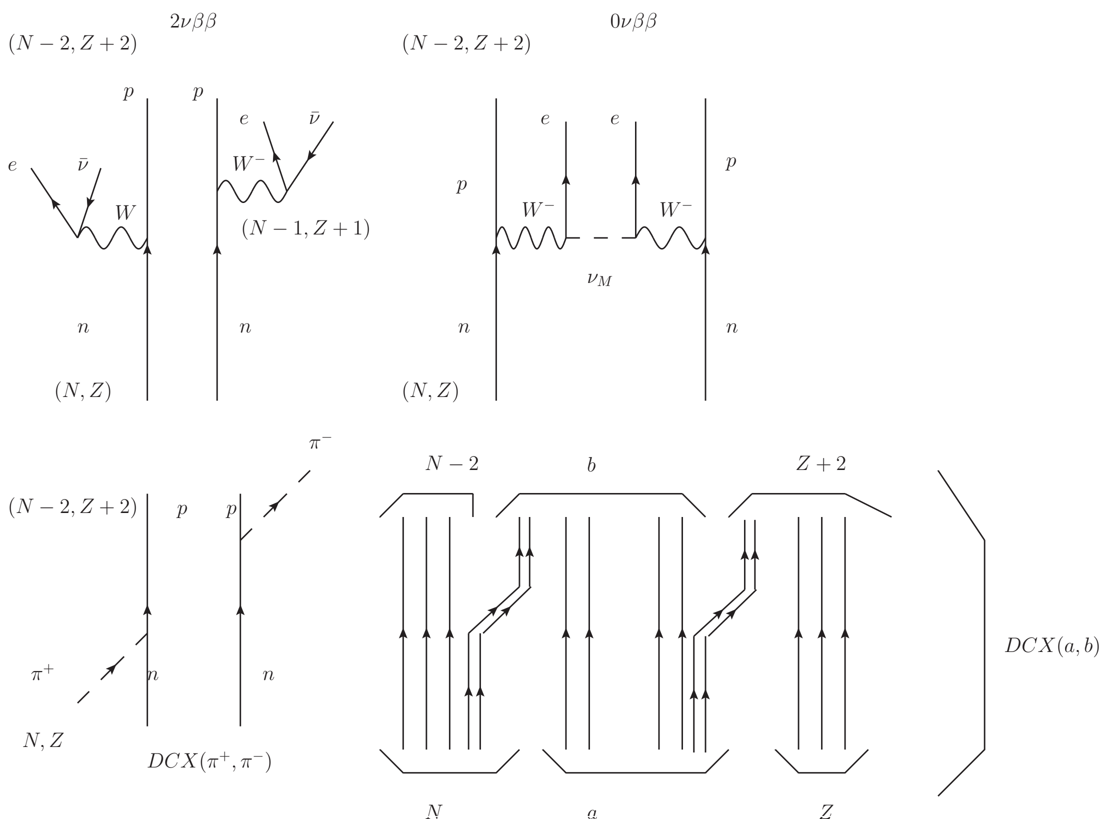
Field theoretical treatment of the double beta decay: GT and pairing vibrational modes.

O. Civitarese Catania, december 1-2, 2015

- Motivations
- Collective treatment of Isospin, Spin and Pairing degrees of freedom
- Renormalization of the GT charge and the Ikeda Sum Rule
- An example: the decay of Te isotopes
- On DCX and $\beta\beta$-decays
- Summary


$2\nu\beta\beta$
$(N-2, Z+2)$
p
p
e
$\bar{\nu}$
e
$\bar{\nu}$
W^-
W
$(N-1, Z+1)$
n
n
(N, Z)
$0\nu\beta\beta$
$(N-2, Z+2)$
e
e
p
p
W^-
W^-
ν_M
n
n
(N, Z)
π^-
$(N-2, Z+2)$
p
p
π^+
n
n
N, Z
$DCX(\pi^+, \pi^-)$
$N-2$
b
$Z+2$
$DCX(a, b)$
N
a
Z

Some references

- Comparison and test of the known matrix elements :
  Phys. Rep.300 (1998) 123; JPG 39 (2012) 124005
- Symmetry violations (and restorations) in the reconstruction of matrix elements:
  PLB 446 (1999)93; PRC 63 (2001) 044323; NPA 705 (2002) 297; NPA 732 (2004) 49; NPA 741 (2004) 60.
- Interplay of Spin and Pairing correlations and their effects upon Gamow-Teller transitions :
  PRC 78 (2008) 014317
- The decay of Te isotopes: testing basic notions :
  PRC 81 (2010) 014315
- Interplay between GT and IVSM and orbital-spin modes:
  PRC 86 (2012) 024314; PRC 89 (2014) 044319

Basic definitions ($2\nu\beta\beta - decays$)

$$\left[t_{1/2}^{(2\nu)}(0_i^+ \to J_f^+)\right]^{-1} = G^{(2\nu)}(J)\left|M^{(2\nu)}(J)\right|^2$$

$$M^{(2\nu)}(J) = \sum_{k_1 k_2} \frac{M_{\mathrm{F}}^{J}(1_{k_1}^+)\langle 1_{k_1}^+|1_{k_2}^+\rangle M_{\mathrm{I}}(1_{k_2}^+)}{(\frac{1}{2}\Delta + \frac{1}{2}[E(1_{k_1}^+) + \tilde{E}(1_{k_1}^+)] - M_i c^2)/m_{\mathrm{e}}c^2} \ .$$

$$\langle J_{k_1}^{\pi}|J_{k_2}^{\pi}\rangle = \sum_{pn}\left[X_{pn}^{J^\pi k_2}\bar{X}_{pn}^{J^\pi k_1} - Y_{pn}^{J^\pi k_2}\bar{Y}_{pn}^{J^\pi k_1}\right] \ .$$

It is a second order term, the intermediate states may be different from each side of the virtual transitions

Matrix elements

$$M_{\mathrm{I}}(1^+_{k_2}) = (1^+_{k_2}\|\sum_n t^-_n \boldsymbol{\sigma}_n\|0^+_i)$$

$$M^J_{\mathrm{F}}(1^+_{k_1}) = (J^+_f\|\sum_n t^-_n \boldsymbol{\sigma}_n\|1^+_{k_1})$$

$$M_{\mathrm{I}}(1^+_{k_2}) = \frac{1}{\sqrt{3}}\sum_{pn}(p\|\boldsymbol{\sigma}\|n)(1^+_{k_2}\|[c^\dagger_p \tilde{c}_n]_1\|0^+_i)$$

$$M^J_{\mathrm{F}}(1^+_{k_1}) = \frac{1}{\sqrt{3}}\sum_{pn}(p\|\boldsymbol{\sigma}\|n)(J^+_f\|[c^\dagger_{p'} \tilde{c}_{n'}]_1\|1^+_{k_1})$$

The use of the $t^{\pm}$ operators may not be correct for open shell systems, due to induced isospin violations

GT strengths

$$\mathsf{GT}_k^- = |(1_k^+\| \sum_n t_n^- \boldsymbol{\sigma}_n \|0_i^+)|^2$$

$$\mathsf{GT}_k^+ = |(1_k^+\| \sum_n t_n^+ \boldsymbol{\sigma}_n \|0_f^+)|^2 ,$$

Other correlations, like IVSM modes, may add (substract) to the strength due to couplings with the GT correlations. GT strength distributions available from data.

$0\nu\beta\beta$

$$
\begin{aligned}
t_{1/2}^{(0\nu)} &= g^{(0\nu)} \left| M^{(0\nu)\prime} \right|^{-2} (|\langle m_\nu \rangle|[\mathrm{eV}])^{-2} \\
\langle m_\nu \rangle &= \sum_j \lambda_j^{\mathrm{CP}} m_j |U_{\mathrm{e}j}|^2 \, . \\
M^{(0\nu)\prime} &= \left(\frac{g_{\mathrm{A}}}{g_{\mathrm{A}}^{\mathrm{b}}} \right)^2 \left[M_{\mathrm{GT}}^{(0\nu)} - \left(\frac{g_{\mathrm{V}}}{g_{\mathrm{A}}} \right)^2 M_{\mathrm{F}}^{(0\nu)} + M_{\mathrm{T}}^{(0\nu)} \right] \\
M_{\mathrm{F}}^{(0\nu)} &= \sum_k (0_f^+ || \sum_{mn} h_{\mathrm{F}}(r_{mn}, E_k) || 0_i^+) \, , \quad r_{mn} = |\mathbf{r}_m - \mathbf{r}_n| \, , \\
M_{\mathrm{GT}}^{(0\nu)} &= \sum_k (0_f^+ || \sum_{mn} h_{\mathrm{GT}}(r_{mn}, E_k)(\boldsymbol{\sigma}_m \cdot \boldsymbol{\sigma}_n) || 0_i^+) \, ,
\end{aligned}
$$

$$h_K(r_{mn}, E_k) = \frac{2}{\pi} R_{\mathrm{A}} \int dq \frac{q h_K(q^2)}{q + E_k - (E_{\mathrm{i}} + E_{\mathrm{f}})/2} j_0(q r_{mn})$$

$$M_K^{(0\nu)} = \sum_{J^\pi, k_1, k_2, J'} \sum_{pp'nn'} (-1)^{j_n + j_{p'} + J + J'} \sqrt{2J'+1} \times \left\{ \begin{array}{ccc} j_p & j_n & J \\ j_{n'} & j_{p'} & J' \end{array} \right\}$$
$$(pp' : J' || \mathcal{O}_K || nn' : J') \times (0_f^+ || \left[c_{p'}^\dagger \tilde{c}_{n'} \right]_J || J_{k_1}^\pi) \langle J_{k_1}^\pi | J_{k_2}^\pi \rangle (J_{k_2}^\pi || \left[c_p^\dagger \tilde{c}_n \right]_J || 0_i^+)$$

$$\mathcal{O}_{\mathrm{F}} = h_{\mathrm{F}}(r, E_k) \ , \quad \mathcal{O}_{\mathrm{GT}} = h_{\mathrm{GT}}(r, E_k) \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \ , \quad r = \left| \mathbf{r}_1 - \mathbf{r}_2 \right| \ ,$$

Transitions between structureless nucleons, momentum cut-off dependent, renormalization effects upon g_A (PLB 725(2013) 153)

$$|J^\pi_k\ M\rangle = \sum_{pn} \left(X^{J^\pi k}_{pn} \left[a^\dagger_p a^\dagger_n\right]_{JM} - Y^{J^\pi k}_{pn} \left[a^\dagger_p a^\dagger_n\right]^\dagger_{JM} \right) |\mathrm{QRPA}\rangle\ ,$$

$$\begin{aligned}
(0^+_f|||[c^\dagger_{p'}\tilde{c}_{n'}]_J||J^\pi_{k_1}) &= \sqrt{2J+1}\left[\bar{v}_{p'}\bar{u}_{n'}\bar{X}^{J^\pi k_1}_{p'n'} + \bar{u}_{p'}\bar{v}_{n'}\bar{Y}^{J^\pi k_1}_{p'n'}\right] \\
(J^\pi_{k_2}|||[c^\dagger_p\tilde{c}_n]_J||0^+_i) &= \sqrt{2J+1}\left[u_p v_n X^{J^\pi k_2}_{pn} + v_p u_n Y^{J^\pi k_2}_{pn}\right]\ ,
\end{aligned}$$

Transition densities for all possible multipoles, not always amenable to comparison with data.

Results (Jastrow)

Calculated ground-state-to-ground-state NMEs for $g_{\mathrm{A}} = 1.25$ using the Jastrow short-range correlations. The last line summarizes the overall magnitude and the associated dispersion of the NMEs of the cited nuclear model (without ^{48}Ca included).

Transition	pnQRPA(J)	IBA-2(J)	ISM(J)	PHFB(J)
^{48}Ca $\rightarrow$ ^{48}Ti	1.67 ± 0.09	2.00	0.61	-
^{76}Ge $\rightarrow$ ^{76}Se	3.83 ± 0.53	5.46	2.30	-
^{82}Se $\rightarrow$ ^{82}Kr	3.15 ± 0.30	4.41	2.18	-
^{96}Zr $\rightarrow$ ^{96}Mo	2.07	2.53	-	2.80 ± 0.10
^{100}Mo $\rightarrow$ ^{100}Ru	2.74	3.73	-	6.19 ± 0.46
^{110}Pd $\rightarrow$ ^{110}Cd	4.15 ± 0.41	3.62	-	7.07 ± 0.58
^{116}Cd $\rightarrow$ ^{116}Sn	3.03	2.78	-	-
^{124}Sn $\rightarrow$ ^{124}Te	3.30 ± 0.92	3.53	2.10	-
^{128}Te $\rightarrow$ ^{128}Xe	3.80 ± 0.37	4.52	2.34	3.59 ± 0.28
^{130}Te $\rightarrow$ ^{130}Xe	3.47 ± 0.37	4.06	2.12	4.01 ± 0.45
^{136}Xe $\rightarrow$ ^{136}Ba	2.36 ± 0.22	3.35	1.76	-
Overall NME	3.19 ± 0.66	3.80 ± 0.86	2.13 ± 0.21	4.73 ± 1.81

Results(UCOM)

Calculated ground-state-to-ground-state NMEs for $g_A = 1.25$ using the UCOM short-range correlations. The last line summarizes the overall magnitude and the associated dispersion of the NMEs of the cited nuclear model (without ^{48}Ca included).

Transition	pnQRPA(U)	EDF(U)	ISM(U)	PHFB(U)
^{48}Ca $\rightarrow$ ^{48}Ti	-	2.37	0.85	-
^{76}Ge $\rightarrow$ ^{76}Se	5.18 ± 0.54	4.60	2.81	-
^{82}Se $\rightarrow$ ^{82}Kr	4.20 ± 0.35	4.22	2.64	-
^{96}Zr $\rightarrow$ ^{96}Mo	3.12	5.65	-	3.32 ± 0.12
^{100}Mo $\rightarrow$ ^{100}Ru	3.93	5.08	-	7.22 ± 0.50
^{110}Pd $\rightarrow$ ^{110}Cd	5.63 ± 0.49	-	-	8.23 ± 0.62
^{116}Cd $\rightarrow$ ^{116}Sn	3.93	4.72	-	-
^{124}Sn $\rightarrow$ ^{124}Te	4.57 ± 1.33	4.81	2.62	-
^{128}Te $\rightarrow$ ^{128}Xe	5.26 ± 0.40	4.11	2.88	4.22 ± 0.31
^{130}Te $\rightarrow$ ^{130}Xe	4.76 ± 0.41	5.13	2.65	4.66 ± 0.43
^{136}Xe $\rightarrow$ ^{136}Ba	3.16 ± 0.25	4.20	2.19	-
Overall NME	4.37 ± 0.86	4.72 ± 0.51	2.63 ± 0.24	5.53 ± 2.09

Isospin and gauge freedom

$$\beta^{(F-)} = \sqrt{2}\tau_1 .$$

$$\begin{aligned}
\beta^{(F-)} &\rightarrow \sqrt{2}\left(D^1_{11}\tau_1 + D^1_{10}\tau_0 + D^1_{1(-1)}\tau_{-1}\right) \\
&= \sqrt{2}D^1_{10}\langle\tau_0\rangle + O\left(T^{-\frac{1}{2}}\right) \\
&= -\sqrt{2T}\,\xi^+ + O\left(T^{-\frac{1}{2}}\right)
\end{aligned}$$

$$
\begin{aligned}
\tau_0 &= \sum_j \tau_{0j} \; ; \qquad \tau_{0j} = \frac{1}{2} \sum_m \left(c^+_{pjm} c_{pjm} - c^+_{njm} c_{njm} \right) \\
\tau_1 &= \sum_j \tau_{1j} \; ; \qquad \tau_{1j} = -\frac{1}{\sqrt{2}} \sum_m c^+_{pjm} c_{njm} \\
\tau_{\bar{1}} &= -\tau_1^+
\end{aligned}
$$

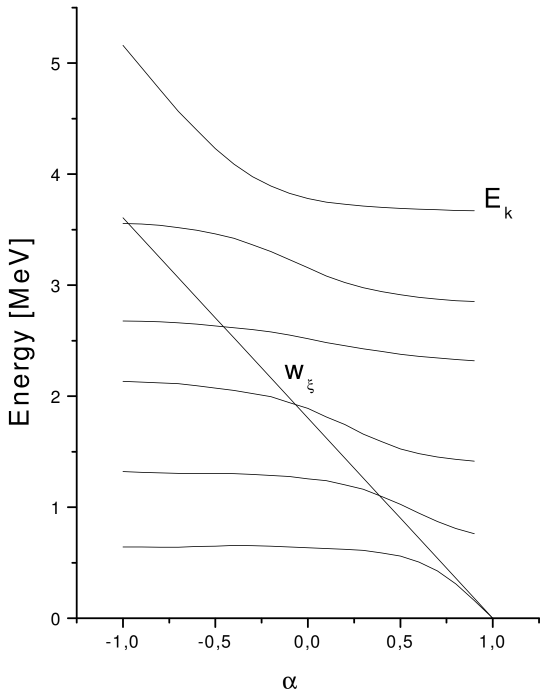
$$\begin{aligned} H &= H_{sp} + H_{TD} + H_{sc} \\ H_{sp} &= \sum_j \left(\epsilon_{aj} + \epsilon_{0j}\right) \tau_{0j} \\ H_{TD} &= \alpha \sum_{k,j} \langle k|\hat{V}|j\rangle \tau_{1k} \tau_{\bar{1}j} \\ H_{sc} &= -\beta \sum_{k,j} \frac{\langle k|\hat{V}|j\rangle}{2} \tau_{0k} \tau_{0j} \end{aligned}$$

$$\epsilon_{0j} = s_j + r_j \qquad s_j = \sum_k \langle j|\hat{V}|k\rangle (2k+1) \qquad r_j = \frac{1}{2} \langle j|\hat{V}|j\rangle$$

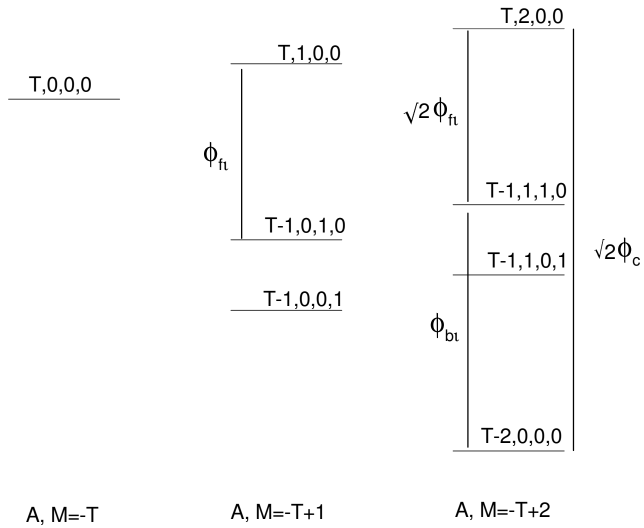
$$
\begin{aligned}
[\tau_1, H] \quad &= \quad -\sum_j \left((1-\alpha)\, s_j \; + \; (1-\beta)\, r_j\right)\; \tau_{1j} \\
&\quad\;\; -(\alpha - \beta) \; \sum_k \langle k|\hat{V}|j\rangle \, \tau_{ik}\tau_{0j}
\end{aligned}
$$

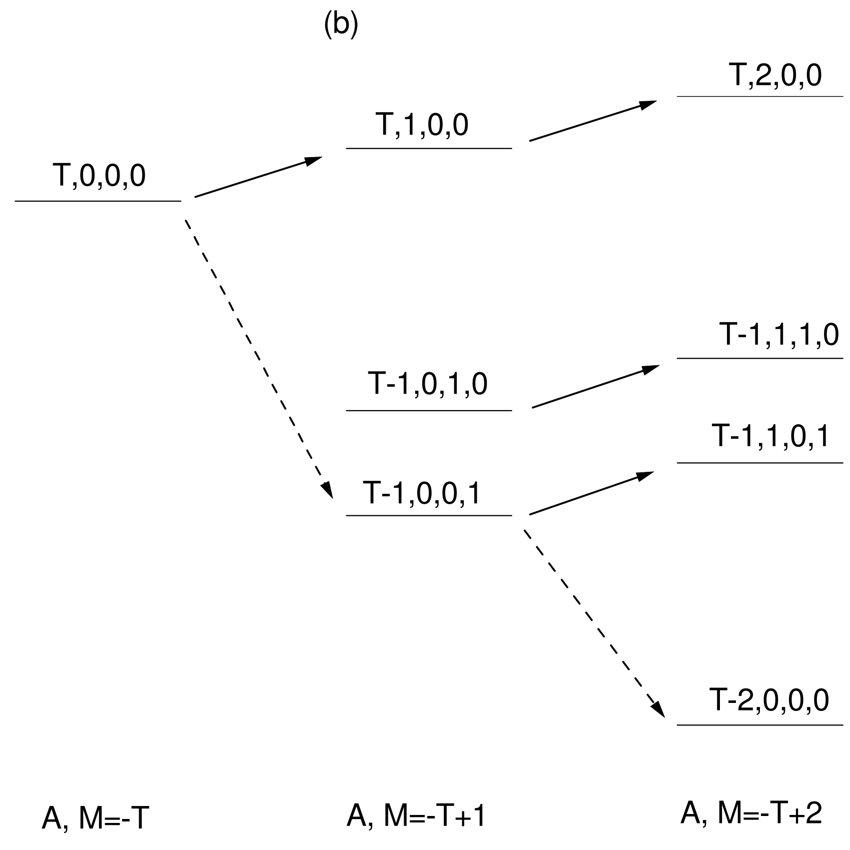
Spectrum

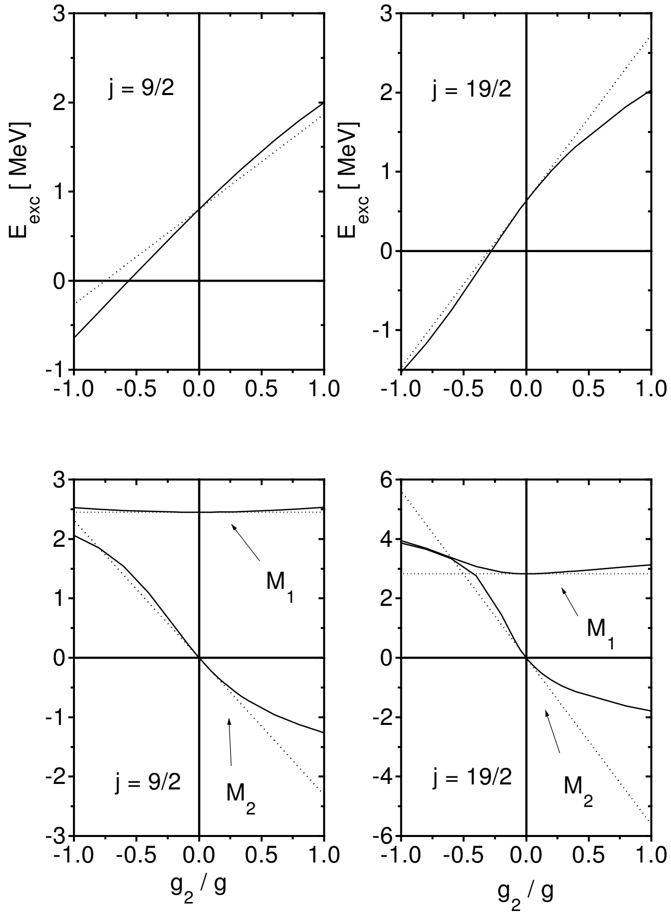
RPA spectrum showing the symmetry restoring mechanism, the collective solution W_ζ is decoupled from the intrinsic solutions at the point $\alpha = 1$, affecting the first excited state of the system (energies E_k)



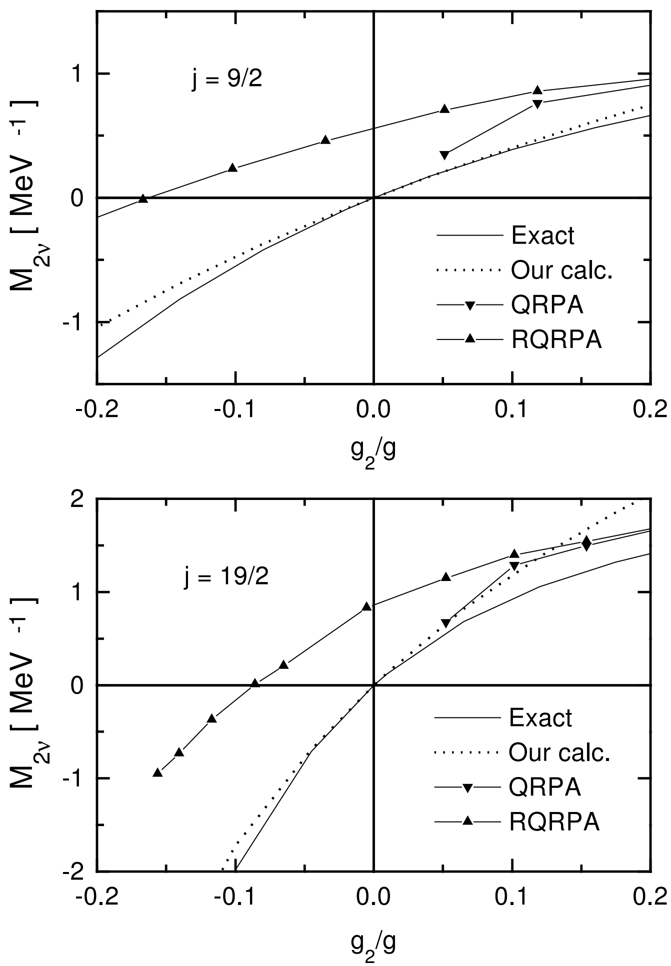
(a)


T,0,0,0
T,1,0,0
ϕ_{ft}
T-1,0,1,0
T-1,0,0,1
T,2,0,0
$\sqrt{2}\phi_{ft}$
T-1,1,1,0
T-1,1,0,1
ϕ_{bt}
T-2,0,0,0
$\sqrt{2}\phi_{c}$
A, M=-T
A, M=-T+1
A, M=-T+2


(b)
T,0,0,0
T,1,0,0
T,2,0,0
T-1,0,1,0
T-1,0,0,1
T-1,1,1,0
T-1,1,0,1
T-2,0,0,0
A, M=-T
A, M=-T+1
A, M=-T+2



j = 9/2
j = 19/2
E_exc [MeV]
M_1
M_2
g_2 / g


j = 9/2
$M_{2\nu}$ [MeV^{-1}]
g_2/g
Exact
Our calc.
QRPA
RQRPA
j = 19/2
$M_{2\nu}$ [MeV^{-1}]
g_2/g
Exact
Our calc.
QRPA
RQRPA

Therefore...

- The interplay between intrinsic and collective coordinates guarantees that the isospin symmetry is restored
- The instabilities of the QRPA (and RPA) approaches are avoided by the explicit elimination of the zero frequency mode
- The appearance of this mode cannot be avoided by any added renormalization, like the RQRPA one
- Use of effective couplings may not be correct due to the presence of the zero frequency mode
- The reconstruction of the matrix element (Fermi sector) cannot be achieved at the level of the standard QRPA or RPA methods

Pairing and GT modes of excitation

- The (^{3}He,t) and (p,n) reactions explore isospin-spin dependent excitations and they have been used as experimental tool for large scale nuclear structure studies.
- A typical energy spectrum of tritons from (^{3}He,t) reactions on medium-heavy nuclei displays the narrow peak corresponding to IAS and the broad distribution of the GTR.
- Studies on ^{58}Cu show, in addition, the existence of four T=1, I=1 states and a high energy spin-dipole resonance. All these excitations participate in low energy charge-exchange, beta-decay and electron-capture processes.

Microscopic phonon structure

$$[a^+_{j_1} a^+_{j_2}]^{I=0\,T=1}_{0T_z} \;; \qquad [a^+_{j_1} a^+_{j_2}]^{10}_{M0} \;; \qquad [a^+_{j_1} a_{j_2}]^{11}_{MT_z} \,.$$

$$\begin{aligned}
P^+_{IM,TT_z,1} &= f^{IT1}_{j_1 j_2} \, [a^+_{j_1} a^+_{j_2}]^{I\,T}_{MT_z} \\
P^+_{1M,1T_z,0} &= f^{110}_{j_1 j_2} \, [a^+_{j_1} a_{j_2}]^{1\,1}_{MT_z} \\
P^+_{IM,TT_z,-1} &= (-1)^{I+T+M+T_z} P_{I(-M),T(-T_z),1}, \\
f^{01(\pm 1)}_{j_1 j_2} &= \delta_{j_1 j_2} \, \hat{j}_1 \;; \; f^{10(\pm 1)}_{j_1 j_2} = f^{110}_{j_1 j_2} = \frac{\langle j_1 || \sigma || j_2 \rangle}{\sqrt{3}}. \\
H^{IT\alpha} &= -\frac{g^{IT\alpha}}{1+\delta_{\alpha 0}} P^+_{IM,TT_z,\alpha} P_{IM,TT_z,\alpha}.
\end{aligned}$$

Collective phonon structure

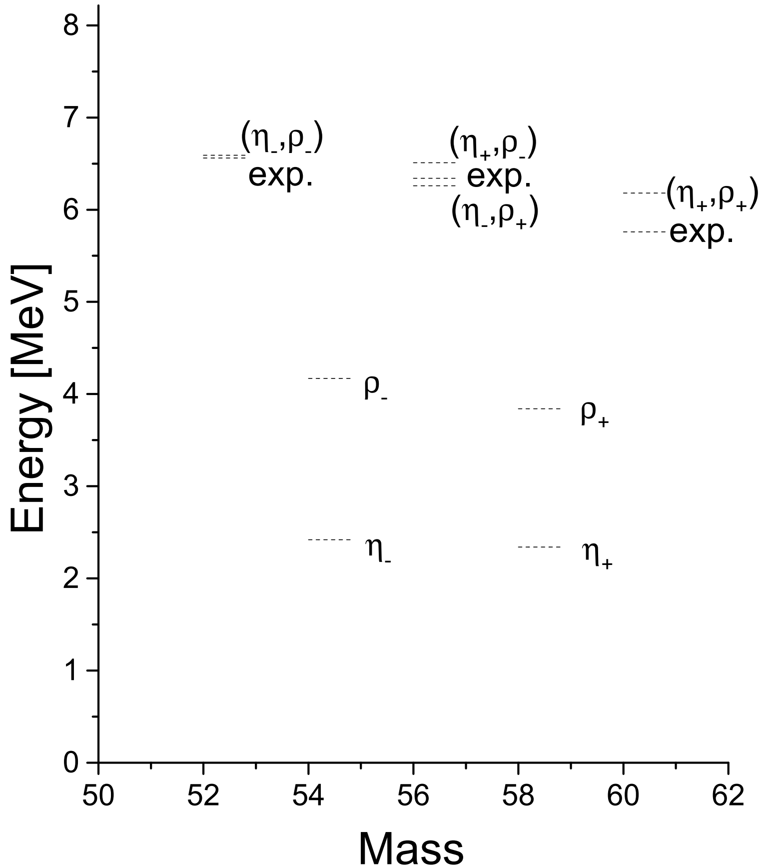
$$
\begin{aligned}
\Gamma^{+}_{IM,TT_z,\alpha;\nu} &= \lambda^{IT\alpha}_{\nu;j_1j_2}\,\gamma^{+}_{IM,TT_z,\alpha;j_1j_2} \\
&\quad -(-1)^{I+M+T+T_z}\,\mu^{IT\alpha}_{\nu;j_1j_2}\,\gamma_{I(-M),T(-T_z);-\alpha;j_1j_2} \\
\left(P^{+}_{IM,TT_z,\alpha}\right)_{coll} &= \frac{\Xi^{IT\alpha}_{\nu}}{g^{IT\alpha}}\,\Gamma^{+}_{IM,TT_z,\alpha;\nu} \\
&\quad + \frac{\Xi^{IT(-\alpha)}_{\nu}}{g^{IT(-\alpha)}}\,(-1)^{I+M+T+T_z}\,\Gamma_{I(-M),T(-T_z),-\alpha;\nu}\;.
\end{aligned}
$$

The collective Hamiltonian

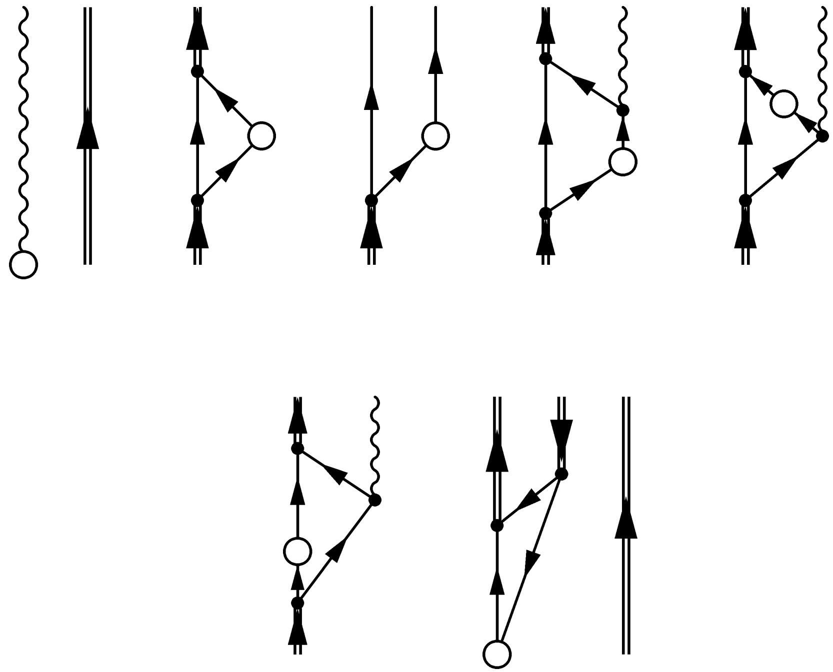
$$
\begin{aligned}
H_{pv} \quad = \quad & -\frac{g^{IT\alpha}}{1+\delta_{\alpha 0}} \left(\left(P^{+}_{IM,TT_z,\alpha} \right)_{coll} P_{IM,TT_z,\alpha} \right. \\
& \left. + P^{+}_{IM,TT_z,\alpha} \left(P_{IM,TT_z,\alpha} \right)_{coll} \right)
\end{aligned}
$$

I	T	α	ν	$g^{IT\alpha}$	$\omega_\nu^{IT\alpha}$	$\Xi_\nu^{IT\alpha}$
0	1	1	1	0.387	2.34	1.42
			2		5.61	0.38
			3		6.83	0.077
0	1	-1	1	0.402	2.42	1.52
			2		15.38	0.69
			3		18.55	0.33
			4		23.32	0.55
1	0	1	1	0.171	3.84	0.50
			2		5.32	0.21
			3		6.38	0.092
			4		6.89	0.036
1	0	-1	1	0.212	4.17	0.58
			2		16.02	0.39
			3		17.94	0.52
			4		19.51	0.26
1	1	0	1	-0.272	7.00	0.53

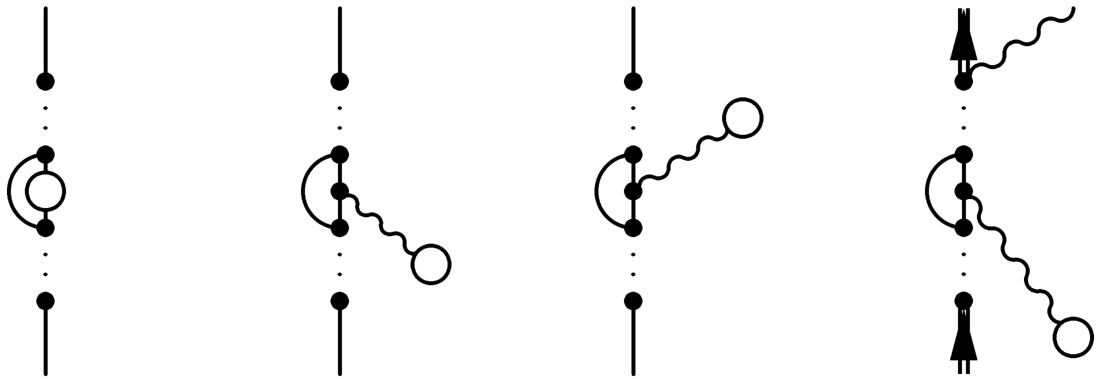
$I^\pi = 1^+$ states states around A=56.



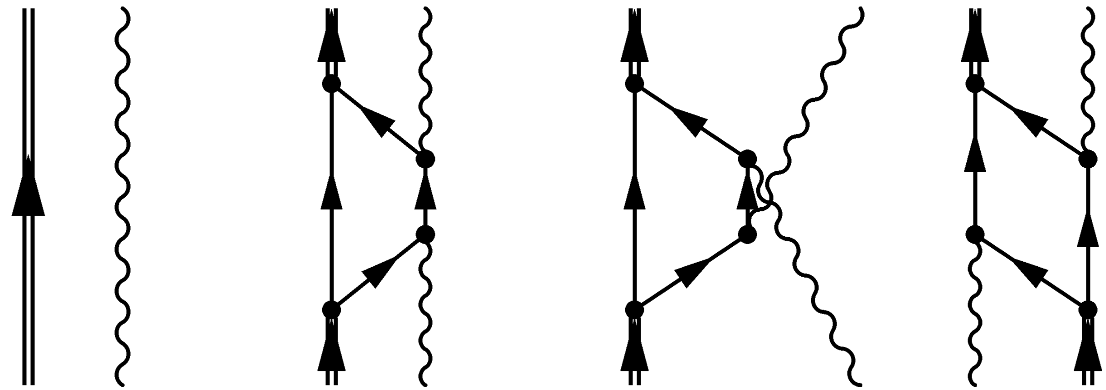
Processes accounted for in the treatment of GT transitions



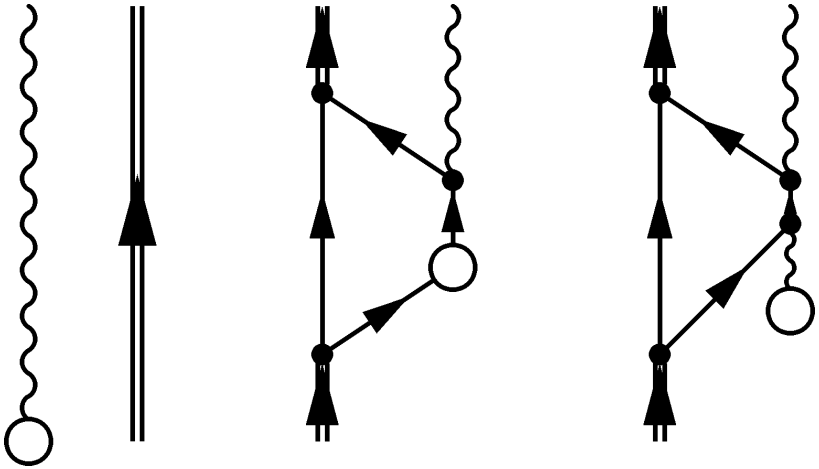
Renormalization.



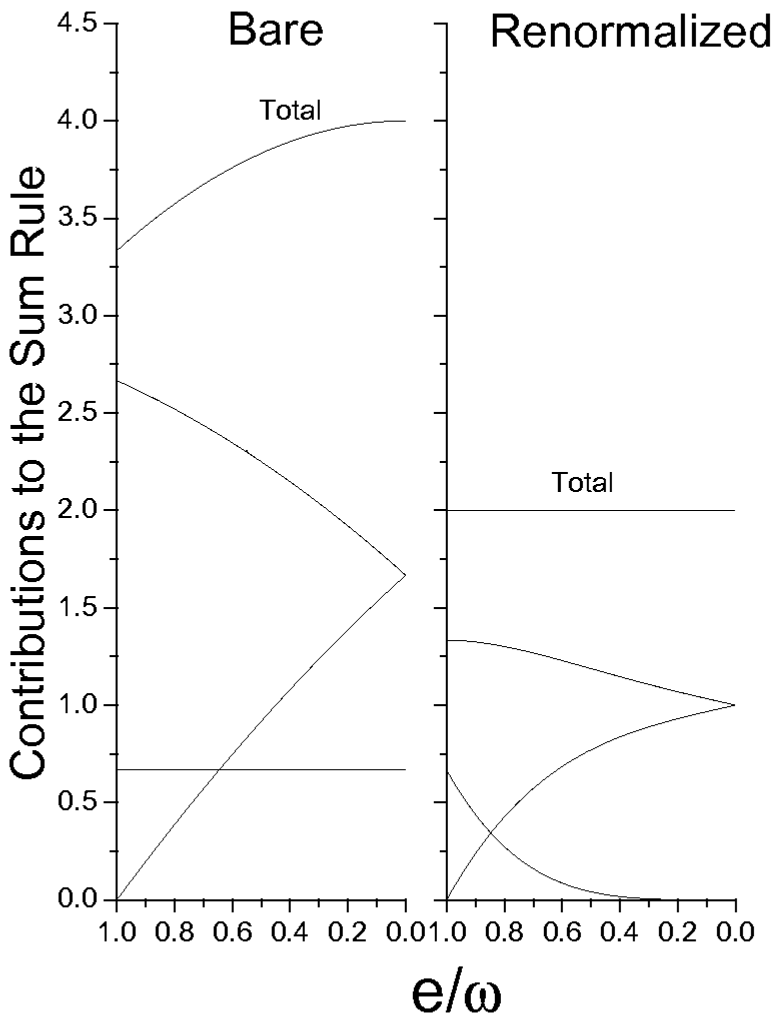
Diagrammatic corrections to the energy of the two-phonon states



Diagrams which illustrate the action of the collective operator (open circles)



The Ikeda sum rule revisited



Thus...

- The validity of the ISP and IVP vibrational model was tested against the low lying spectrum of even and odd-mass nuclei around the double-closed shell.
- In the example, the ground state of ^{58}Ni was interpreted as the low-lying one-phonon IVP excitation of the double closed shell nucleus ^{56}Ni. Final 1^+ states in ^{58}Cu included both one ISP phonon and a superposition of GT and pairing phonons.
- The procedure to calculate renormalization diagrams, including those cases in which the GTR was populated yields an effective charge for each transition.
- The departure from a single effective charge for all the shell is essential to preserve the Ikeda sum rule, as was shown exactly for schematic models.

Matrix elements for the g.s to g.s $2\nu\beta^-\beta^-$ decay of Te isotopes

- Experimental constraints for theoretical results on double-beta decay transitions e.g.: experimentally extracted occupancies and energies.
- Study of single- and double-beta decay transitions in nuclei near closed shells (or with few particles outside the closed shell). e.g: to handle fewer degrees of freedom
- Fragility of the pn-QRPA results due to failures of the BCS for two and four proton configurations.e.g: pairs and pairing-phonons rather than quasiparticles
- Interplay between Pairing and Gamow-Teller vibrational modes e.g: fragmentation of the GT strength
- Renormalization of the Gamow-Teller strength. : States with energy greater (smaller) than the GTR produce opposite effects

- The decay involves only two protons in the initial state and four in the final state in Xe.
- The large neutron-excess contributes to the decoupling between neutron and proton mean fields and thus to the preservation of good isospin properties.
- The coupling between GT and pairing modes must be accounted for (see our previous calculation in 58Ni).

Phenomenology

- Elementary degrees of freedom and phenomenology from ($^3He, n$), (n, p), (p, n) and ($^3He, t$) reactions.

Refs: (Alford et al)
NPA 281, pag 389, NPA 323, pag 339, NPA 304, pag 520
for (^{3}He,n) reactions in this region.

The empirical evidence for describing the motion of neutrons in Sn isotopes as a superfluid system is well documented in the literature. The main features of this description are:

- Nuclear masses vary smoothly but for the odd-even pairing effect.
- Energies associated with the ground state of even superfluid systems may be grouped together in terms of rotations in gauge space, that is the number-angle space, with Δ as a deformation parameter
- The enhanced, specific operators for superfluid systems are the two-body transfer operators P^+, P, where

$$P^+ = \sum_j \hat{j}[c_j^+ c_j^+]^0$$

These operators may be realized by means of (t,p) or (p,t) reactions. The ground state to ground state transitions are expected to stay fairly constant along the rotational band and to be much larger than the transitions to excited $I^\pi = 0^+$ states.

Evidences on proton-pair and pairing phonons:I

- Distorted wave calculations relate the experimental cross sections to the predicted ones. We write

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{exp}} = \epsilon\, C^2\, S\, \left(\frac{d\sigma}{d\Omega}\right)_{\text{code}},$$

where C is an isospin Clebsch-Gordan coefficient and S, a two-proton spectroscopic factor. The "enhancement" factor ϵ is adjusted to reproduce the data. In the present discussion we concentrate on the values of ϵ derived in refs (Alford et al) Refs:(NPA 281, pag 389),(NPA 323, pag 339), (NPA 304, pag 520), for (^{3}He,n) reactions in this region.

Evidences on proton-pair and pairing phonons: II

- It is inherent to the vibrational description that the population of the pairing vibrational state of Sn^{A} should be identical to the transition populating the ground state of Te^{A+2}.
- Another consequence of the model is that strengths of ground to ground state transitions should be proportional to the number of bosons in the initial state for removal phonons, and in the final state for addition phonons.

Microscopic phonon structure (Pairing sector)

Pairs of single-particle operators coupled to good angular momentum I and isospin T

$$\left[a^{+}_{j_1} a^{+}_{j_2}\right]^{I=0\,T=1}_{0T_z} \;; \qquad \left[a^{+}_{j_1} a^{+}_{j_2}\right]^{10}_{M0} \;; \qquad \left[a^{+}_{j_1} a_{j_2}\right]^{11}_{MT_z}$$

Here a^{+}_{j} is either $= b^{+}_{j}$ or b^{+}_{h}. We assume separable residual interactions of the form

$$H^{IT\alpha} = -\frac{g^{IT\alpha}}{1+\delta_{\alpha 0}} P^{+}_{IM,TT_z,\alpha} P_{IM,TT_z,\alpha}.$$

The coupled pairing phonons

$$\Gamma^{+}_{IM,TT_z,\alpha;\nu} = \lambda^{IT\alpha}_{\nu;j_1j_2}\,\gamma^{+}_{IM,TT_z,\alpha;j_1j_2} - (-1)^{I+M+T+T_z}\,\mu^{IT\alpha}_{\nu;j_1j_2}\,\gamma_{I(-M),T(-T_z);-\alpha;j_1j_2}\ .$$

The $P^{+}_{IM,TT_z,\alpha}$ operators have a collective version, which is obtained through the inversion of the previous equations

$$\left(P^{+}_{IM,TT_z,\alpha}\right)_{coll} = \frac{\Xi^{IT\alpha}_{\nu}}{g^{IT\alpha}}\,\Gamma^{+}_{IM,TT_z,\alpha;\nu} + \frac{\Xi^{IT(-\alpha)}_{\nu}}{g^{IT(-\alpha)}}\,(-1)^{I+M+T+T_z}\,\Gamma_{I(-M),T(-T_z),-\alpha;\nu}\ .$$

Particle-vibration couplings

$$H_{pv} = -\frac{g^{IT\alpha}}{1+\delta_{\alpha 0}}\left((P^{+}_{IM,TT_z,\alpha})_{coll}P_{IM,TT_z,\alpha} + P^{+}_{IM,TT_z,\alpha}\,(P_{IM,TT_z,\alpha})_{coll}\right) \; .$$

Experimental ratios of the factor ϵ.

N	$\frac{\sum \epsilon_i}{\epsilon(\text{Te(g.s)})}$	$\frac{\epsilon(\text{Pd}^{A-4}\rightarrow\text{Cd}^{A-2})}{\epsilon(\text{Cd}^{A-2}\rightarrow\text{Sn}^{A})}$	N	$\frac{\epsilon(\text{Te}^{A+2}\rightarrow\text{Xe}^{A+4})}{\epsilon(\text{Sn}^{A}\rightarrow\text{Te}^{A+2})}$
58	0.95	1.4	70	1.7
62	0.75	1.4	72	
64	0.85	1.9	74	2.1
66	1.23			
68	1.11			

e.g: Total strength (second column) should be closer to 1 and the ratios between the cross sections (third and fifth-columns) should be closer to 2

Energies of the proton-pairing phonon and of the pairing vibrations

N	$\omega_{A+2,Z+2}$	$\omega_{A-2,Z-2}$	$\omega^{(pv)}_{A,Z}$	$\omega^{(exp)}_{A,Z}$
52		1.74		
54	1.51	1.26	2.77	
56	1.49	1.10	2.59	
58	1.19	0.91	2.10	3.49
60	1.55	0.98	2.53	
62	1.36	1.11	2.47	2.88
64	1.30	1.24	2.54	2.90
66	1.41	1.26	2.67	3.02
68	1.32	1.20	2.53	2.58
70	1.30	1.21	2.51	
72	1.33	1.37	2.70	
74	1.46	1.52	2.98	
76	1.67	1.96	3.63	
78	1.94	2.33	4.27	

Energies of the two phonon states

N	$\omega_{A+4,Z+4}$	$\frac{\omega_{A+4,Z+4}}{\omega_{A+2,Z+2}}$	$\omega_{A-4,Z-4}$	$\frac{\omega_{A-4,Z-4}}{\omega_{A-2,Z-2}}$
52			3.14	1.80
54			2.41	1.91
56	2.65	1.78	2.14	1.95
58	2.44	2.06	1.97	2.16
60	2.29	1.47	2.01	2.05
62	2.25	1.66	2.02	1.82
64	2.16	1.66	1.99	1.60
66	2.01	1.43	1.93	1.53
68	1.87	1.42	1.92	1.60
70	1.88	1.45	2.00	1.65
72	2.19	1.64	2.48	1.81
74	2.36	1.61	3.31	2.18
76	2.81	1.68	3.73	1.93
78	3.36	1.73		

Theoretical steps

- Nuclear Structure of the initial and final states.
- The Hamiltonian of the model (Pairing and Gamow-Teller excitations)
- Gamow-Teller transitions (two-neutrinos).

Single-proton energies and Wood-Saxon parameters

N	$\epsilon_{7/2}+\epsilon_{9/2}$	$\epsilon_{5/2}-\epsilon_{7/2}$	V_{ls}/V	a
76	4.18	0.491	-0.44	0.59
78	4.28	0.645	-0.44	0.58

V	=	-51 MeV		(protons)
V	=	(-51+66 T/A) MeV		(neutrons)
V_{ls}	=	-0.44V		(neutrons)
R	=	r_0 A$^{1/3}$	r_0=1.27 fm	(protons and neutrons)
a	=	0.67 fm (neutrons)		

Pair-Operators

$$
\begin{aligned}
P^{+}_{1q} &= \frac{\langle j_1||\sigma||j_2\rangle}{\sqrt{3}}\ \left[b^{+}_{j_1} c^{+}_{j_2}\right]^1_q \\
&= \frac{\langle k||\sigma||j\rangle}{\sqrt{3}}\ \left(U_j\ \left[b^{+}_{k}\alpha^{+}_{j}\right]^1_q - V_j\ \left[b^{+}_{k}\alpha_{j}\right]^1_q\right) \\
&\quad + \frac{\langle j||\sigma||h\rangle}{\sqrt{3}}\ \left(-U_j\ \left[\alpha^{+}_{j} b^{+}_{h}\right]^1_q + V_j\ \left[\alpha_{j} b^{+}_{h}\right]^1_q\right) \\
P^{+}_{\bar{1}q} &= (-1)^{1+q}(P^{+}_{1\bar{q}})^{+} = \frac{\langle j_1||\sigma||j_2\rangle}{\sqrt{3}}\ \left[c_{j_1} b_{j_2}\right]^1_q \\
&= \frac{\langle j||\sigma||h\rangle}{\sqrt{3}}\ \left(V_j\ \left[\alpha^{+}_{j} b_{h}\right]^1_q + U_j\ \left[\alpha_{j} b_{h}\right]^1_q\right) \\
&\quad - \frac{\langle k||\sigma||j\rangle}{\sqrt{3}}\ \left(V_j\ \left[b_{k}\alpha^{+}_{j}\right]^1_q + U_j\ \left[b_{k}\alpha_{j}\right]^1_q\right)
\end{aligned}
$$

Gamow-Teller Operators

$$
\begin{aligned}
Q_{1q} &= -\frac{\langle j_1||\sigma||j_2\rangle}{\sqrt{3}}\ \left[b^{+}_{j_1} c_{j_2}\right]^1_q \\
&= -\frac{\langle k||\sigma||j\rangle}{\sqrt{3}}\ \left(V_j\ \left[b^{+}_{k}\alpha^{+}_{j}\right]^1_q + U_j\ \left[b^{+}_{k}\alpha_{j}\right]^1_q\right) \\
&\quad + \frac{\langle j||\sigma||h\rangle}{\sqrt{3}}\ \left(U_j\ \left[\alpha^{+}_{j} b^{+}_{h}\right]^1_q + V_j\ \left[b^{+}_{h}\alpha_{j} b^{+}_{h}\right]^1_q\right) \\
Q_{\bar{1}q} &= (-1)^{1+q}(Q_{1\bar{q}})^{+} = \frac{\langle j_1||\sigma||j_2\rangle}{\sqrt{3}}\ \left[c^{+}_{j_1} b_{j_2}\right]^1_q \\
&= \frac{\langle j||\sigma||h\rangle}{\sqrt{3}}\ \left(U_j\ \left[\alpha^{+}_{j} b_{h}\right]^1_q - V_j\ \left[\alpha_{j} b_{h}\right]^1_q\right) \\
&\quad + \frac{\langle k||\sigma||j\rangle}{\sqrt{3}}\ \left(V_j\ \left[\alpha^{+}_{j} b_{k}\right]^1_q - U_j\ \left[\alpha_{j} b_{k}\right]^1_q\right)
\end{aligned}
$$

The Hamiltonian

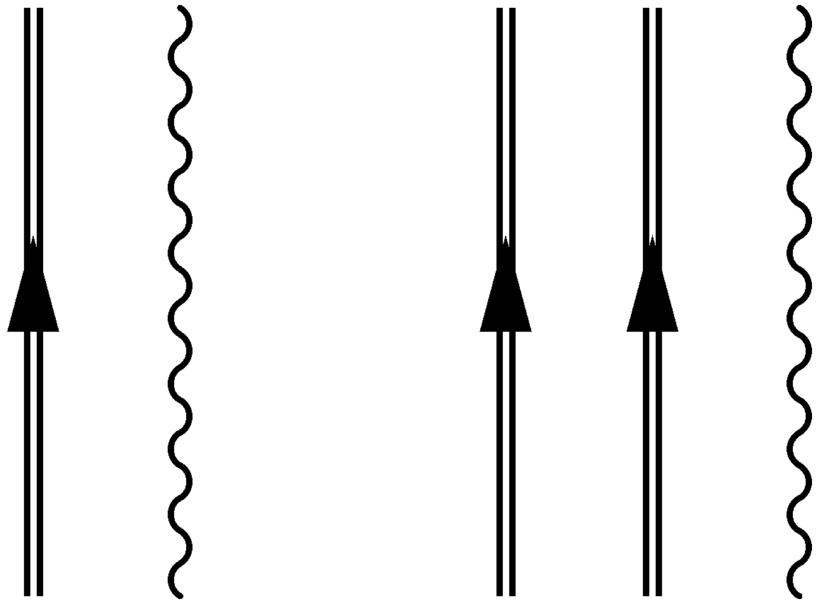
$$H = -g_p\sqrt{3}[P_1^+ P_{\bar{1}}^+]^0 - g\sqrt{3}[Q_1 Q_{\bar{1}}]^0$$

The basis

$$|a,n\rangle = \Gamma^+_{00,11,1}\,\Gamma^+_{1q,11,n}\,|0\rangle$$

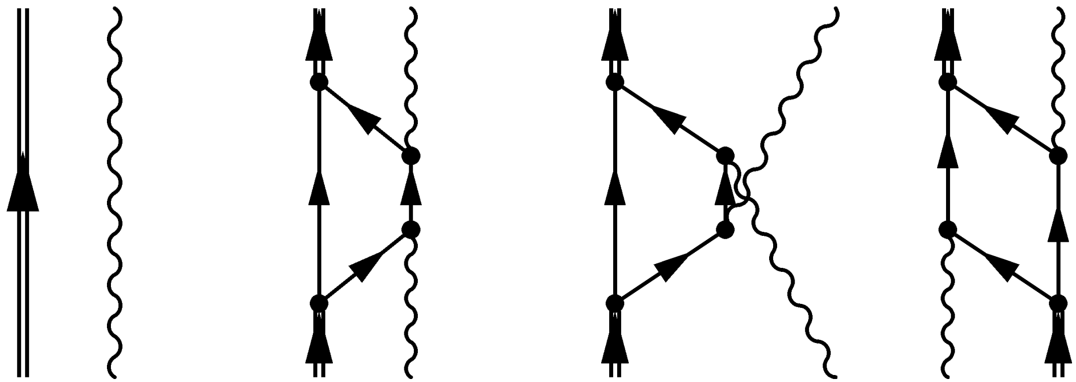
$$|b,n\rangle = \frac{1}{\sqrt{2}}(\Gamma^+_{00,11,1})^2\,\Gamma^+_{1q,1\bar{1},n}\,|0\rangle$$

Intermediate States



Pairing phonons ⊗GT phonons

Energy of the intermediate states



Collective pairs of fermions

$$
\begin{aligned}
(P^+_{1q})_{coll} &= -\frac{\langle k||\sigma||j\rangle}{\sqrt{3}} U_j \gamma^+_{kjq} + (-1)^{1+q} \frac{\langle j||\sigma||h\rangle}{\sqrt{3}} V_j \gamma_{jh\bar{q}} \\
(P^+_{\bar{1}q})_{coll} &= \frac{\langle j||\sigma||h\rangle}{\sqrt{3}} V_j \gamma^+_{jhq} + (-1)^{q} \frac{\langle k||\sigma||j\rangle}{\sqrt{3}} U_j \gamma_{kj\bar{q}} \\
(Q_{1q})_{coll} &= \frac{\langle k||\sigma||j\rangle}{\sqrt{3}} V_j \gamma^+_{kjq} + (-1)^{1+q} \frac{\langle j||\sigma||h\rangle}{\sqrt{3}} U_j \gamma_{jh\bar{q}} \\
(Q_{\bar{1}q})_{coll} &= \frac{\langle j||\sigma||h\rangle}{\sqrt{3}} U_j \gamma^+_{jhq} + (-1)^{1+q} \frac{\langle k||\sigma||j\rangle}{\sqrt{3}} V_j \gamma_{kj\bar{q}} \\
(H_{pn})_{RPA} &= -g_p\sqrt{3}[(P^+_1)_{coll}(P^+_{\bar{1}})_{coll}]^0 - g\sqrt{3}[(Q_1)_{coll}(Q_{\bar{1}})_{coll}]^0 \\
(H_{sp})_{RPA} &= \epsilon_{kj}\, \gamma^+_{kj,q1}\gamma_{kj,q1} + \epsilon_{jh}\, \gamma^+_{jh,q\bar{1}}\gamma_{jh,q\bar{1}} \\
\epsilon_{kj} &= e_k + E_j \\
\epsilon_{jh} &= e_j + E_h
\end{aligned}
$$

Phonons

$$
\begin{aligned}
\Gamma^{+}_{1nq} &= \lambda_{kjn}\gamma^{+}_{kjq} + (-1)^{q}\mu_{jhn}\gamma_{jh\bar{q}} \\
\Gamma^{+}_{\bar{1}nq} &= \lambda_{jhn}\gamma^{+}_{jhq} + (-1)^{q}\mu_{kjn}\gamma_{kjn} \\
\gamma^{+}_{kjq} &= \left[b^{+}_{k}\alpha^{+}_{j}\right]^{1}_{q} \\
\gamma^{+}_{jhq} &= \left[\alpha^{+}_{j}b^{+}_{h}\right]^{1}_{q}
\end{aligned}
$$

$b \equiv$ proton-particle(hole)-operators

$\alpha \equiv$ neutron-quasiparticle-operators

Two-Mode RPA Equations

$$
\begin{aligned}
0 &= (g_p^{-1} + X_{\nu n})\,\Lambda_{\nu n} \;-\; \nu Z_{\nu n}\,\Xi_{\nu n} \\
0 &= -\nu Z_{\nu n}\,\Lambda_{\nu n} \;+\; (g^{-1} + Y_{\nu n})\,\Xi_{\nu n} \\
X_{\nu n} &= -\frac{\langle k||\sigma||j\rangle^2}{3}\,\frac{U_j^2}{\epsilon_{kj} - \nu\omega_{\nu n}} - \frac{\langle j||\sigma||h\rangle^2}{3}\,\frac{V_j^2}{\epsilon_{jh} + \nu\omega_{\nu n}} \\
Y_{\nu n} &= -\frac{\langle k||\sigma||j\rangle^2}{3}\,\frac{V_j^2}{\epsilon_{kj} - \nu\omega_{\nu n}} - \frac{\langle j||\sigma||h\rangle^2}{3}\,\frac{U_j^2}{\epsilon_{jh} + \nu\omega_{\nu n}} \\
Z_{\nu n} &= -\frac{\langle k||\sigma||j\rangle^2}{3}\,\frac{\nu U_j V_j}{\epsilon_{kj} - \nu\omega_{\nu n}} + \frac{\langle j||\sigma||h\rangle^2}{3}\,\frac{\nu U_j V_j}{\epsilon_{jh} + \nu\omega_{\nu n}} \\
\lambda_{kjn} &= \frac{\langle k||\sigma||j\rangle}{\sqrt{3}}\,\frac{(\Lambda_{1n}U_j - \Xi_{1n}V_j)}{\epsilon_{kj} - \omega_{1n}} \\
\mu_{jhn} &= -\frac{\langle j||\sigma||h\rangle}{\sqrt{3}}\,\frac{(\Lambda_{1n}V_j + \Xi_{1n}U_j)}{\epsilon_{jh} + \omega_{in}} \\
\lambda_{jhn} &= -\frac{\langle j||\sigma||h\rangle}{\sqrt{3}}\,\frac{(\Lambda_{\bar{1}n}V_j + \Xi_{\bar{1}n}U_j)}{\epsilon_{jh} - \omega_{\bar{1}n}} \\
\mu_{kjn} &= \frac{\langle k||\sigma||j\rangle}{\sqrt{3}}\,\frac{\sigma_{kj}(\Lambda_{\bar{1}n}U_j - \Xi_{\bar{1}n}V_j)}{\epsilon_{kj} + \omega_{\bar{1}n}}
\end{aligned}
$$

Collective Picture

$$\left(P^{+}_{\nu q}\right)_{coll} = \frac{1}{g_p}\left(-\Lambda_{\nu n}\Gamma^{+}_{\nu n q} + (-1)^q \Lambda_{\bar{\nu} n}\Gamma_{\bar{\nu} n \bar{q}}\right)$$

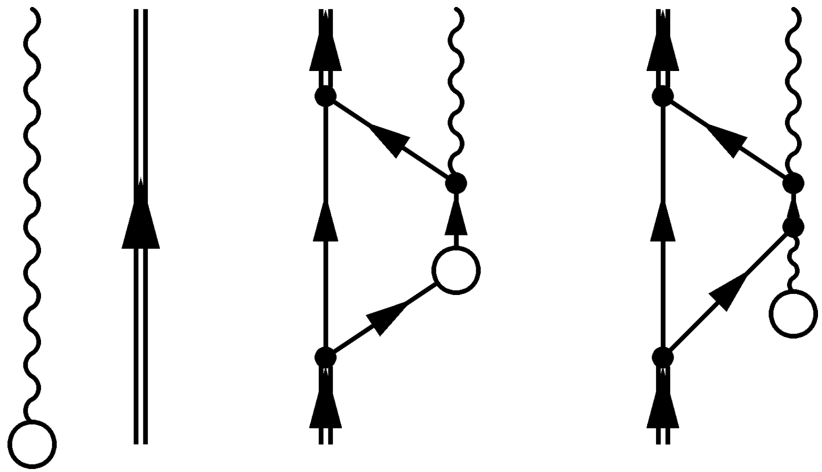
$$\left(Q_{\nu q}\right)_{coll} = \frac{1}{g}\left(-\Xi_{\nu n}\Gamma^{+}_{\nu n q} + (-1)^q \Xi_{\bar{\nu} n}\Gamma_{\bar{\nu} n \bar{q}}\right)$$

$$\begin{aligned} H_{pv} = & -g_p\sqrt{3}\left([(P^{+}_{1})_{coll}\, P^{+}_{\bar{1}}]^0 + [P^{+}_{1}\,(P^{+}_{\bar{1}})_{coll}]^0\right) \\ & - g\sqrt{3}\left([(Q_1)_{coll}\, Q_{\bar{1}}]^0 + [Q_1\,(Q_{\bar{1}})_{coll}]^0\right) \end{aligned}$$

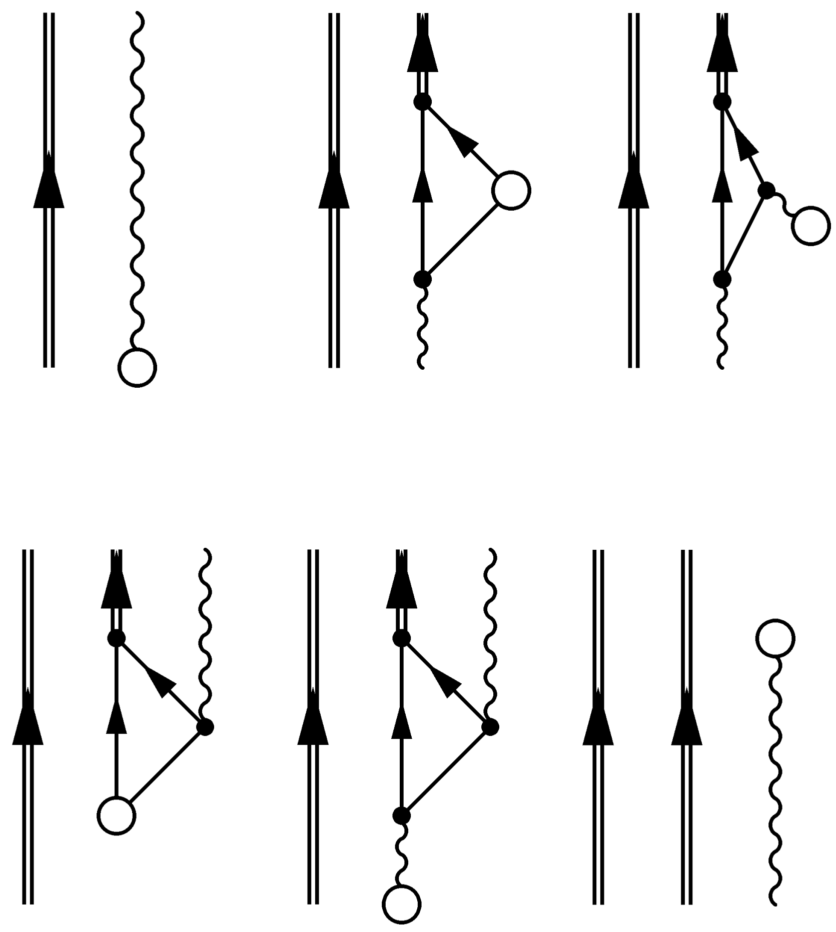
The dependence of the proton pairing phonon on the isospin strength

b_t	ω_a	ω_r	$\omega_a + \omega_r$	$\epsilon_{7/2} + \epsilon_{9/2}$	Λ_0	g_0
70	-0.59	4.45	3.86	4.24	1.02	0.19
75	0.54	3.21	3.75	4.21	1.03	0.20
80	1.67	1.96	3.63	4.18	1.04	0.20
85	2.80	0.72	3.52	4.15	1.06	0.20
90	3.92	-0.53	3.39	4.12	1.07	0.20

Action of the collective operators



Transitions



GT-renormalization factors

$$
\begin{aligned}
f_j^U(1n) &= U_j - g^{-1}\sum_{n'} \frac{(\Lambda_{1n'}V_j + \Xi_{1n'}U_j)\,\Xi_{1n'}}{(\omega_0 - \omega_{1n} - \omega_{1n'})} \\
&\quad - g^{-1}\sum_{n'} \frac{(\Lambda_{\bar{1}n'}V_j + \Xi_{\bar{1}n'}U_j)\,\Xi_{\bar{1}n'}}{(-\omega_0 + \omega_{1n} - \omega_{\bar{1}n'})} \\
f_j^V(1n) &= V_j + g^{-1}\sum_{n'} \frac{(\Lambda_{1n'}U_j - \Xi_{1n'}V_j)\,\Xi_{1n'}}{(\omega_0 - \omega_{1n} - \omega_{1n'})} \\
&\quad + g^{-1}\sum_{n'} \frac{(\Lambda_{\bar{1}n'}U_j - \Xi_{\bar{1}n'}V_j)\,\Xi_{\bar{1}n'}}{(-\omega_0 + \omega_{1n} - \omega_{\bar{1}n'})} \\
f_j^U(\bar{1}n) &= U_j - g^{-1}\sum_{n'} \frac{(\Lambda_{1n'}V_j + \Xi_{1n'}U_j)\,\Xi_{1n'}}{(\omega_0 + \omega_{\bar{1}n} - \omega_{1n'})} \\
&\quad - g^{-1}\sum_{n'} \frac{(\Lambda_{\bar{1}n'}V_j + \Xi_{\bar{1}n'}U_j)\,\Xi_{\bar{1}n'}}{(-\omega_0 - \omega_{\bar{1}n} - \omega_{\bar{1}n'})} \\
f_j^V(\bar{1}n) &= V_j + g^{-1}\sum_{n'} \frac{(\Lambda_{1n'}U_j - \Xi_{1n'}V_j)\,\Xi_{1n'}}{(\omega_0 + \omega_{\bar{1}n} - \omega_{1n'})} \\
&\quad + g^{-1}\sum_{n'} \frac{(\Lambda_{\bar{1}n'}U_j - \Xi_{\bar{1}n'}V_j)\,\Xi_{\bar{1}n'}}{(-\omega_0 - \omega_{\bar{1}n} - \omega_{\bar{1}n'})}
\end{aligned}
$$

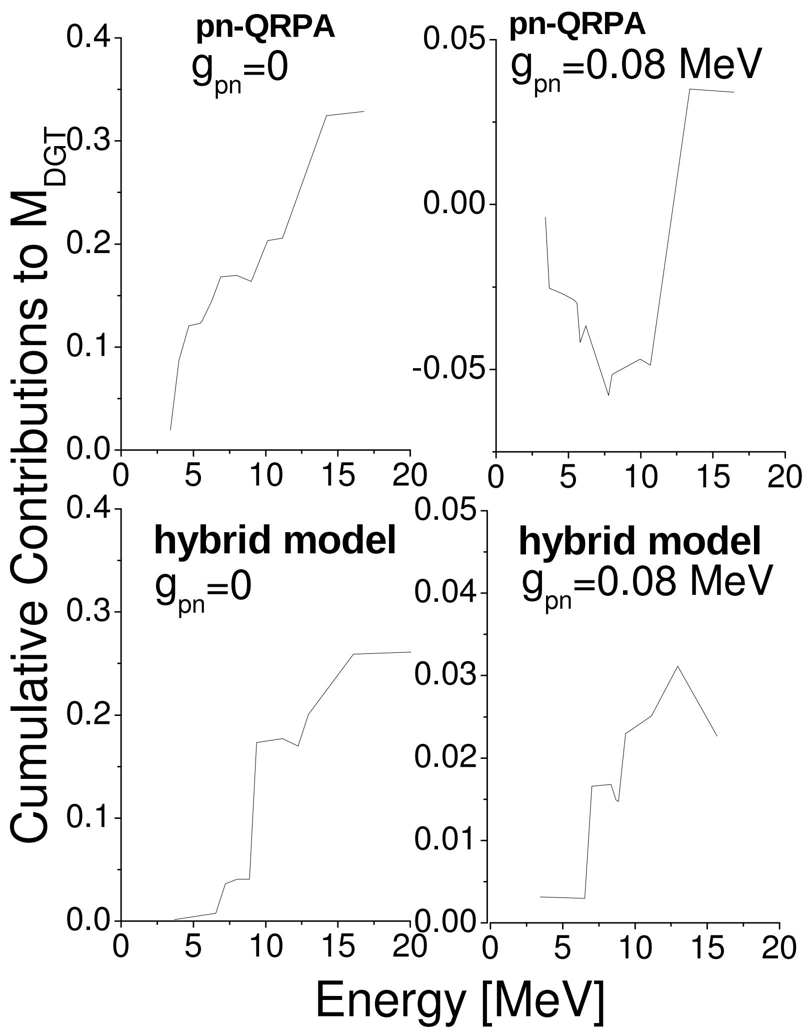
$2\nu\beta^-\beta^-$ *transitions in* ^{128,130}*Te.*

$$M_{DGT}^{exp} = \left(F_0 T_{1/2}^{(2\nu\beta\beta)}\right)^{-1/2}$$

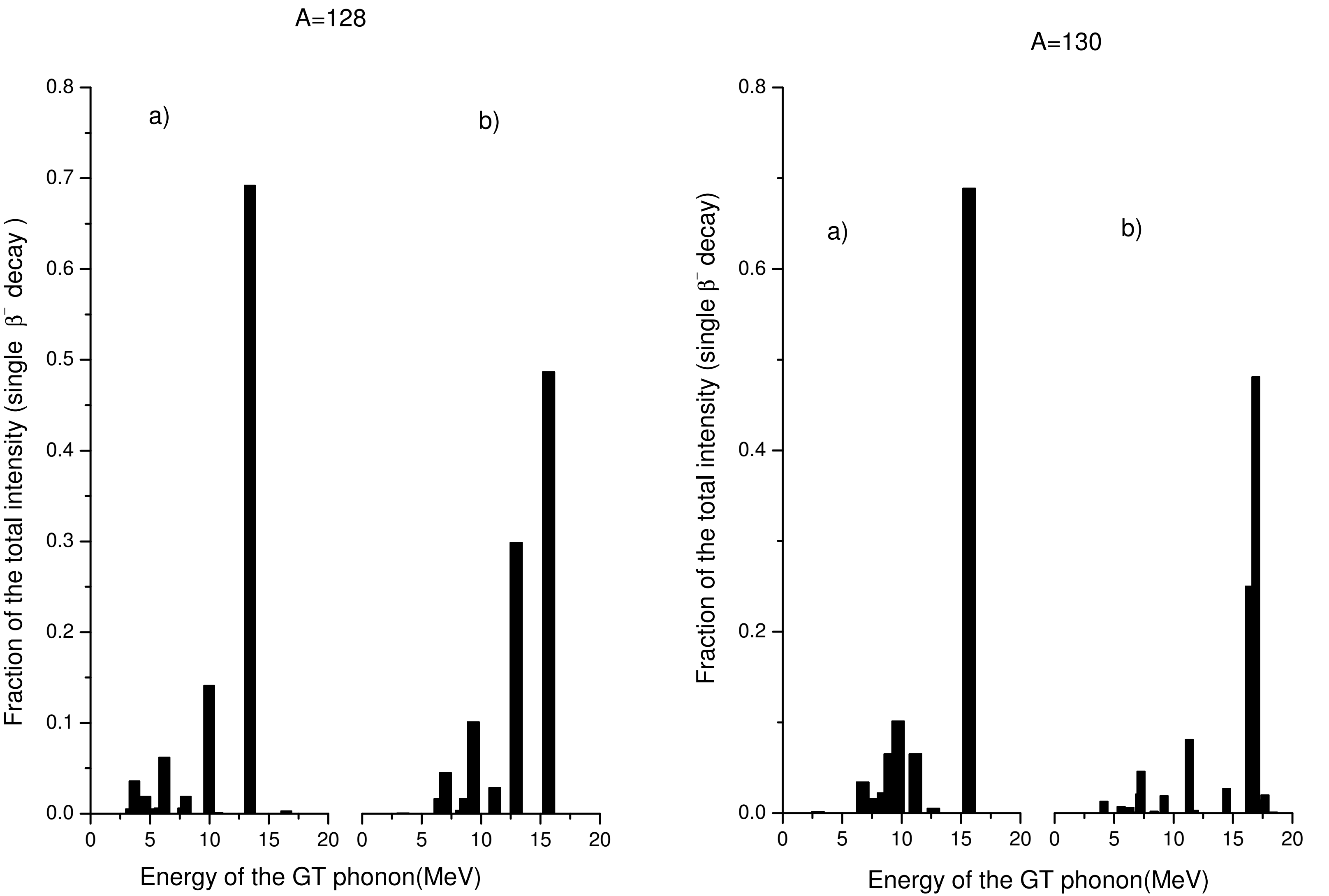
Mass	F_0 [yrs $^{-1}$]	$T_{1/2}^{(2\nu\beta\beta)}$ [yrs]
128	8.5 10^{-22}	2.5 $\pm$ 0.3 10^{24}
130	4.8 10^{-18}	0.9 $\pm$ 0.1 10^{21}
		7.6 $\pm$ 1.5 (stat.) $\pm$ 0.8(syst.) 10^{20}

Mass	Experimental value	Theory (this work)
128	0.025 $\pm$ 0.005	0.016
130	0.015 $\pm$ 0.001	0.012
	0.016 $\pm$ 0.002	

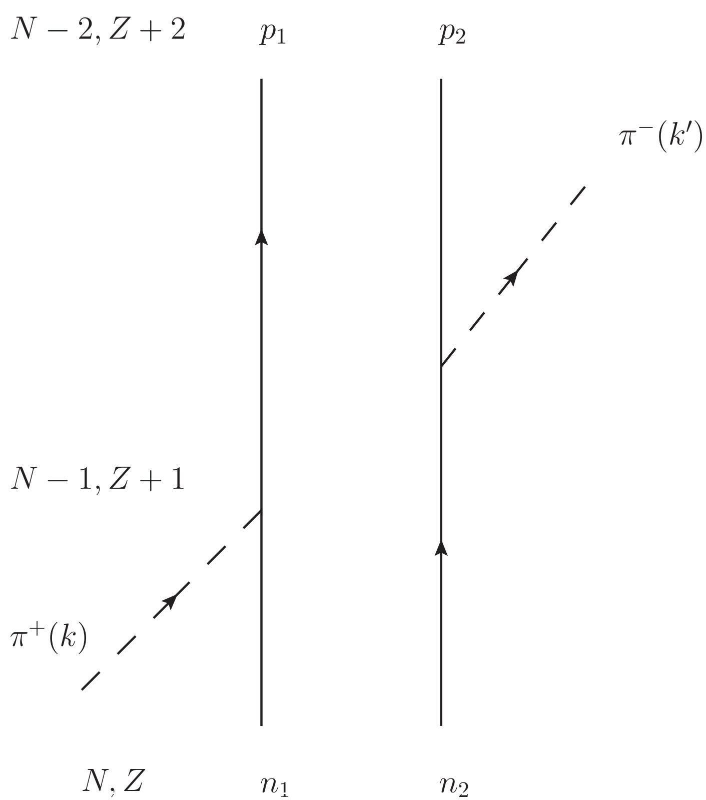
A=128



pn-QRPA
g_{pn}=0
pn-QRPA
g_{pn}=0.08 MeV
hybrid model
g_{pn}=0
hybrid model
g_{pn}=0.08 MeV
Cumulative Contributions to M_{DGT}
Energy [MeV]


A=128
A=130
a)
b)
a)
b)
Fraction of the total intensity (single β⁻ decay)
Fraction of the total intensity (single β⁻ decay)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0 5 10 15 20
Energy of the GT phonon(MeV)
Energy of the GT phonon(MeV)

DCX reactions with low-energy pions



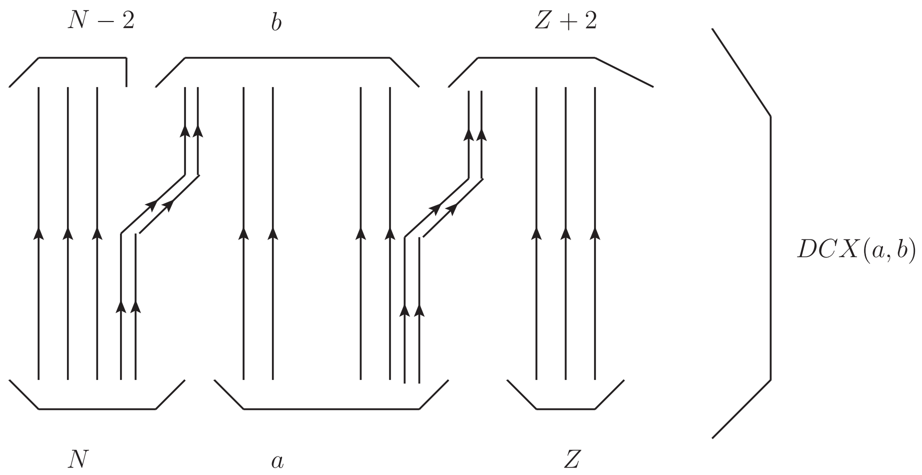
$$F_\lambda(k,k') = \sum_I \frac{\langle f, \pi^-(k') \mid O_\lambda \mid I\rangle\langle I \mid O_\lambda \mid i, \pi^+(k)\rangle}{E_{gs} + E_k - E_I}$$

- The operators are constructed from the plane wave expansion of the pion wave functions, and they are of the orbital-isospin and spin-orbital-isospin type, as well as momentum dependent
- The differential cross section is angular dependent:

$$\frac{d\sigma}{d\Omega} = \mid F(k,k') \mid^2$$
$$F(k,k') = \sum_{\lambda} F_{\lambda}(k,k')$$

- The reaction involves the same set of nuclear wave functions of the neutrinoless-double-beta decay, but very different operators

DCX with heavy ions



$a = \alpha n, \beta p \qquad b = \alpha n + 2, \beta p - 2$

$$\langle N-2, Z+2, b \mid O_{dcx} \mid N, Z, a \rangle \approx \langle N-2, \alpha n + 2 \mid T_n t_n^+ \mid N, \alpha n \rangle \langle Z+2, \beta p - 2 \mid T_p^+ t_p \mid Z, \beta p \rangle$$

- The radial dependence of the particle-exchange (both for neutrons and protons) is unknown, intermediate states are different (with respect to the DBD)

Summary

- Study of the role of pairing and isospin correlations on the matrix elements for double-beta-decay transitions.
- NME for the two-neutrino mode of Te isotopes in a phenomenological approach, which is parameter-free
- Differences between DCX and DBD mostly at the level of the radial dependence of the operators
- DCX reactions may test the structure of the final states involved in DBD decays
- DCX operators are, generally speaking, different from the DBD ones, so are their matrix elements