Rényi entropy and conformal defects

based on 1511.06713 with M. Meineri, R. Myers, M. Smolkin
and work in progress with S. Chapman, X. Dong, D. Galante, M. Meineri, R. Myers

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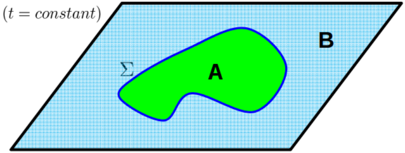
Florence, New frontiers in theoretical physics

Rényi entropy



$$\rho_A = \mathrm{Tr}_B(\rho)$$

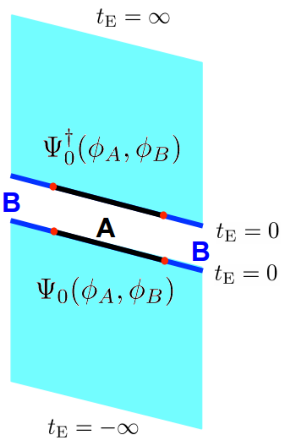
$$S_n = \frac{1}{1-n} \log \mathrm{Tr}(\rho_A^n)$$

Entanglement entropy

$$\lim_{n\to 1} S_n = S_{\mathrm{EE}} = -\mathrm{Tr}(\rho_A \log \rho_A)$$

Rényi entropy

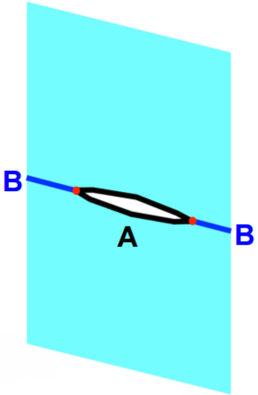
- Path-integral representation of ground state wave-function $\Psi_0(\phi_A, \phi_B)$



Rényi entropy

- Path-integral representation of ground state wave-function $\Psi_0(\phi_A, \phi_B)$
- Tracing over ϕ_B to obtain the reduced density matrix ρ_A

$$\rho_A(\phi_A^+, \phi_A^-) = \mathrm{Tr}_{\phi_B} \Psi(\phi_A^-, \phi_B)\Psi^\dagger(\phi_A^+, \phi_B)$$



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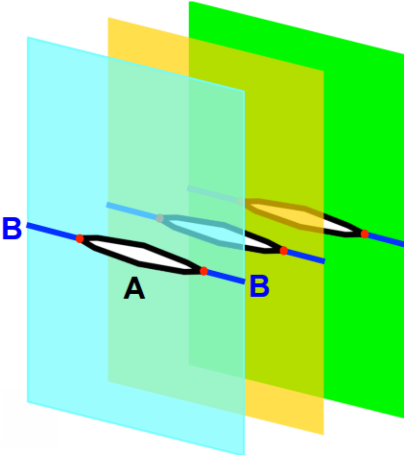
- Evaluate $\mathrm{Tr}\rho_a^n$

$$\mathrm{Tr}\rho_a^n = \frac{Z_n}{Z_1^n}$$



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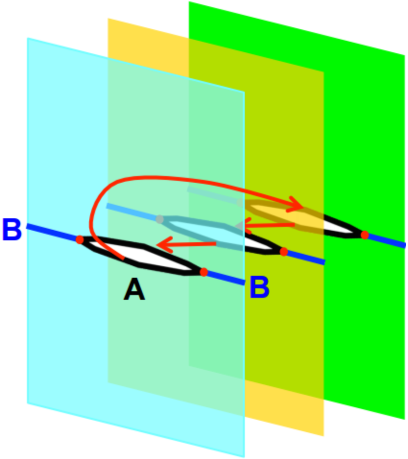
- Z_n is the partition function evaluated over a n-sheeted surface $\rightarrow$ HARD



Replica trick

Move the problem to target space

$$Z_n = \int [\mathcal{D}\phi]_{\mathcal{R}} \exp\left[\int_{\mathcal{R}} d^2x\, \mathcal{L}[\phi](x,t)\right]$$

$$\downarrow$$

$$Z_n = \int_{\mathcal{C}_{u,v}} [\mathcal{D}\phi_1 \ldots \mathcal{D}\phi_n] \exp\left[S^{(n)}[\phi_1, \ldots \phi_n]\right]$$

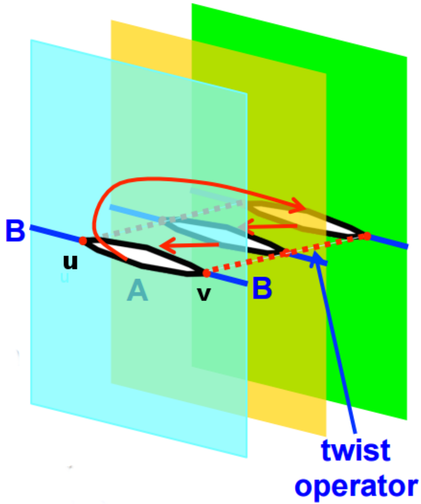
- Restricted path integral

$$\phi_i(x,0^+) = \phi_{i+1}(x,0^-) \qquad x \in [u,v]$$

- n copies of the same theory

$$S^{(n)}[\phi_1, \ldots \phi_n] = S[\phi_1] + \ldots + S[\phi_n]$$

$$S[\phi_i] = \int_{\mathbb{C}} d^2x \mathcal{L}[\phi_i](x)$$

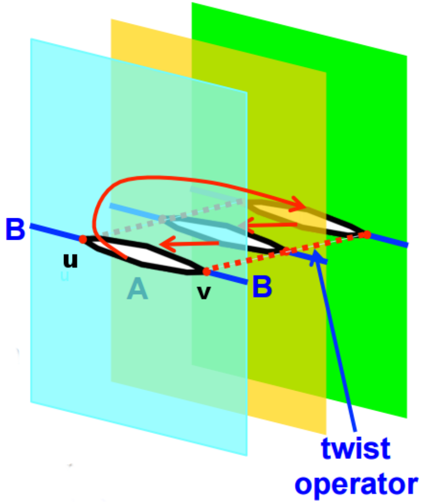


Twist operators

Location of the cut meaningless: local effect

↓

Insertion of local operators in u and v



Rényi entropy

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Higher dimensions

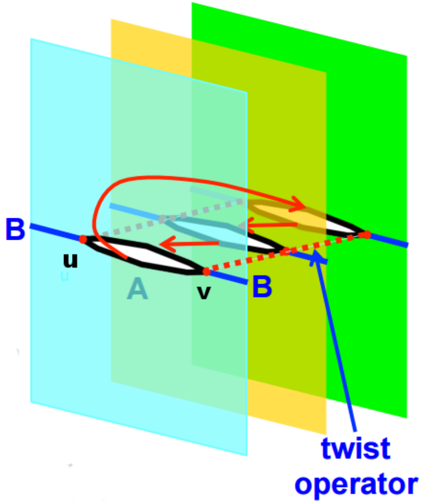
Codimension-two extended operator defined on the boundary of A.

$$\langle \tau_n \rangle \equiv \frac{Z_n}{Z_1^n} = e^{(1-n)S_n}.$$

Preserved symmetry

$$SO(d-1,1) \times U(1)_n$$

Conformal defect



Conformal defects

- The defect breaks translation invariance

$$\sum_{\mathrm{m}=1}^{n} \partial_\mu T^{\mu a}_{(m)}(x^\nu) = \delta_\Sigma(x)\ D^a(x^i),$$

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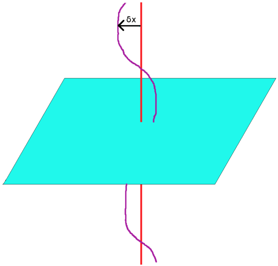

- $D^a(x^i)$ is the displacement operator
- It implements small modifications of the defect

$$\delta \langle X \rangle_n = -\int d^{d-2}x\, \delta x^a(x^i)\, \langle D^a(x^i) X \rangle_n$$

- Its two-point function is fixed by symmetry

$$\langle D^a(x^i)\, D^b(0) \rangle_n = C_D(n)\ \frac{\delta^{ab}}{|x^i|^{2(d-1)}}\,.$$

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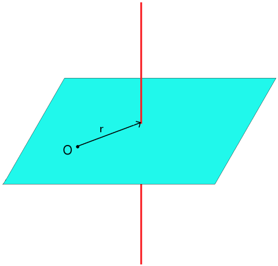


- Local operators acquire a non-vanishing one-point function.
- The kinematics is fixed by conformal invariance

$$\langle O \rangle_n \equiv \frac{\langle \tau_n\, O \rangle}{\langle \tau_n \rangle} = \frac{C_O}{r^\Delta}$$

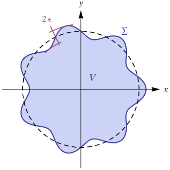
- For the stress tensor

$$\langle T_{ij} \rangle_n = -\frac{h_n}{2\pi n} \frac{\delta_{ij}}{r^d} \qquad \langle T_{ab} \rangle_n = \frac{h_n}{2\pi n} \frac{(d-1)\,\delta_{ab} - d\; n_a n_b}{r^d}$$

Three different conjectures

- Entropy across a deformed sphere [Mezei, 2014]

$$\delta S_n \sim c(d) h_n$$



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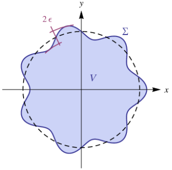
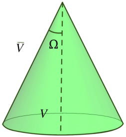


- Entropy across a cone [Bueno, Myers,Witczak-Krempa 2015]

$$S_n^{\text{univ}} \overset{\Omega\to\pi/2}{\sim} \begin{cases} a(d)\frac{h_n}{n(n-1)}(\Omega-\frac{\pi}{2})^2\log(\ell/\delta) & d \text{ odd} \\ b(d)\frac{h_n}{n(n-1)}(\Omega-\frac{\pi}{2})^2\log^2(\ell/\delta) & d \text{ even} \end{cases}$$



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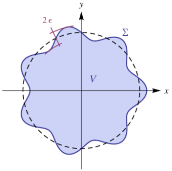

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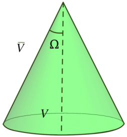

- Entropy in 4d for arbitrary smooth entangling surface [Lee, McGough, Safdi, 2014]

$$S_n = \left(-\frac{f_a(n)}{2\pi}\int_{\partial A} R_\Sigma - \frac{f_b(n)}{2\pi}\int_{\partial A}\tilde{K}^a_{ij}\tilde{K}^a_{ij} + \frac{f_c(n)}{2\pi}\int_{\partial A}\gamma^{ij}\gamma^{kl}C_{ikjl}\right)\log(\mu\ell)+\lambda_n\ ,$$

$$f_b(n) = f_c(n)$$

Are they true?

The three conjectures are equivalent to: [LB, Meineri, Myers, Smolkin, 2015]

$$C_D(n) = d\,\Gamma\left(\tfrac{d+1}{2}\right)\left(\frac{2}{\sqrt{\pi}}\right)^{d-1} h_n \qquad \partial_n C_D|_{n=1} = d\,\Gamma\left(\tfrac{d+1}{2}\right)\left(\frac{2}{\sqrt{\pi}}\right)^{d-1} \partial_n h_n|_{n=1} = \frac{2\pi^2}{d+1} C_T$$

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Let us focus on 4d

$$S_n^{\text{univ}} = \left(-\frac{f_a(n)}{2\pi}\int_{\partial A} R_\Sigma - \frac{f_b(n)}{2\pi}\int_{\partial A}\tilde{K}^a_{ij}\tilde{K}^a_{ij} + \frac{f_c(n)}{2\pi}\int_{\partial A}\gamma^{ij}\gamma^{kl}C_{ikjl}\right)\log(\mu\ell) + \lambda_n\,,$$

Related to the conformal anomaly of a two-dimensional surface embedded in 4d.
Matching the anomaly [LB, Meineri, Myers, Smolkin, 2015]

$$f_c(n) = \frac{3\pi}{2}\frac{h_n}{n-1}\,, \qquad f_b(n) = \frac{\pi^2}{16}\frac{C_D(n)}{n-1}$$

Holographically [Dong, 2016]

$$f_b(n) \neq f_c(n)$$

Holographic setup [Dong, 2016]

Flat entangling surface in flat spacetime

$$ds^2 = \rho^2 d\tau^2 + d\rho^2 + dy^i dy_i$$

$$\tau \sim \tau + 2\pi n$$

$$\langle T^{ij} \rangle_n = -\frac{h_n}{2\pi n} \frac{\delta^{ij}}{\rho^4}$$

Map to the hyperboloid $\mathbb{H}^{d-1} \times S^1_n$

$$ds^2 = d\tau^2 + \frac{d\rho^2 + dy^i dy_i}{\rho^2}$$

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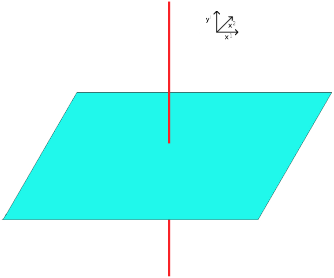
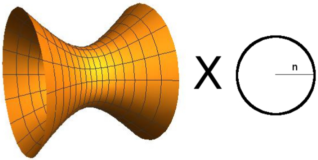
$$\langle T^{ij} \rangle_n = \rho^2 P_n \delta^{ij} \qquad P_n - P_1 = \frac{h_n}{2\pi n}$$





Holographic setup [Dong, 2016]

Deformed entangling surface

$$ds^2 = \rho^2 d\tau^2 + d\rho^2 + G_{ij} dy^i dy^j + O(\rho^2)$$

$$G_{ij} = \gamma_{ij} + 2\, K^a_{ij} x^a$$

$$\langle T^{ij} \rangle_n = -\frac{h_n}{2\pi n \rho^4} \delta^{ij} + \frac{C_D}{16\, n \rho^4} \tilde{K}^{ij}_a x^a + \ldots$$

Deformed hyperboloid $\tilde{H}$

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Metric on the bulk

$$ds^2_{bulk} = \frac{dr^2}{f(r)} + f(r) d\tau^2 + \frac{r^2}{\rho^2} \left[d\rho^2 + (\delta_{ij} + 2k(r) K_{aij}) dy^i dy^j \right] + \ldots$$

$$f(r) = r^2 - 1 - \frac{r_h^2 (r_h^2 - 1)}{r^2} \qquad n = \frac{r_h}{2\, r_h^2 - 1}$$

Einstein equations yield a second-order differential equation for $k(r)$ and from the metric one can extract the stress-tensor one-point function [de Haro, Solodukhin, Skenderis, 2000]

Holographic C_D in any dimension [LB, Chapman, Dong, Galante, Meineti, Myers]

In any d

$$\delta \langle T^{ij}(z)\rangle = -\epsilon \int d^{d-2}w \, \langle D_a(w) T^{ij}(z)\rangle \, f^a(w^i) + \mathcal{O}(\epsilon^2)$$

$$\langle T^{ij}(x,y)\rangle_{\tilde{H}} = \frac{1}{|x|^d}\left(-\frac{h_n}{2\pi n}\delta^{ij} + A_n \tilde{K}_a^{ij} x^a + B_n \delta^{ij} K_a x^a + \ldots\right)$$

$$\langle T^{ai}(x,y)\rangle_{\tilde{H}} = \frac{1}{|x|^{d-2}}\,\partial^i K_b \left(D_n \delta^{ab} + C_n n^a n^b\right) + \ldots$$

$$A_n = \frac{(d-1)\Gamma\left(\frac{d}{2}-1\right)\pi^{\frac{d}{2}-2}}{2\Gamma(d+1)}\frac{C_D}{n} + \frac{d-4}{d-2}\frac{h_n}{2\pi n} \qquad B_n = \frac{d+4}{d-2}\frac{h_n}{4\pi n}$$

$$C_n = \frac{(d-1)\Gamma\left(\frac{d}{2}-1\right)\pi^{\frac{d}{2}-2}}{(d-2)2\Gamma(d+1)}\frac{C_D}{n} - \frac{d^2}{(d-2)^2}\frac{h_n}{4\pi n} \qquad D_n = \frac{d}{d-2}\frac{h_n}{4\pi n}$$

Conclusion

- We developed a field theoretic framework for calculating the dependence of Rényi entropies on the shape of the entangling surface in a conformal field theory.
- We unified different conjectures, which were proven for the entanglement entropy [Faulkner, Leigh, Parrikar, 2015], but were recently shown to fail holographically for any n in 4d [Dong, 2016].
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THANK YOU