γ_5 in Dimensional Regularization: a Novel Approach

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A new dimensional Regularization of γ_5 is proposed. Cyclicity and Lorentz covariance are enforced. The extension to generic dimension is based on the integral representation of the trace of gamma's, presented in a previous paper (arXiv:1403.4212) .

1. Introduction

Analytic continuation in the dimension D is very important for model construction and explicit calculations. We focus on the Trace . See [1]

$$Tr(\not{p}_1 \not{p}_2 \ldots \not{p}_{N-1} \not{p}_N) = \int d^N \bar{c} \exp \Big(\sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \Big)$$

$$Tr(\not{p}_1 \not{p}_2 \ldots \not{p}_{N-1} \not{p}_N \gamma_\chi) = i^{\frac{D(D-1)}{2}} \int d^D \xi \; d^N \bar{c}$$

$$\exp \Big(\sum_{i=1}^{N} \bar{c}_i (p_i)_\mu \xi_\mu + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j \Big),$$

For trace and Pfaffian Ref. [2]. Integral representation of the Pfaffian on Grassmannian variables [3].

2. Grassmann Variables

$$\int d\xi_\mu = 0, \quad \int d\xi_\mu \xi_\nu = \delta_{\mu\nu}, \quad \int d\bar{c}_i = 0, \quad \int d\bar{c}_i \bar{c}_j = \delta_{ij}.$$

Thus, for example, we have for any integer D

$$Tr(\not{p}_1 \ \not{p}_2) = \int d\bar{c}_2 \, d\bar{c}_1 \exp\left[\bar{c}_1 (p_1, p_2) \bar{c}_2\right] = (p_1, p_2)$$

and for $D = 4$

$$Tr(\not{p}_1 \ \not{p}_2 \ \not{p}_3 \ \not{p}_4 \gamma_\chi) \ = i^{\frac{D(D-1)}{2}}$$
$$\int d^D\xi \, ((p_1)_\mu \xi_\mu (p_2)_\nu \xi_\nu (p_3)_\rho \xi_\rho (p_4)_\sigma \xi_\sigma)(-)^{\frac{D(D-1)}{2}}|_{D=4}$$
$$= -\epsilon_{\mu\nu\rho\sigma} (p_1)_\mu (p_2)_\nu (p_3)_\rho (p_4)_\sigma \ .$$

Nice Interpolation on Integer D ! See Part I

3. Many γ_χ's

For instance <u>two</u> γ_χ's are represented by

$$
\begin{aligned}
&Tr(\not{p}_1 \ldots \not{p}_k \gamma_\chi \;\cdots\; \not{p}_N \gamma_\chi\;) = (-)^{\frac{D(D-1)}{2}} \int d^D\eta \;\; d^{(N-k)}\bar{c}\, d^D\xi \;\; d^k\bar{c} \\
&\exp\,(\sum_{i=1}^{N} \bar{c}_i (p_i)_\mu \eta_\mu \;+ \xi_\mu \eta_\mu \;+ \xi_\mu \sum_{i=k+1}^{N} \bar{c}_i (p_i)_\mu + \sum_{i=1}^{k} \bar{c}_i (p_i)_\mu \xi_\mu \\
&+ \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j) \qquad (1)
\end{aligned}
$$

Cyclicity and Lorentz covariance are OK! .

PART I: Integer D

4. Pairing and Algebra

It is possible to integrate over the variables $\xi, \eta, ...$ two by two (pairing). One gets for eq. (1)

$$Tr(\not{p}_1 \ldots \not{p}_k \gamma_\chi \cdots \not{p}_N \gamma_\chi) = (-)^{(D-1)(N-k)} \, Tr(\not{p}_1 \ldots \not{p}_k \cdots \not{p}_N)$$

General proof can be given for the algebra

$$\gamma_\chi \gamma_\mu = (-)^{D-1} \gamma_\mu \gamma_\chi$$
$$\gamma_\chi^2 = 1. \qquad (2)$$

which interpolates over the integer values of D. But this result cannot be continued to complex values of D .

5. No Continuation for the Integer D Algebra

If

$$\begin{aligned} &\gamma_\chi \gamma_\mu = q \gamma_\mu \gamma_\chi \\ &\Longrightarrow \gamma_\chi \gamma_\mu^2 = q^2 \gamma_\mu^2 \gamma_\chi \\ &\Longrightarrow q^2 = 1 \end{aligned} \tag{3}$$

Then the relations in eq. (2) cannot be continued in D:

$$(-)^{D-1} = \exp(i\pi(D-1)) \Longrightarrow \exp(i2\pi(D-1)) \neq 1.$$

However successful calculations have been done using dimensional renormalization! Ref. [4] and [5].

PART II: Generalized Trace

6. Leaving Out the Integration over ξ

We split the integration over $\bar{c}$ from the one over ξ. For a single γ_χ we introduce the Generalized Trace

$$R(\not{p}_1 \not{p}_2 \ldots \not{p}_{N-1} \not{p}_N|\xi) = \int d^N \bar{c} \exp\big(\sum_{i=1}^{N} \bar{c}_i (p_i)_\mu \xi_\mu + \sum_{i<j=1}^{N} \bar{c}_i (p_i p_j) \bar{c}_j\big) \tag{4}$$

Then the conventional trace is

$$(i)^{\frac{D(D-1)}{2}} \int d^D \xi R(\not{p}_1 \not{p}_2 \ldots \not{p}_{N-1} \not{p}_N|\xi) = Tr(\not{p}_1 \not{p}_2 \ldots \not{p}_{N-1} \not{p}_N \gamma_\chi)$$

7. Dimensional Renormalization on the Generalized Trace

Ansatz. **The dimensional renormalization is performed on the Generalized Trace, i.e. before the integration over ξ. This means that the value of D is chosen after the pole subtraction: at the moment we integrate over the γ_χ-Grassmannian variable. At that moment the completely antisymmetric tensor might emerge if one γ_χ survived the pairing process.**

8. Clifford Algebra, Cyclicity and Lorentz Covariance

The Generalized Trace has many properties of the conventional trace. Some of them are slightly modified

1. Clifford algebra of the γ is valid

2. Lorentz covariance is valid

3. Cyclicity works in the form

$$R(\not p_1 \ldots \not p_{N-1}|\xi|\not p_N) = (-)^N \, R(\not p_N \not p_1 \ldots \not p_{N-1}|\xi)$$

and

$$R(\xi|\not p_1 \ldots \not p_{N-1}|\eta|\not p_N) = R(\not p_1 \ldots \not p_{N-1}|\eta|\not p_N|-\xi).$$

9. Pairing on Generalized Trace

We consider the most general pairing setup. We evaluate

$$Tr(\mathcal{A}\gamma_\chi\mathcal{B}\gamma_\chi) = (i)^{(D-1)D} \int d^D\xi \, d\mathcal{B} \, d^D\eta \, d\mathcal{A}$$
$$\exp\left((\mathcal{B}+\eta+\mathcal{A})\xi + \mathcal{B}*\mathcal{B} + (\eta+\mathcal{A})\mathcal{B} + \mathcal{A}\eta + \mathcal{A}*\mathcal{A}\right) \qquad (5)$$

Thus we get the integration over the pair i.e. the elements of $\mathcal{B}$ encapsulated between two γ_χ change sign in the pairing.

$$R(\mathcal{A}|\eta|\mathcal{B}|\xi) = \delta_P e^{\eta\xi} R(\mathcal{A}(-\mathcal{B})) \qquad (6)$$

Moreover, when the final integral is taken

$$\int d^D\xi \, d^D\eta \, e^{\eta\xi} = (-)^{\frac{D(D-1)}{2}}. \qquad (7)$$

10. Expansion in Powers of ξ

The Generalized Trace has no explicit dependence on D. Thus we can expand it in powers of ξ by using

$$\exp\Big(\sum_{i=1}^{N}\bar{c}_i(p_i)\xi\Big) = \prod_{i=1}^{N} e^{\bar{c}_i(p_i\xi)} = \prod_{i=1}^{N}(1+\bar{c}_i(p_i\xi)) = 1$$
$$+\sum_{i=1}^{N}\bar{c}_i(p_i\xi) + \sum_{i<j=1}^{N}\bar{c}_i(p_i\xi)\bar{c}_j(p_j\xi) + \sum_{i<j<k=1}^{N}\bar{c}_i(p_i\xi)\bar{c}_j(p_j\xi)\bar{c}_k(p_k\xi)$$
$$+\dots . \tag{8}$$

In the last integration we fix D i.e. one of the above terms. Momenta appear as simple factors. No "completely anti-symmetric tensor". All p_j's appearing in (p_j,ξ) are taken out of trace (emerging from the integration over $\bar{c}$).

11. Fundamental Formula

After the integration on $\bar{c}$ present in the eq. (4) a typical expansion in terms of powers of ξ is given by

$$R(\not{p}_1 \not{p}_2 \ldots \not{p}_{N-1} \not{p}_N|\xi) = \sum_{\mathcal{P}} \delta_{\mathcal{P}} Tr(\not{p}_{i_1} \ldots \not{p}_{i_{N-K}})(p_{j_1}, \xi) \ldots (p_{j_K}, \xi) \tag{9}$$

where the sum is over all partitions $\mathcal{P}$ of the N integers in two mutually disjoint ordered sets $(i_1, \ldots, i_{N-K})$ and $(j_1, \ldots, j_K)$. The parity $\delta_{\mathcal{P}}$ counts the permutations needed to perform the integrations over $d\bar{c}_{j_1}, \ldots, d\bar{c}_{j_K}$. The quantity in eq. (9) is the perfect tool to be continued in D and renormalized .

12. Conclusions

Good News: γ_χ survives Dimensional Regularization!

But: is the procedure consistent?

References

[1] R. Ferrari, "Managing γ_5 in Dimensional Regularization and ABJ Anomaly," arXiv:1403.4212 [hep-th].

[2] S. Fubini and E. R. Caianiello, "On the Algorithm of Dirac spurs," Nuovo Cim. 9, 1218 (1952).

[3] A. M. Jaffe, A. Lesniewski and J. Weitsman, "Pfaffians On Hilbert Space," J. Funct. Anal. 83, 348 (1989).

[4] R. Ferrari, "Managing γ_5 in Dimensional Regularization II: the Trace with more γ_5," arXiv:1503.07410 [hep-th].

[5] R. Ferrari and M. Raciti, "On effective Chern-Simons Term induced by a Local CPT-Violating Coupling using γ_5 in Dimensional Regularization," arXiv:1510.04666 [hep-th].