Green-Schwarz superstring on the lattice

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Based on 1601.04670 and 1605.01726
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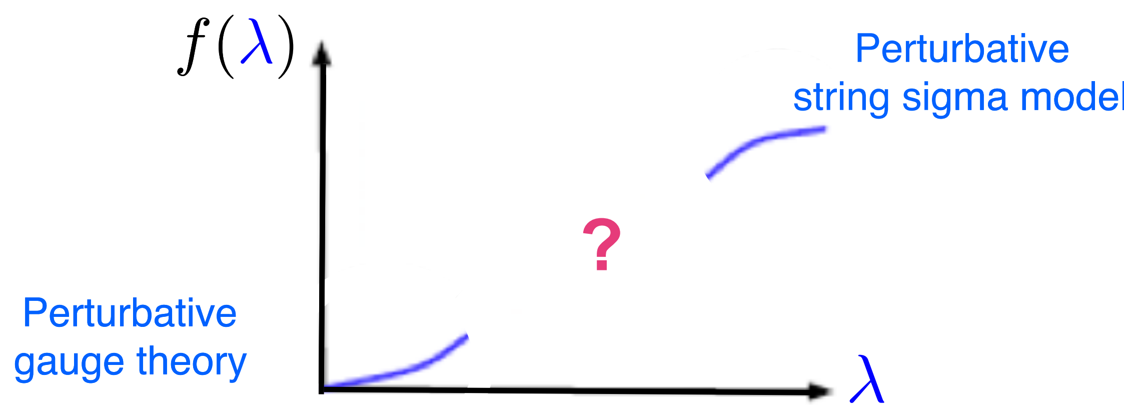
Motivation

Beautiful progress in obtaining exact results within AdS/CFT

- from integrability
- from supersymmetric localization



$$f(\lambda) = c\sqrt{\lambda} + d + e\,\frac{1}{\sqrt{\lambda}} + \cdots$$

$$f(\lambda) = a\,\lambda + b\,\lambda^2 + \cdots$$

$$\sqrt{\lambda} = \sqrt{g_{\rm YM}^2 N} = \frac{R^2}{\alpha'}$$

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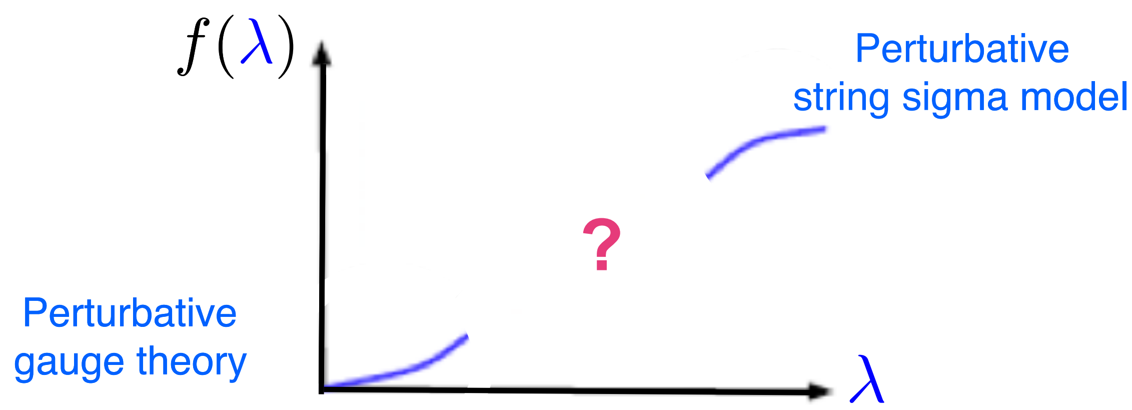
Beautiful progress in obtaining exact results within AdS/CFT

- from integrability (assumed)
- from supersymmetric localization (BPS observable)

$$f(\lambda) = a\,\lambda + b\,\lambda^2 + \cdots$$



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$$\sqrt{\lambda} = \sqrt{g^2_{\rm YM} N} = \frac{R^2}{\alpha'}$$

In the world-sheet string theory integrability only classically, localization not formulated.

Superstrings in $AdS_5 \times S^5$ with RR fluxes: complicated interacting 2d field theory

$$S_{\rm IIB} = \frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \Big[\partial_a X \partial^a X + \bar\theta\,\Gamma^a (D + F_5)\,\theta\,\partial_a X + \bar\theta\theta\bar\theta\theta\,\partial_a X \partial^a X + \cdots\Big]$$

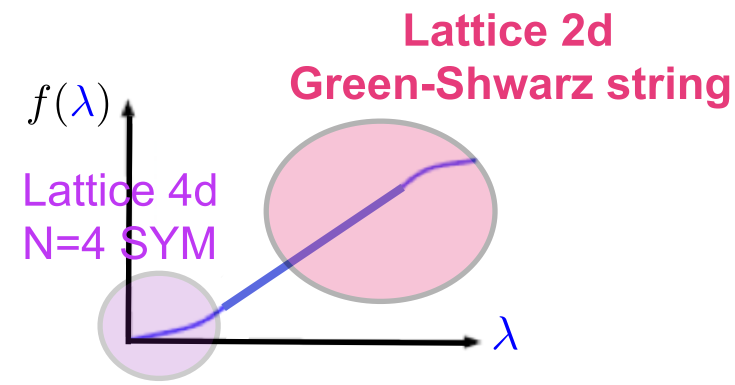
under control perturbatively (and with some caveats).

Is there a genuine 2d QFT way
to cover the finite-coupling region?

Motivation

Lattice techniques in AdS/CFT: existing program on 4d gauge theory, good results at weak coupling.

[Catterall et al.]



Lattice for superstring world-sheet in $AdS_5 \times S^5$

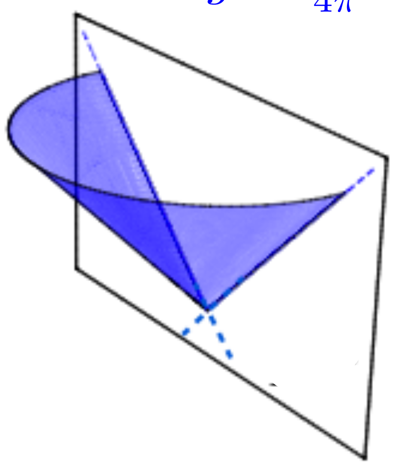
[McKeown Roiban, 2013]

- 2d: computationally cheap
- no world-sheet susy (Green-Schwarz)
- all gauge symmetries are fixed (no formulation à la Wilson), only scalar fields (also anti-commuting)

Non-trivial 2d qft with strong coupling analytically known, finite-coupling (numerical) prediction.

The cusp anomaly of $\mathcal{N}=4$ SYM from string theory

$g = \frac{\sqrt{\lambda}}{4\pi}$



Completely solved via integrability. [Beisert Eden Staudacher 2006]

Expectation value of a light-like cusped Wilson loop

$$\langle W[C_{\rm cusp}]\rangle \sim e^{-f(g)\,\phi\,\ln\frac{L_{\rm IR}}{\epsilon_{\rm UV}}}$$

AdS/CFT

$$Z_{\rm cusp} = \int [D\delta X][D\delta\theta]\, e^{-S_{\rm IIB}(X_{\rm cusp}+\delta X,\delta\theta)} = e^{-\Gamma_{\rm eff}} \equiv e^{-f(g)\,V_2}$$

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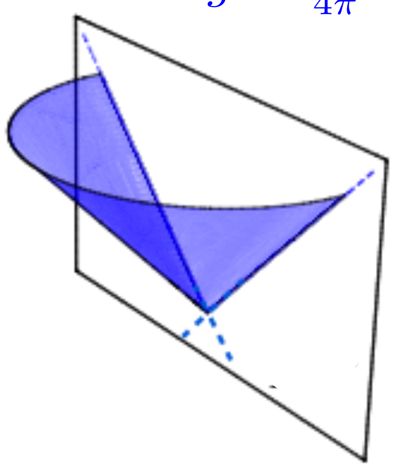

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String partition function with "cusp" boundary conditions, evaluated perturbatively

$$f(g)|_{g\to 0} = 8g^2\left[1-\frac{\pi^2}{3}g^2+\frac{11\,\pi^4}{45}g^4-\left(\frac{73}{315}+8\,\zeta_3\right)g^6+\ldots\right]$$ [Bern et al. 2006]

$$f(g)|_{g\to\infty} = 4g\left[1-\frac{3\ln 2}{4\pi}\frac{1}{g}-\frac{K}{16\pi^2}\frac{1}{g^2}+\ldots\right]$$ [Giombi et al. 2009]

A lattice approach prefers expectation values

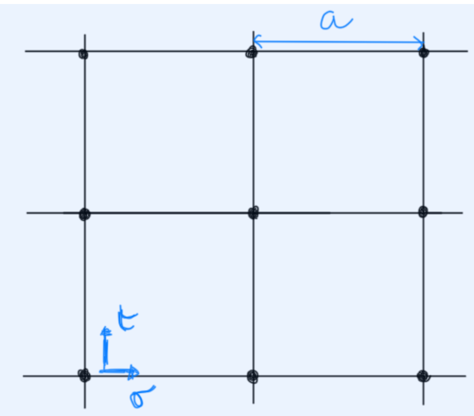
$$\langle S_{\text{cusp}}\rangle = \frac{\int [D\delta X][D\delta\Psi]\, S_{\text{cusp}}\, e^{-S_{\text{cusp}}}}{\int [D\delta X][D\delta\Psi]\, e^{-S_{\text{cusp}}}} = -g\,\frac{d\ln Z_{\text{cusp}}}{dg} \equiv g\,\frac{V_2}{8}\,f'(g)$$

Simulations in lattice QFT

Spacetime grid with lattice spacing a, size $L = N\, a$,
points $\xi = (an_1, an_2) \equiv a\, n$ and fields $\phi \equiv \phi_n$

a) natural cutoff for the momenta, $-\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}$

b) path integral measure $[D\phi] = \prod_n d\phi_n$.



Then $\int \prod_n d\phi_n\, e^{-S_{\mathrm{discr}}}$ can be studied via Monte Carlo methods.
Ensamble of configurations $\{\Phi_1, \ldots, \Phi_K\}$, with $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]}}{Z}$:

Ensemble average $\langle A \rangle = \int [D\Phi]\, P[\Phi]\, A[\Phi] = \frac{1}{K} \sum_{i=1}^{K} A[\Phi_i] + \mathcal{O}(\frac{1}{\sqrt{K}})$

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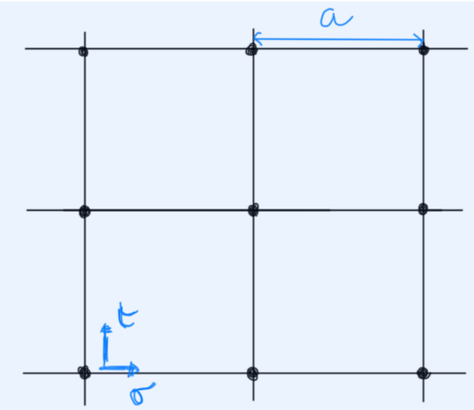

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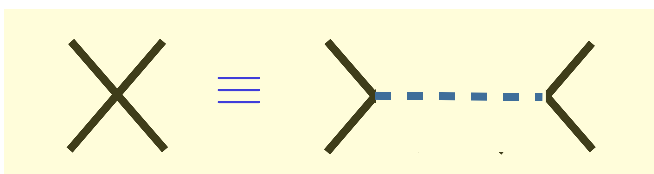
Graßmann-odd fields are formally integrated out: $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]}\det \mathcal{O}_F}{Z}$

- action must be quadratic in fermions (linearization via auxiliary fields):



- determinant must be definite positive

$$\det O_F \longrightarrow \sqrt{\det(\mathcal{O}_F\, \mathcal{O}_F^\dagger)} = \int D\zeta\, D\bar{\zeta}\;\, e^{-\int d^2\xi\, \bar{\zeta}(\mathcal{O}_F \mathcal{O}_F^\dagger)^{-\frac{1}{4}}\zeta}$$

potential ambiguity!

Green-Schwarz string in the null cusp background

After linearization the Lagrangian reads ($m \sim P_+$)

$$\mathcal{L}_{\text{cusp}} = |\partial_t x + \frac{m}{2}x|^2 + \frac{1}{z^4}|\partial_s x - \frac{m}{2}x|^2 + (\partial_t z^M + \frac{m}{2}z^M)^2 + \frac{1}{z^4}(\partial_s z^M - \frac{m}{2}z^M)^2 + \frac{1}{2}\phi^2 + \frac{1}{2}(\phi_M)^2 + \psi^T O_F \psi \,,$$

- 8 bosonic coordinates: x, x^*, z^M $(M = 1, \cdots, 6)$, $z = \sqrt{z_M z^M}$;
- 7 auxiliary fields ϕ, ϕ^M $(M = 1, \cdots, 6)$);
- 8 fermionic variables, $\psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$, and $\theta^i = (\theta_i)^\dagger$, $\eta^i = (\eta_i)^\dagger$, $i = 1, 2, 3, 4$

$$O_F = \begin{pmatrix} 0 & i\partial_t & -\mathrm{i}\rho^M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} & 0 \\ \mathrm{i}\partial_t & 0 & 0 & -\mathrm{i}\rho^\dagger_M\left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} \\ \mathrm{i}\frac{z^M}{z^3}\rho^M\left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) & i\partial_t - A^T \\ 0 & \mathrm{i}\frac{z^M}{z^3}\rho^\dagger_M\left(\partial_s - \frac{m}{2}\right) & \mathrm{i}\partial_t + A & -2\frac{z^M}{z^4}\rho^\dagger_M\left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}z^2}\phi_M \rho^{MN} z_N - \frac{1}{\sqrt{2}z}\phi + \mathrm{i}\,\frac{z_N}{z^2}\rho^{MN}\,\partial_t z^M$$

and ρ^M are off-diagonal blocks of SO(6) Dirac matrices $\gamma^M \equiv \begin{pmatrix} 0 & \rho^\dagger_M \\ \rho^M & 0 \end{pmatrix}$.

Manifest global symmetry is $SO(6) \times SO(2)$.

Discretization and lattice perturbation theory

A naive discretization $p_\mu \to \overset{\circ}{p}_\mu \equiv \frac{1}{a}\sin(a\,p_\mu)$ leads to fermion doublers,

i.e. identical propagator at 2^d points: $(0,0)$, $(\frac{\pi}{a},0)$,$(0,\frac{\pi}{a})(\frac{\pi}{a},\frac{\pi}{a})$

$$\det K_F = \Big(\frac{\sin^2(p_1\,a)}{a^2} + \frac{\sin^2(p_2\,a)}{a^2} + \frac{m^2}{4}\Big)^8$$

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Add to the action a "Wilson term", $K_F + W \equiv K_F^W$ such that

- SO(6) invariance is maintained
- No (additional) complex phase is introduced
- For $a \to 0$ continuum perturbation theory is reproduced

Using its determinant in the one-loop effective action $\Gamma^{(1)}_{\rm LAT} = \ln\frac{\det K_B}{\det K_F^W}$

$$\Gamma^{(1)}_{\rm LAT} = \frac{V_2}{2\,a^2}\int_{-\pi}^{+\pi}\frac{d^2p}{(2\pi)^2}\,\ln\Big[\frac{4^8(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2})^5(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2}+\frac{M^2}{8})^2(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2}+\frac{M^2}{4})}{\Big(\sin^2 p_0+\sin^2 p_1+\frac{M^2}{4}+4\sin^4\frac{p_0}{2}+4\sin^4\frac{p_1}{2}\Big)^8}\Big]$$

$$\xrightarrow{a\to 0}\; -\frac{3\ln 2}{8\pi}\,V_2\,m^2\,,$$ cusp anomaly at strong coupling $\quad(\,|r|=1, M=m\,a.)$

Line of constant physics

In the continuum, “effective” masses undergo a *finite* renormalization

$$m_x^2(g) = \frac{m^2}{2}\left(1 - \frac{1}{8\,g} + \mathcal{O}(g^{-2})\right) \qquad (\star)$$

The dimensionless physical quantity to keep constant when $a \to 0$ is

$$L^2\, m_x^2 = \text{const}\,, \qquad \text{leading to} \qquad (L\,m)^2 \equiv (NM)^2 = \text{const}\,,$$

if $(\star)$ is still true on the lattice and g is not (infinitely) renormalized.

Continuum limit $a \to 0$

We assume that, on the lattice, no further scale but a is present.

A generic observable

$$F_{\rm LAT} = F_{\rm LAT}(g, N, M) = F(g) + \mathcal{O}\Big(\frac{1}{N}\Big) + \mathcal{O}\left(e^{-MN}\right)$$

where

$$g = \frac{\sqrt{\lambda}}{4\pi}\,, \qquad N = \frac{L}{a}\,, \qquad M = a\,m\,.$$

Recipe:

- fix g
- fix MN, large enough so to to keep small finite volume effects
- evaluate $F_{\rm LAT}$ for $N = 6, 8, 10, 12, 16, \cdots$
- obtain $F(g)$ extrapolating to $N \to \infty$.

Measure I: mass of x boson

From the correlator of the x fields

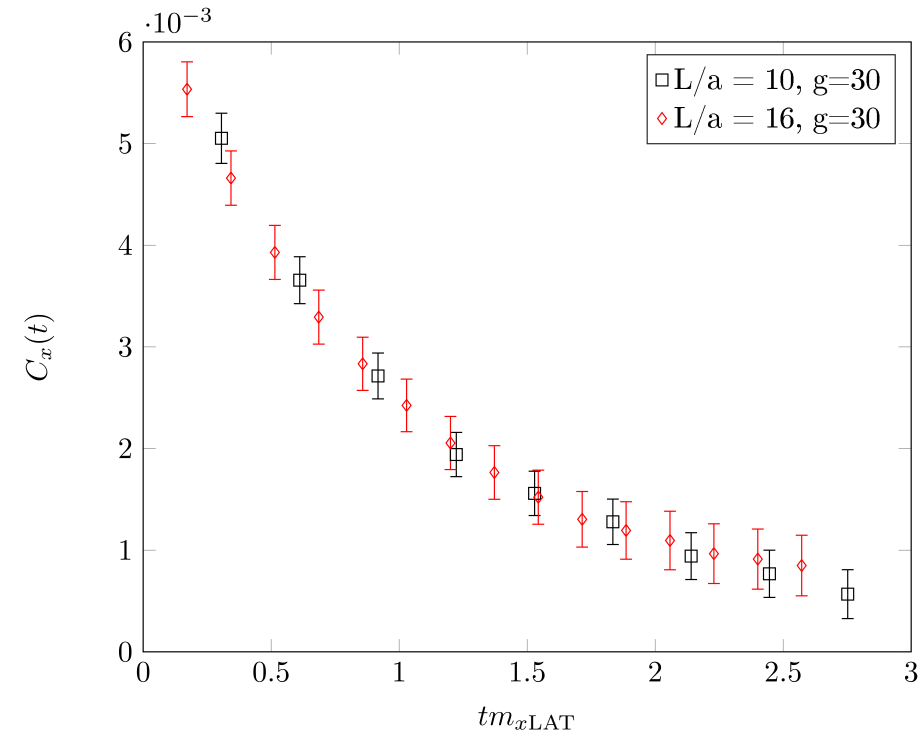
$$C_x(t) = \sum_{s_1,\, s_2} \langle x(t,\, s_1) x^*(0,\, s_2) \rangle$$

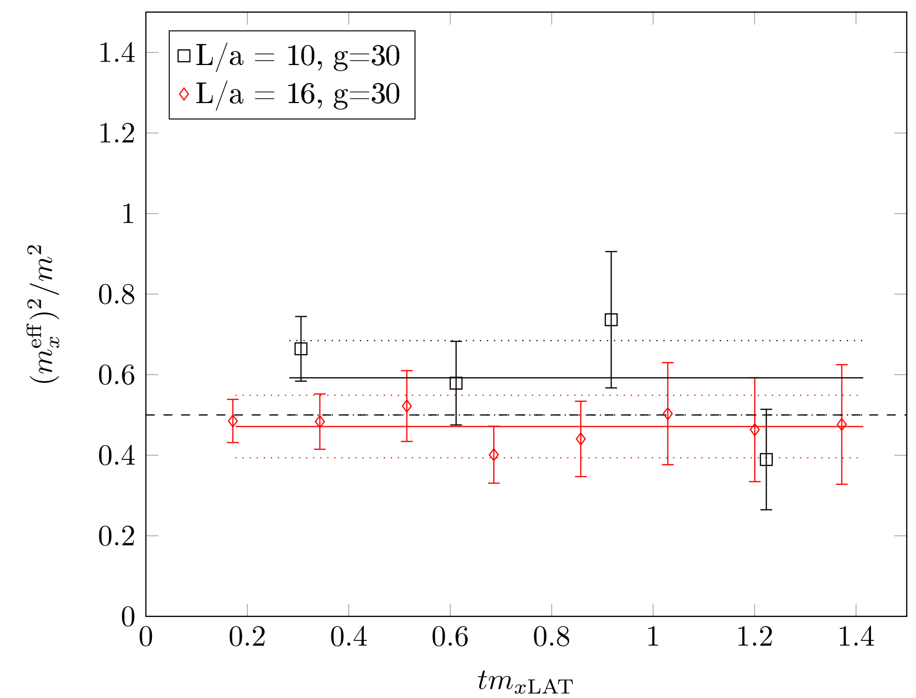
$$= c_0\, e^{-t\, m_{x\mathrm{LAT}}} + \dots$$

extract the x-mass

$$m_{x\mathrm{LAT}} = \lim_{T,\, t \to \infty} m^{\mathrm{eff}_x}$$

$$\equiv \lim_{T,\, t \to \infty,} \frac{1}{a} \log \frac{C_x(t;\, 0)}{C_x(t+a;\, 0)}$$

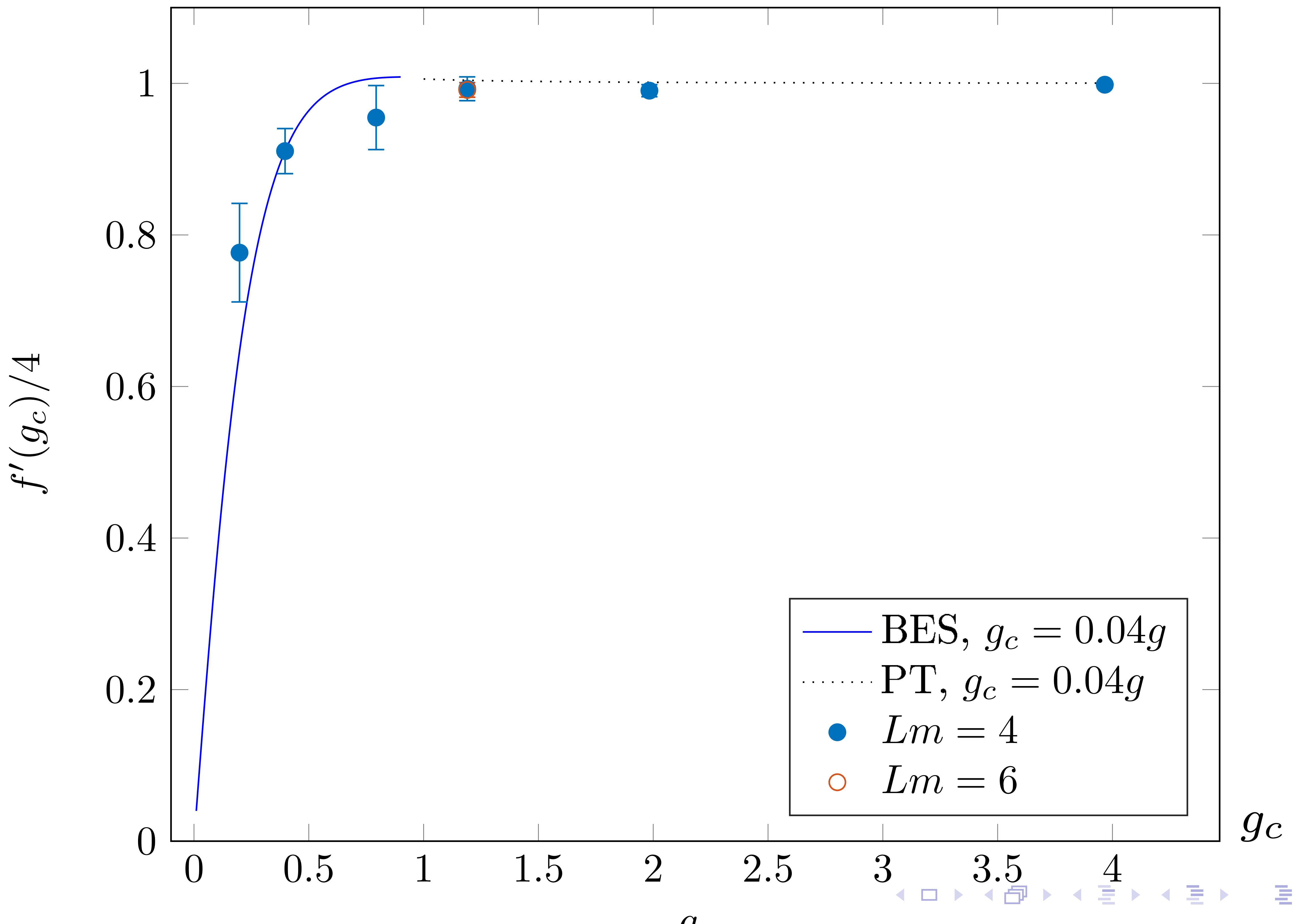




No infinite renormalization occurring, no need of tuning m to adjust for it.
This corroborates our choice of line of constant physics.

Measure II: (derivative of the) cusp anomaly

Having subtracted quadratic divergences in $\langle S_{\mathrm{LAT}} \rangle \sim f'(g)_{\mathrm{LAT}} + c(g)\, N^2$ (set to zero in dim. reg.), assume $g = \alpha\, g_c$: then from $f'(g) = f'(g_c)_c$ is $g_c = 0.04g$.



The phase

After linearization $\mathcal{L}_F = \psi^T\, \mathcal{O}_F\, \psi$, integrating fermions leads to a complex Pfaffian $\mathrm{Pf}\, O_F = |(\det O_F)^{\frac{1}{2}}|\, e^{i\theta}$.

The phase is encoded in the linearization

$$e^{-\int (i\,\eta\rho\eta)^2} \sim e^{-\frac{b^2}{4\,a}} = \int dx\, e^{-a\,x^2 + i\,b\,x}$$

and can be treated via reweighting: incorporate the non positive part of the Boltzmann weight into the observable

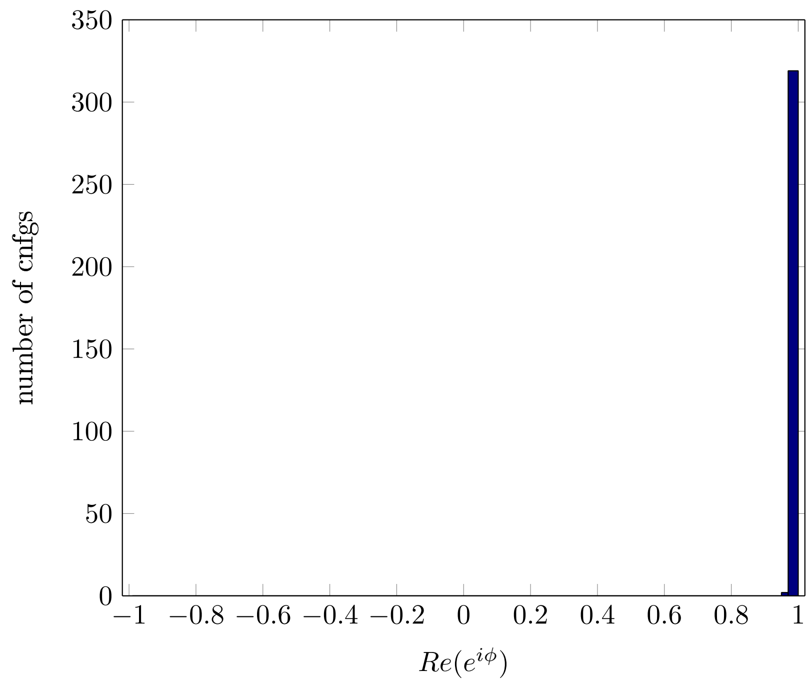
$$\langle \mathcal{O} \rangle_{\text{reweight}} = \frac{\langle \mathcal{O}\, e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}}$$

It gives meaningful results as long as the phase does not average to zero.

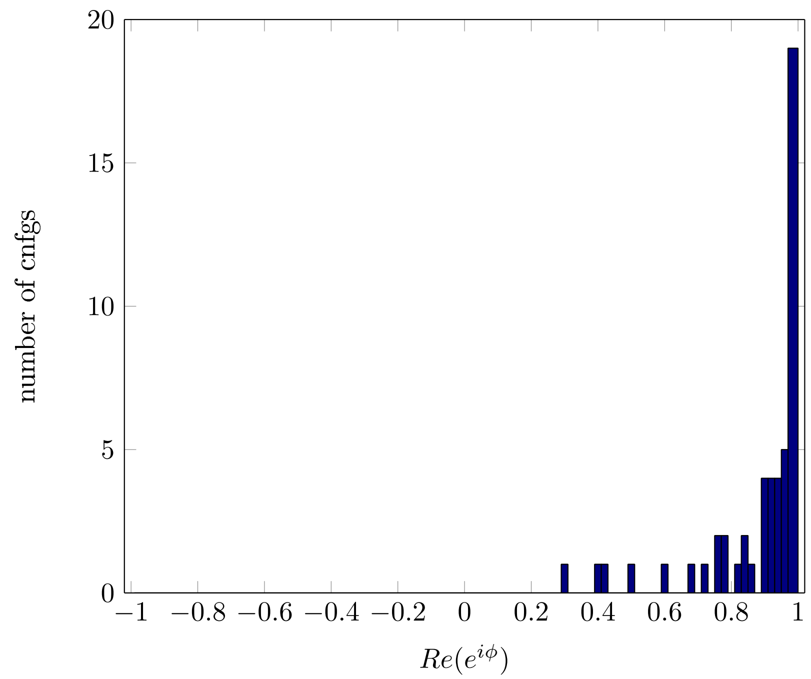
The phase

In the interesting ($g = 1$) region the phase has a flat distribution.

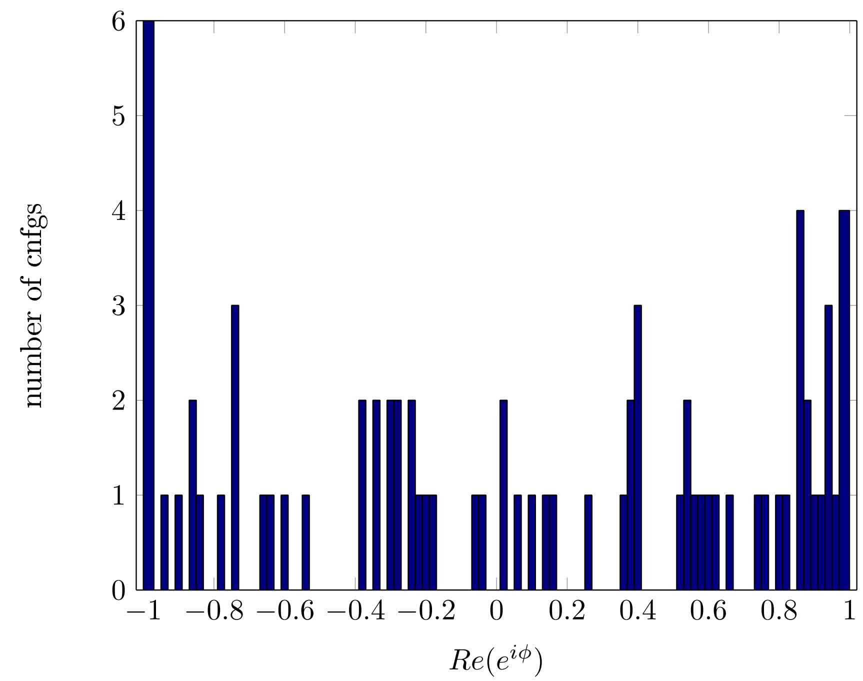
g=30



g=5



g=1



Alternative algorithms: active field of study, no general proof of convergence.

Alternative linearization: in progress.

Conclusions

Solving a non-trivial 4d QFT is hard reduce the problem via AdS/CFT:

solve a non-trivial 2d QFT.

Lattice simulation of gauge-fixed Green-Schwarz string,
two discretizations and Rational Hybrid Monte Carlo:

- Observables measured are in good agreement with expectation at large g;
- At small g, complex phase and related sign problem .

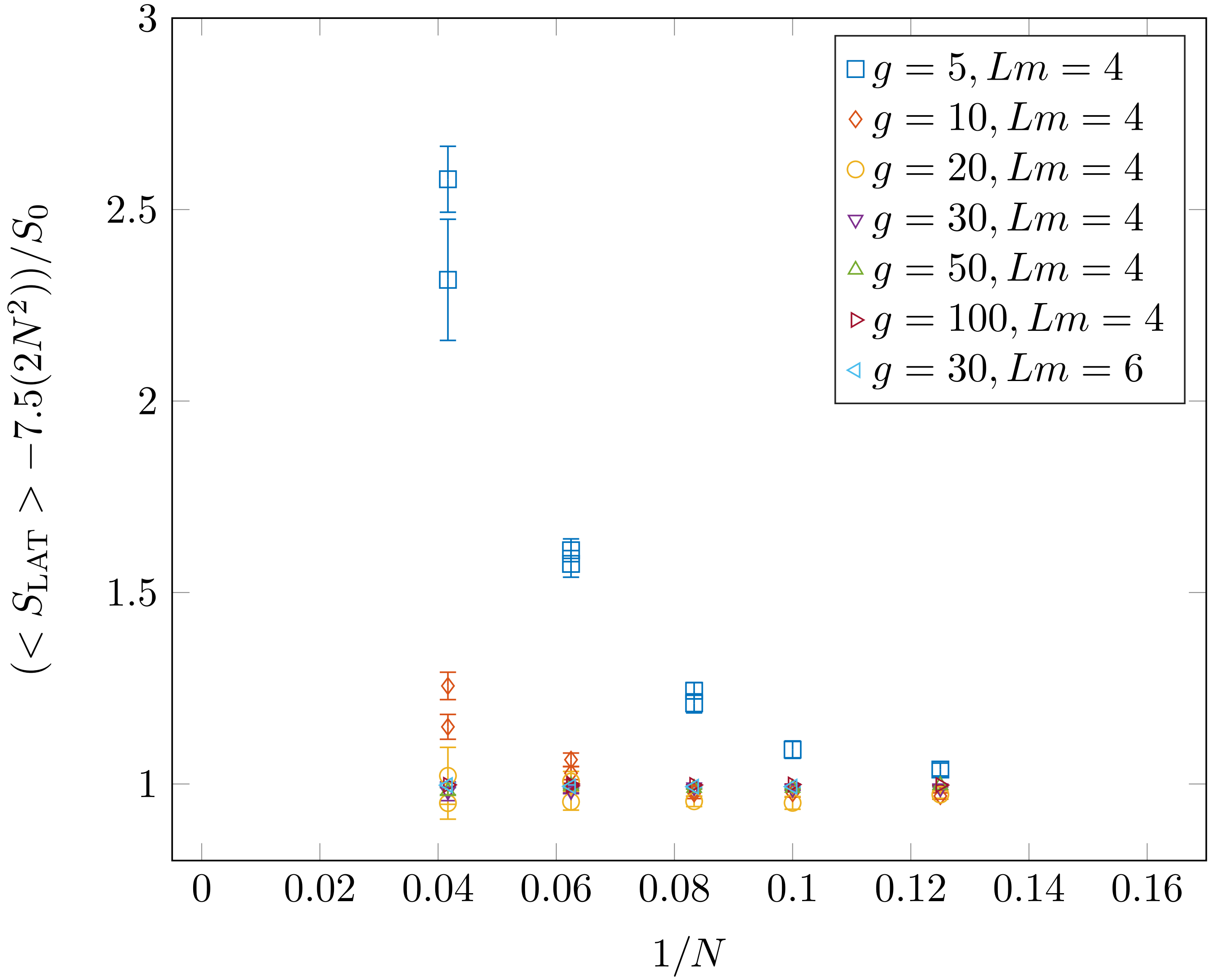
Next

- Alternative linearization, phase-free
- Further observables, different backgrounds (AdS_4/CFT_3)
- Correlators of string vertex operators (gauge theory 3-point functions)

Extra slides

Simulation: the cusp action

In measuring $\langle S_{\rm cusp}\rangle \equiv g\,\frac{V_2\, m^2}{8}\, f'(g)$ quadratic divergences appear.



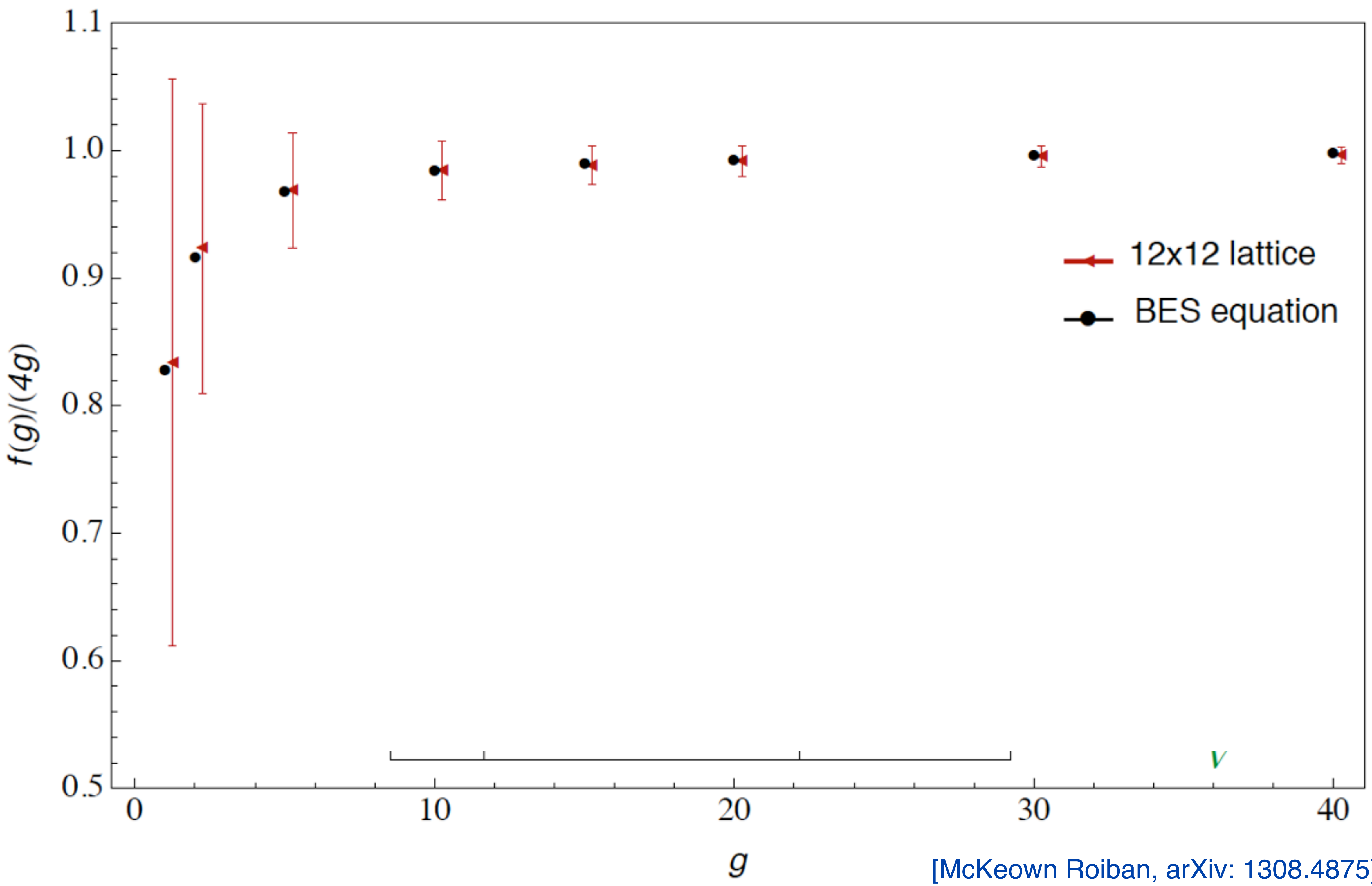
$$\langle S_{\rm LAT}\rangle = S_0\,\frac{f'(g)_{\rm LAT}}{4} + \frac{c(g)}{2}(2N^2)$$

$$S_0 = g\,N^2\,M^2$$

In continuum perturbation theory dim. reg. set them to zero.
Here, expected mixing of the Lagrangian with lower dimension operator

$$\mathcal{O}(\phi(s))_r = \sum_{\alpha:[O_\alpha]\leq D} Z_\alpha\,\mathcal{O}_\alpha(\phi(x))\,, \qquad Z_\alpha \sim \Lambda^{(D-[\mathcal{O}_\alpha])} \sim a^{-(D-[\mathcal{O}_\alpha])}$$

Roiban McKeown 2013



[McKeown Roiban, arXiv: 1308.4875]