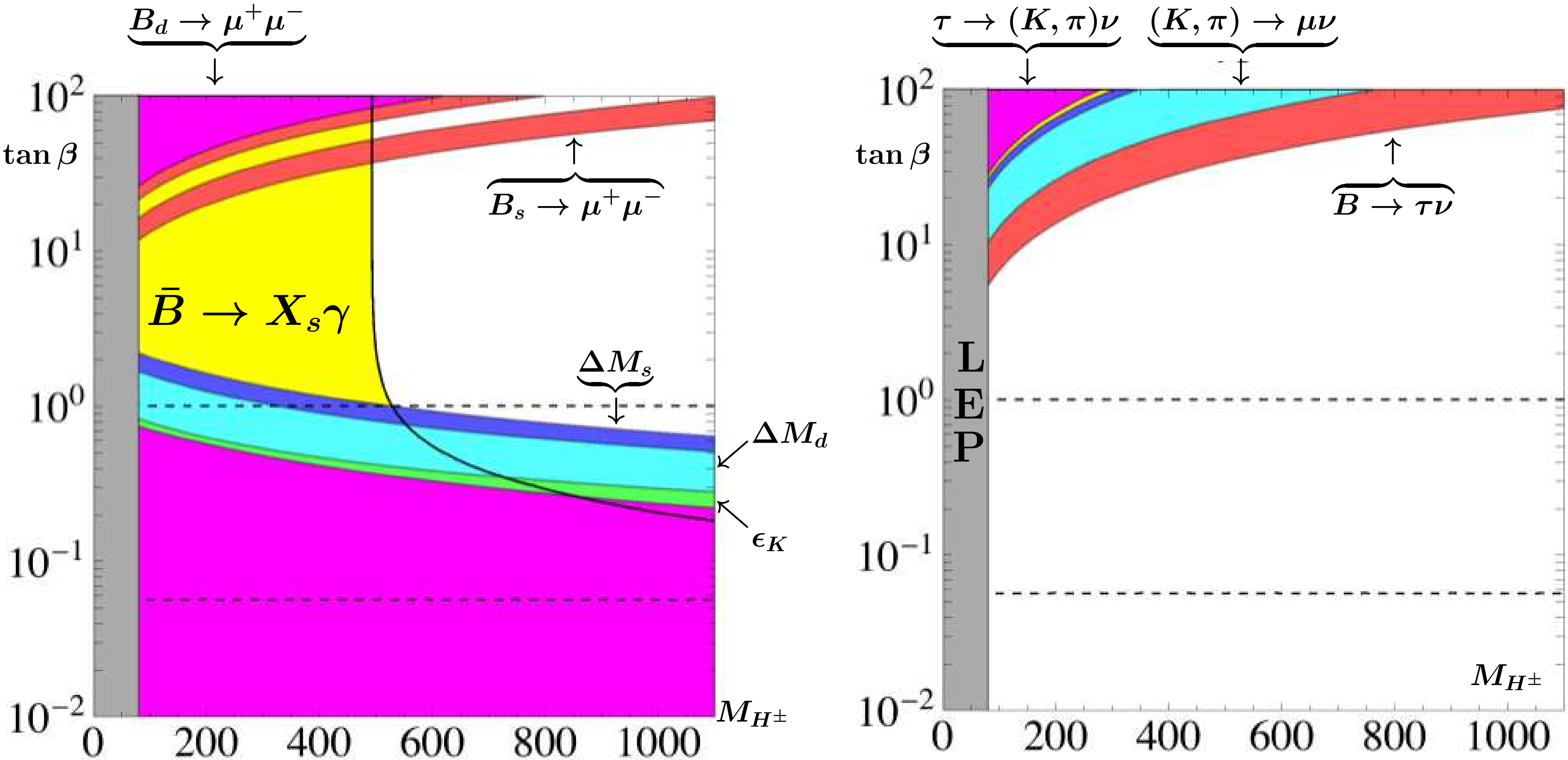
$\Rightarrow$ Strong bounds on the $H^\pm$ mass in the Two-Higgs-Doublet-Model II:

(a) $M_{H^\pm} > 480\,$GeV at 95%C.L.

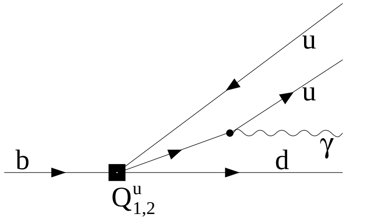
(b) $M_{H^\pm} > 540\,$GeV at 95%C.L.

Current flavour-physics bounds in the $M_{H^\pm} - \tan\beta$ plane of the 2HDM-II

[from T. Enomoto and R. Watanabe, arXiv:1511.05066v2]



$\bar{B} \to X_d \gamma$



$$\mathcal{L}_{\rm eff} \;\sim\; V_{td}^* V_{tb} \left[\sum_{i=1}^{8} C_i Q_i + \kappa_d \sum_{i=1}^{2} C_i (Q_i - Q_i^u) \right]$$

$$\kappa_d = (V_{ud}^* V_{ub})/(V_{td}^* V_{tb}) \;=\; \left(0.007^{+0.015}_{-0.011}\right) + i \left(-0.404^{+0.012}_{-0.014}\right)$$

$$\left.\begin{array}{l} \mathcal{B}_{d\gamma}^{\rm SM} = \left(1.73^{+0.12}_{-0.22}\right) \times 10^{-5} \\ \mathcal{B}_{d\gamma}^{\rm exp} = (1.41 \pm 0.57) \times 10^{-5} \end{array}\right\} \text{ for } E_0 = 1.6\,\text{GeV}$$

- $\mathcal{B}_{d\gamma}^{\rm SM}$ is rough: m_b/m_q varied between $10 \sim m_B/m_K$ and $50 \sim m_B/m_\pi$ $\Longrightarrow$ 2% to 11% of $\mathcal{B}_{d\gamma}$.
- Fragmentation functions give a similar range [H. M. Asatrian and C. Greub, arXiv:1305.6464].
- Collinear logarithms and isolated photons

The ratio R_γ

$$R_\gamma^{\rm SM} \equiv \left(\mathcal{B}_{s\gamma}^{\rm SM} + \mathcal{B}_{d\gamma}^{\rm SM} \right) / \mathcal{B}_{c\ell\nu} = (3.31 \pm 0.22) \times 10^{-3}$$

Generic (but CP-conserving) beyond-SM effects:

$$\mathcal{B}_{s\gamma} \times 10^4 = (3.36 \pm 0.23) - 8.22\,\Delta C_7 - 1.99\,\Delta C_8,$$
$$R_\gamma \times 10^3 = (3.31 \pm 0.22) - 8.05\,\Delta C_7 - 1.94\,\Delta C_8.$$

Summary

- **Some of the $\mathcal{O}(\alpha_s^2, \alpha_{\rm em})$ corrections to $C_9(\mu_b)$ remain unknown.**

- **The non-parametric uncertainties in $\overline{\mathcal{B}}_{s\mu}^{\rm SM}$ have been reduced to the $\pm 1.5\%$ level.**

- **The dominant NNLO corrections to $\mathcal{B}_{s\gamma}$ are now known not only in the large m_c limit, but also at $m_c = 0$. However, no reduction of uncertainties with respect to the 2006 estimate is possible, except for the parametric one.**

- **Completing the calculation of $K_{17}^{(2)}$ and $K_{27}^{(2)}$ for arbitrary $z = m_c^2/m_b^2$ is necessary to further reduce the perturbative uncertainties in $\mathcal{B}_{s\gamma}$.**

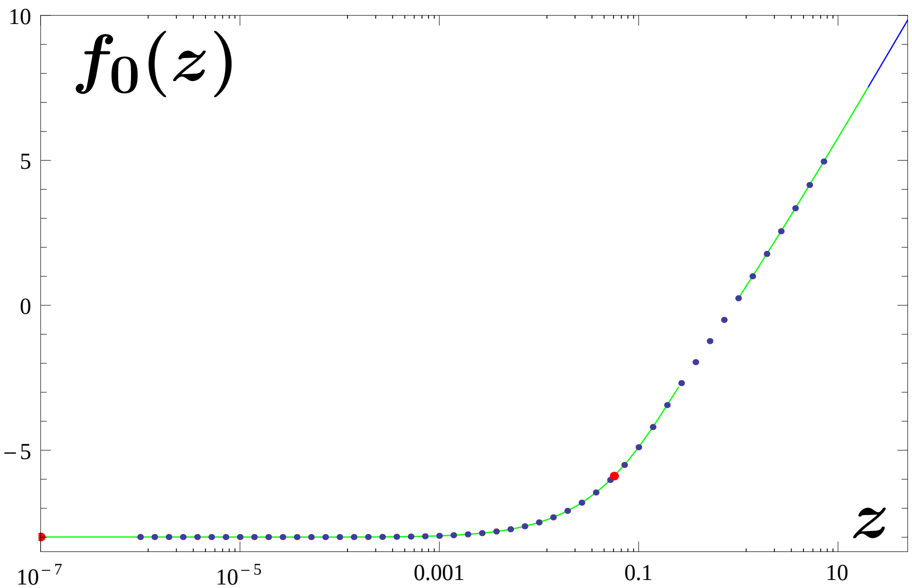
BACKUP SLIDES

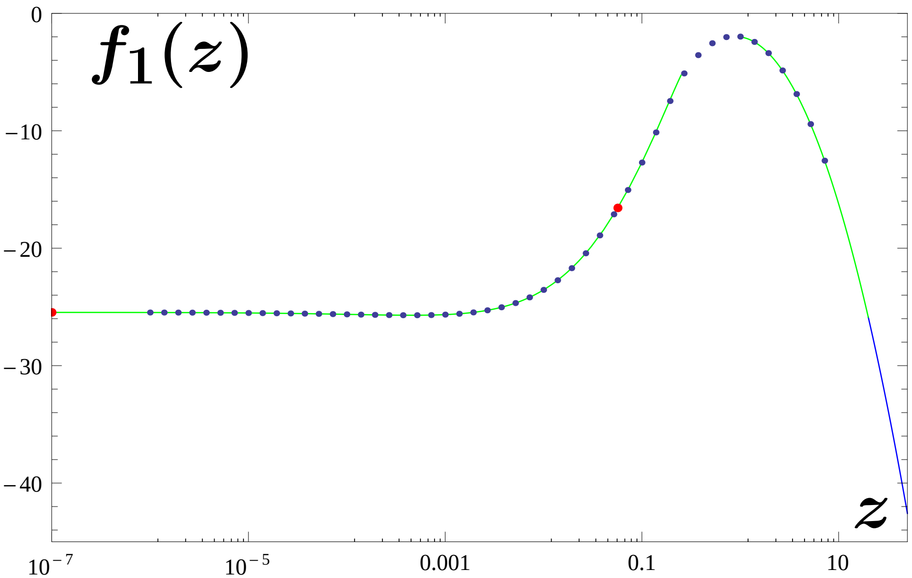
Outlook: generalizing the K_{27} NNLO calculation to arbitrary $z = m_c^2/m_b^2$.

Method: differential equations in z for the master integrals.

Results for the bare NLO contributions up to $\mathcal{O}(\epsilon)$:

$$\tilde{G}_{27}^{(1)2P} = -\frac{92}{81\epsilon} + f_0(z) + \epsilon f_1(z) \xrightarrow{z\to 0} -\frac{92}{81\epsilon} - \frac{1942}{243} + \epsilon\left(-\frac{26231}{729} + \frac{259}{243}\pi^2\right)$$

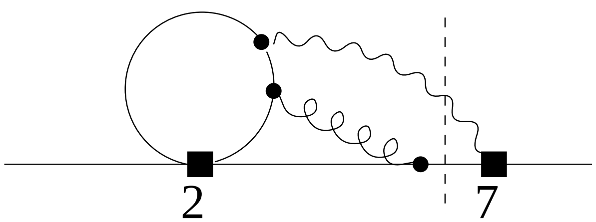




Dots: solutions to the differential equations and/or the exact $z \to 0$ limit.

Lines: large- and small-z asymptotic expansions

Large-z expansions of the 11 master integrals are from M. Steinhauser.



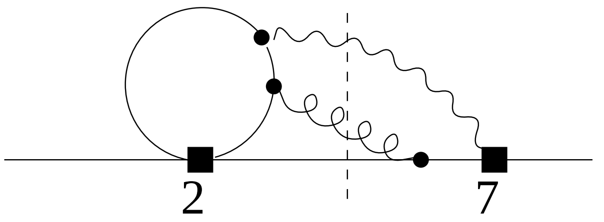
Small-z expansions of $\tilde{G}_{27}^{(1)2P}$:

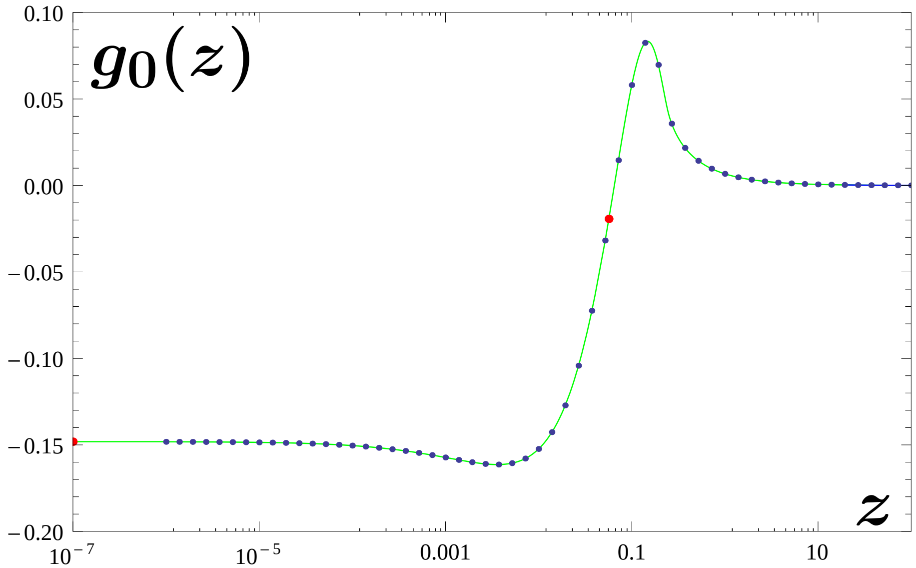
f_0 from C. Greub, T. Hurth, D. Wyler, hep-ph/9602281, hep-ph/9603404, A. J. Buras, A. Czarnecki, MM, J. Urban, hep-ph/0105160,

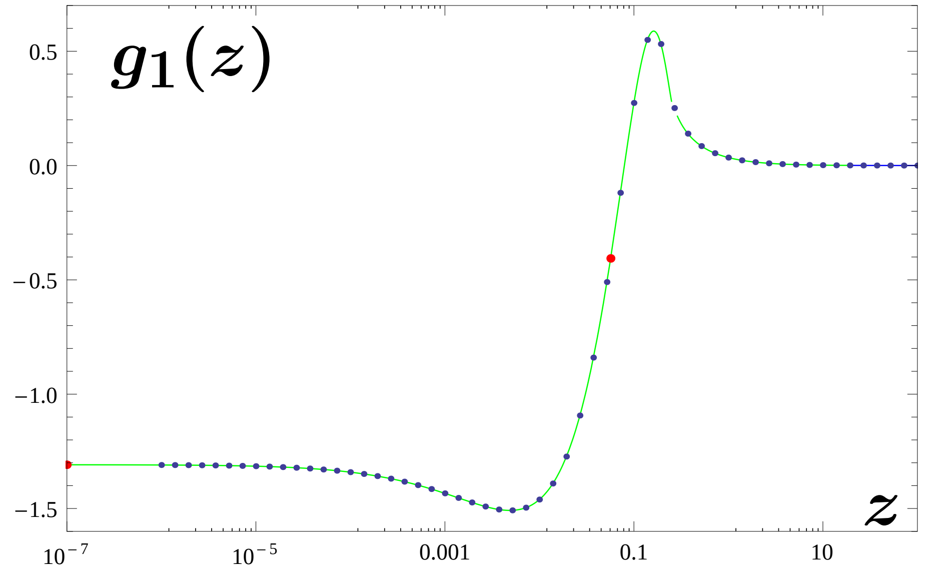
f_1 from H.M. Asatrian, C. Greub, A. Hovhannisyan, T. Hurth and V. Poghosyan, hep-ph/0505068.

Analogous results for the 3-body final state contributions ($\delta = 1$):

$$\tilde{G}_{27}^{(1)3P} = g_0(z) + \epsilon g_1(z) \xrightarrow{z\to 0} -\frac{4}{27} - \frac{106}{81}\epsilon$$





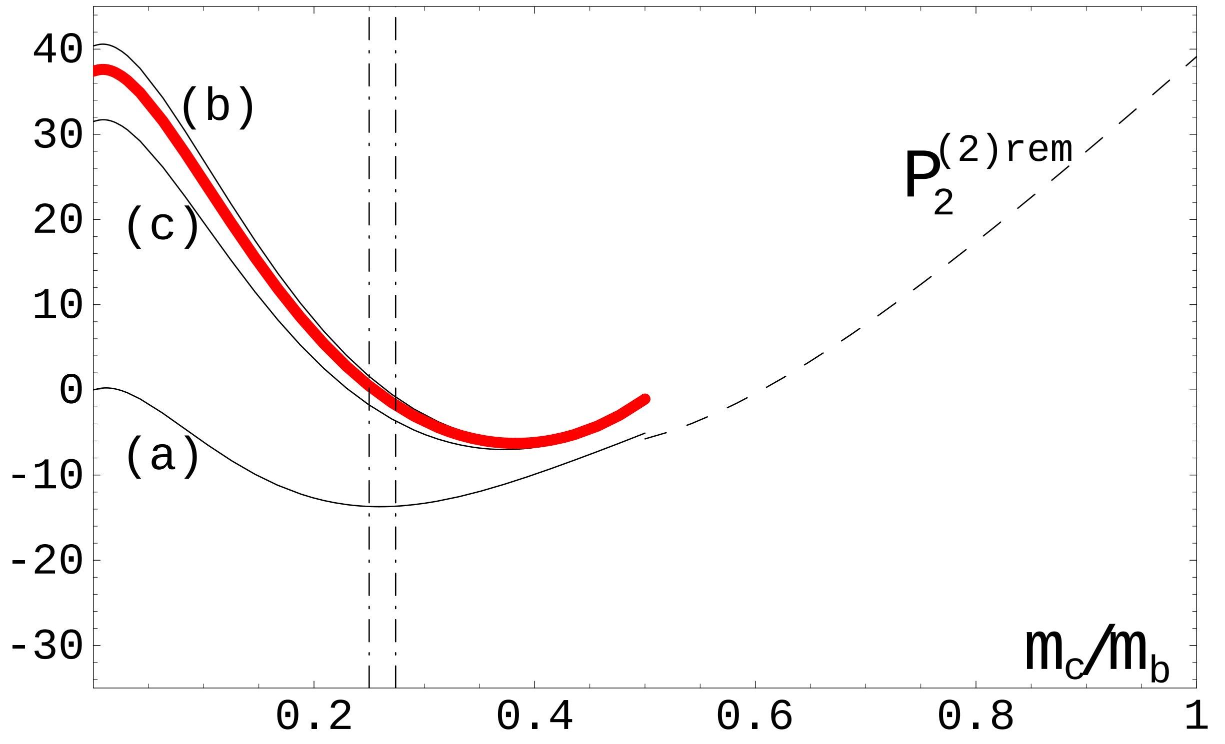


Dots: solutions to the differential equations and/or the exact $z \to 0$ limit.
Lines: exact result for g_0, as well as large- and small-z asymptotic expansions for g_1 from A. Rehman.

$$g_0(z) = \begin{cases} -\frac{4}{27} - \frac{14}{9}z + \frac{8}{3}z^2 + \frac{8}{3}z(1-2z)\, s\, L + \frac{16}{9}z(6z^2-4z+1)\left(\frac{\pi^2}{4} - L^2\right), & \text{for } z \le \frac{1}{4}, \\ -\frac{4}{27} - \frac{14}{9}z + \frac{8}{3}z^2 + \frac{8}{3}z(1-2z)\, t\, A + \frac{16}{9}z(6z^2-4z+1)\, A^2, & \text{for } z > \frac{1}{4}, \end{cases}$$

where $s = \sqrt{1-4z}$, $L = \ln(1+s) - \frac{1}{2}\ln 4z$, $t = \sqrt{4z-1}$, and $A = \arctan(1/t)$.

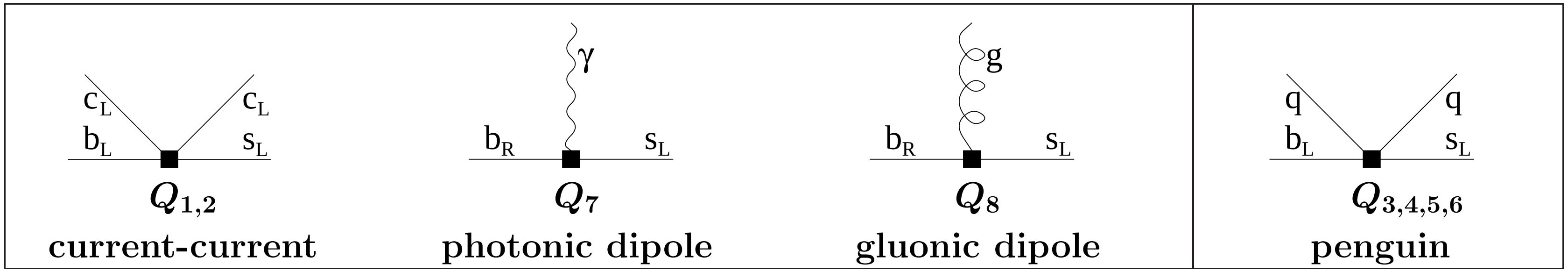
Comparison to the interpolation in hep-ph/0609241



Decoupling of W, Z, t, H^0 $\Rightarrow$ effective weak interaction Lagrangian:

$$L_{\rm weak} \sim \sum_i C_i Q_i$$

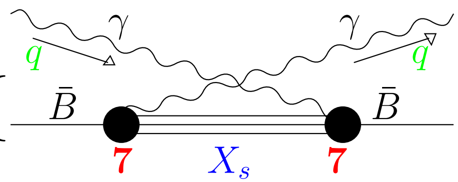
Eight operators Q_i matter for $\mathcal{B}_{s\gamma}^{\rm SM}$ when the NLO EW and/or CKM-suppressed effects are neglected:



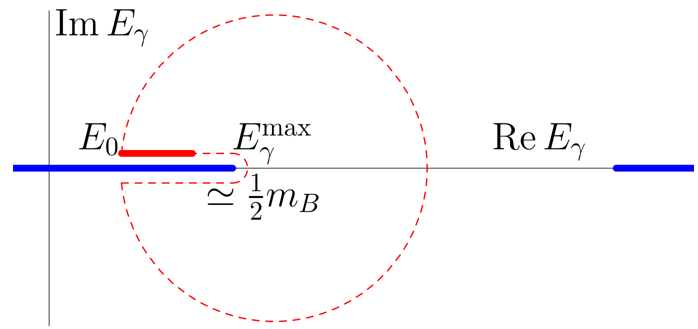
$$\Gamma(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = |C_7(\mu_b)|^2\, \Gamma_{77}(E_0) + (\text{other}) \qquad (\mu_b \sim m_b/2)$$

Optical theorem:

$$\frac{d\Gamma_{77}}{dE_\gamma} \sim \text{Im}\{\ \ \} \equiv \text{Im}\, A$$



Integrating the amplitude A over E_γ:

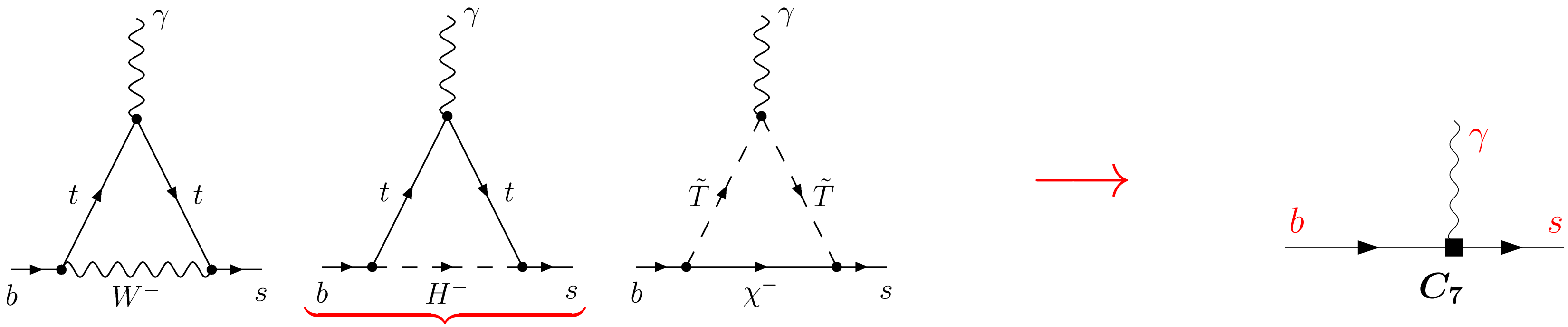


OPE on the ring $\Rightarrow$ **Non-perturbative corrections to $\Gamma_{77}(E_0)$ form a series in $\frac{\Lambda_{\rm QCD}}{m_b}$ and α_s that begins with**

$$\frac{\mu_\pi^2}{m_b^2}, \frac{\mu_G^2}{m_b^2}, \frac{\rho_D^3}{m_b^3}, \frac{\rho_{LS}^3}{m_b^3}, \ldots;\ \frac{\alpha_s \mu_\pi^2}{(m_b - 2E_0)^2}, \frac{\alpha_s \mu_G^2}{m_b(m_b - 2E_0)}; \ldots,$$

where $\mu_\pi, \mu_G, \rho_D, \rho_{LS} = \mathcal{O}(\Lambda_{\rm QCD})$ are extracted from the semileptonic $\bar{B} \to X_c e \bar{\nu}$ spectra and the B–$B^\star$ mass difference.

Information on electroweak-scale physics in the $b \to s\gamma$ transition is encoded in an effective low-energy local interaction:



$\Rightarrow M_{H^\pm} > \sim 500\,\text{GeV}$
in the 2HDM-II

$b \in \bar{B} \equiv (\bar{B}^0$ or $B^-)$

The inclusive $\bar{B} \to X_s\gamma$ decay rate for $E_\gamma > E_0$ is well approximated by the corresponding perturbative decay rate of the b-quark:

$$\Gamma(\bar{B} \to X_s\gamma) \;=\; \Gamma(b \to X_s^p\gamma) \;+ \left(\begin{array}{c}\text{non-perturbative effects}\\ (3\pm5)\%\end{array}\right)$$

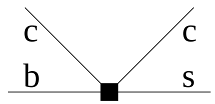
[G. Buchalla, G. Isidori and S.-J. Rey, Nucl. Phys. B511 (1998) 594]
[M. Benzke, S.J. Lee, M. Neubert and G. Paz, JHEP 1008 (2010) 099]

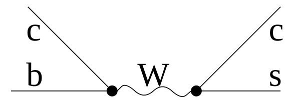
provided E_0 is large ($E_0 \sim m_b/2$)
but not too close to the endpoint $(m_b - 2E_0 \gg \Lambda_{QCD})$.

Conventionally, $E_0 = 1.6\,\text{GeV} \simeq m_b/3$ is chosen.

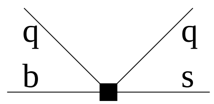
Resummation of $\left(\alpha_s \ln M_W^2/m_b^2\right)^n$ is most conveniently performed in the framework of an effective theory that arises from the SM after decoupling of the heavy electroweak bosons and the top quark. The Lagrangian of such a theory reads:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm QCD\times QED}(u,d,s,c,b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu)\, Q_i + \left(\begin{array}{c}\text{EW-suppressed,}\\ \text{higher-dimensional,}\\ \text{on-shell vanishing,}\\ \text{evanescent}\end{array}\right).$$

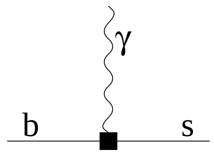
$Q_{1,2} =$ 

$= (\bar{s}\Gamma_i c)(\bar{c}\Gamma'_i b)$, **from** 

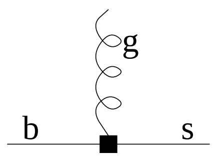
, $|C_i(m_b)| \sim 1$

$Q_{3,4,5,6} =$ 

$= (\bar{s}\Gamma_i b)\sum_q(\bar{q}\Gamma'_i q)$, $|C_i(m_b)| < 0.07$

$Q_7 =$ 

$= \dfrac{e m_b}{16\pi^2}\, \bar{s}_L \sigma^{\mu\nu} b_R F_{\mu\nu}$, $C_7(m_b) \simeq -0.3$

$Q_8 =$ 

$= \dfrac{g m_b}{16\pi^2}\, \bar{s}_L \sigma^{\mu\nu} T^a b_R G^a_{\mu\nu}$, $C_8(m_b) \simeq -0.15$

Three steps of the calculation:

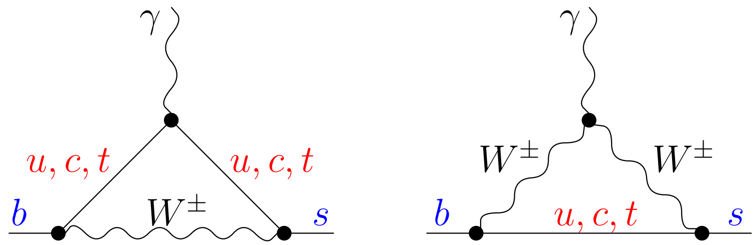
Matching: **Evaluating $C_i(\mu_0)$ at $\mu_0 \sim M_W$ by requiring equality of the SM and the effective theory Green functions.**

Mixing: **Deriving the effective theory Renormalization Group Equations** $\left(C_j^{\rm bare} = C_i Z_{ij}\right)$ **and evolving $C_i(\mu)$ from μ_0 to $\mu_b \sim m_b$.**

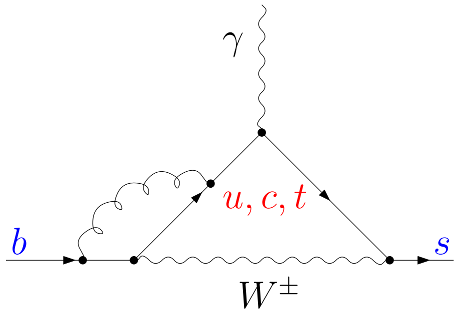
Matrix elements: **Evaluating the on-shell amplitudes at $\mu_b \sim m_b$.**

Examples of SM diagrams for the matching of $C_7(\mu_0)$

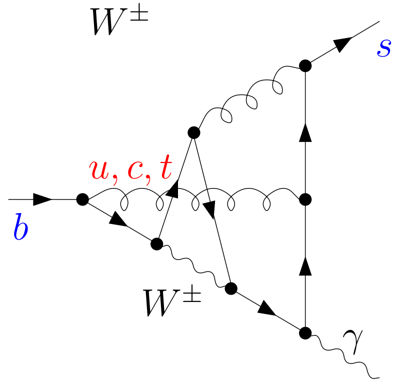
LO:
[Inami, Lim, 1981]



NLO:
[Adel, Yao, 1993]



NNLO:
[Steinhauser, MM, 2004]



NNLO method:

- Taylor expansion in the off-shell external momenta is applied before integration.
- The UV and spurious IR divergences are regulated dimensionally.
- ⇒ In the effective theory, only tree-level diagrams survive (tree vertices and UV counterterms). The UV renormalization constants are known from former anomalous-dimension calculations.
- All the $1/\epsilon$ poles cancel in the matching equation, i.e. in the difference between the effective theory and the full SM Green functions.
- At the 3-loop level, the difference $m_t - M_W$ is taken into account with the help of expansions in y^n and $(1-y^2)^n$ up to $n = 8$, where $y = M_W/m_t$.

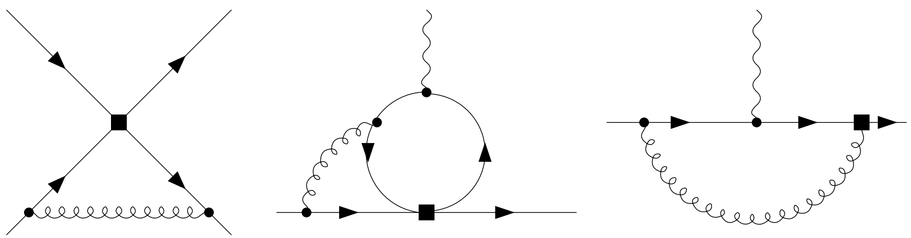
Resummation of large logarithms $\left(\alpha_s \ln \frac{M_W^2}{m_b^2}\right)^n$ in the $b \to s\gamma$ amplitude.

RGE for the Wilson coefficients: $$\mu \frac{d}{d\mu} C_j(\mu) = C_i(\mu)\gamma_{ij}(\mu)$$

The anomalous dimension matrix γ_{ij} is found from the effective theory renormalization constants, e.g.:

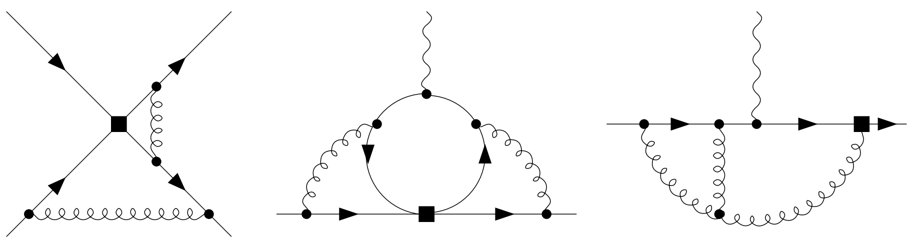
Z_{22} Z_{27} Z_{87}

LO



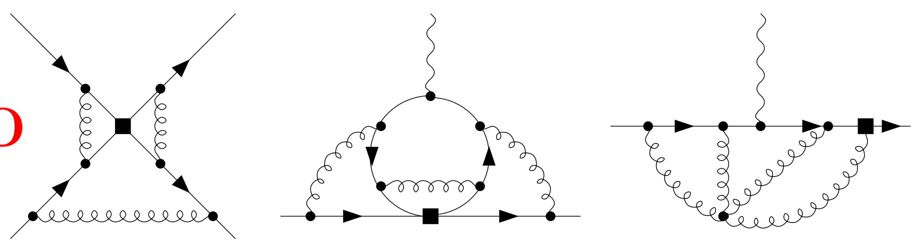
[Gaillard, Lee, 1974] [Altarelli, Maiani, 1974] — [Grinstein *et al.*, 1990] — [Shifman *et al.*, 1978] [Grigjanis *et al.*, 1988]

NLO



[Altarelli *et al.*, 1981] [Buras, Weisz, 1990] — [Chetyrkin, MM, Münz, 1997] — [MM, Münz, 1995]

NNLO



[Gorbahn, Haisch, 2004] — [Czakon, Haisch, MM, 2006] — [Gorbahn, Haisch, MM, 2005]

$\sim 2 \times 10^4$ **diagrams,**
-4% **effect in the BR**

All the Wilson coefficients $C_1(\mu_b), \ldots, C_8(\mu_b)$ are known at the NNLO in the SM.

Massless integrals for the boundary conditions: