Dual formulation of lattice field theory

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Why dual lattice field theory? And what is it?

Quantum field theory on the lattice

- In the lattice formulation we give the Feynman path integral a mathematically precise meaning by introducing a space time lattice Λ.

$$\langle O \rangle \;=\; \frac{1}{Z} \int D[\phi]\; e^{-S[\phi]}\; O[\phi]$$

$D[\phi] = \prod_{x \in \Lambda} d\phi(x)$
$S[\phi]$ .... discretized action.

- In a Monte Carlo simulation one generates a finite number of field configurations $\phi^{(n)}, n = 1, 2 \ldots N$ with probability

$$P[\phi^{(n)}] \;=\; \frac{e^{-S[\phi^{(n)}]}}{Z}$$

Observables assume the form of mean values

$$\langle O \rangle \;=\; \frac{1}{N} \sum_{n=1}^{N} O[\phi^{(n)}] \;+\; \mathcal{O}(1/\sqrt{N})$$

Complex action problem

- In general, lattice field theories with finite chemical potential μ or a topological term have actions $S[\phi]$ with an imaginary part.

- The Boltzmann factor

$$e^{-S[\phi]} \in \mathbb{C}$$

  thus has a complex phase and cannot be used as a probability weight.

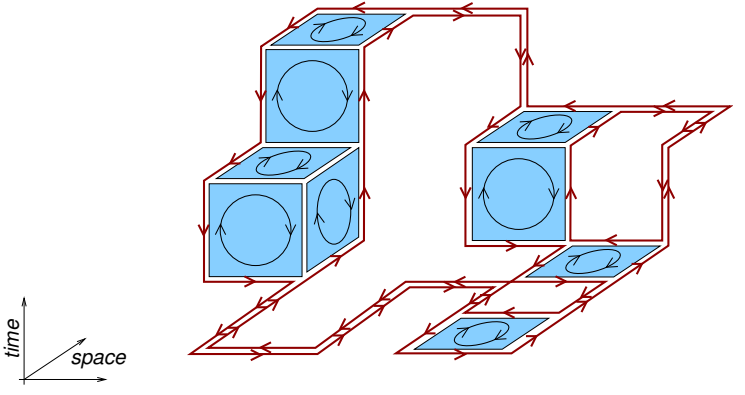
- Standard Monte Carlo simulation techniques are not available for a non-perturbative analysis.

  "Complex action problem" or "Sign problem"

- Generic feature of finite density field theories both, on the lattice and in the continuum, for bosonic and fermionic theories.

Solving the complex action problem with dual variables

- For some systems one can use strong coupling expansion techniques to rewrite the partition function in terms of "dual variables" such that it is a sum over real and positive terms.
- The dual variables are world lines for matter fields and world sheets for gauge fields.
- The Monte Carlo simulation then is done in terms of the dual variables.
- New Monte Carlo techniques for systems with constraints.



How does dualization work?

... example of a charged scalar field with chemical potential

Dual representation for the charged scalar ϕ^4 field

- Lattice action: ($\phi_x \in \mathbb{C}$, $M^2 = 8 + m^2$)

$$S = \sum_x \left[M^2 \, |\phi_x|^2 \; + \lambda \, |\phi_x|^4 \right] - \sum_{x,\nu} \left[e^{-\mu \, \delta_{\nu 4}} \, \phi_x^\star \, \phi_{x+\widehat{\nu}} \; + \; e^{\mu \, \delta_{\nu 4}} \, \phi_x \, \phi_{x+\widehat{\nu}}^\star \right]$$

- Expand the nearest neighbor terms of e^{-S}:

$$\prod_{x,\nu} \exp\left(e^{-\mu \, \delta_{\nu 4}} \, \phi_x^\star \, \phi_{x+\widehat{\nu}}\right) \times \exp\left(e^{\mu \, \delta_{\nu 4}} \, \phi_x \, \phi_{x+\widehat{\nu}}^\star\right)$$

$$= \prod_{x,\nu} \sum_{j_{x,\nu}=0}^{\infty} \frac{\left(e^{-\mu \, \delta_{\nu 4}}\right)^{j_{x,\nu}}}{j_{x,\nu} \, !} \left(\phi_x^\star \, \phi_{x+\widehat{\nu}}\right)^{j_{x,\nu}} \times \sum_{\bar{j}_{x,\nu}=0}^{\infty} \frac{\left(e^{\mu \, \delta_{\nu 4}}\right)^{\bar{j}_{x,\nu}}}{\bar{j}_{x,\nu} \, !} \left(\phi_x \, \phi_{x+\widehat{\nu}}^\star\right)^{\bar{j}_{x,\nu}}$$

$$= \sum_{\{j,\bar{j}\}} e^{-\mu \sum_x (j_{x,4} - \bar{j}_{x,4})} \prod_{x,\nu} \frac{1}{j_{x,\nu}! \, \bar{j}_{x,\nu}!} \prod_x \phi_x^{\sum_\nu (\bar{j}_{x,\nu} + j_{x-\hat{\nu},\nu})} \, \phi_x^{\star \, \sum_\nu (j_{x,\nu} + \bar{j}_{x-\hat{\nu},\nu})}$$

- The $j_{x,\nu}$ will be the new, "dual" degrees of freedom.

- The fact that we can reorder the fields in the last step is crucial.

Dual representation - integrating out the fields

- Integral over ϕ_x at site x: ($\Sigma_j, \overline{\Sigma}_j$ are sums of $j_{y,\nu}, \overline{j}_{y,\nu}$ connected to x)

$$\int_{\mathbb{C}} d\,\phi_x \;\; e^{-M^2|\phi_x|^2 - \lambda|\phi_x|^4} \; (\phi_x)^{\Sigma_j} \; (\phi_x^\star)^{\overline{\Sigma}_j}$$

- Polar coordinates $\phi_x = re^{i\theta}$ to separate radial and U(1) parts (symmetry):

$$\int_0^\infty dr \; r^{\,\Sigma_j + \overline{\Sigma}_j + 1} \; e^{-M^2 r^2 - \lambda r^4} \int_{-\pi}^{\pi} d\theta \; e^{\,i\theta\,(\Sigma_j - \overline{\Sigma}_j)} \;\; = \;\; \mathcal{I}(\Sigma_j + \overline{\Sigma}_j) \; \delta(\Sigma_j - \overline{\Sigma}_j)$$

- At every site there is a weight factor $\mathcal{I}(\Sigma_j + \overline{\Sigma}_j)$ and a constraint.

- The constraint $\delta(\Sigma_j - \overline{\Sigma}_j)$ enforces vanishing flux of $d_{x,\nu} = j_{x,\nu} - \overline{j}_{x,\nu}$ at each site.

Dual representation - final form

- The original partition function is mapped exactly to a sum over configurations of the dual variables $j_{x,\nu}, \bar{j}_{x,\nu} \in \mathbb{N}_0$ with $d_{x,\nu} = j_{x,\nu} - \bar{j}_{x,\nu}$.

$$Z \;=\; \sum_{\{j,\bar{j}\}} \mathcal{W}[\,j,\bar{j}\,]\,\mathcal{C}[\,d\,]$$

- Weight factor from radial d.o.f. and combinatorics:

$$\mathcal{W}[\,j,\bar{j}\,] = e^{-\mu \sum_x (j_{x,4} - \bar{j}_{x,4})} \prod_{x,\nu} \frac{1}{j_{x,\nu}!\,\bar{j}_{x,\nu}!} \prod_x \mathcal{I}\Big(\sum_\nu \Big[j_{x,\nu} + j_{x-\widehat{\nu},\nu} + \bar{j}_{x,\nu} + \bar{j}_{x-\widehat{\nu},\nu} \Big] \Big)$$

- Constraint from integrating over the symmetry group ($d_{x,\nu} = j_{x,\nu} - \bar{j}_{x,\nu}$):

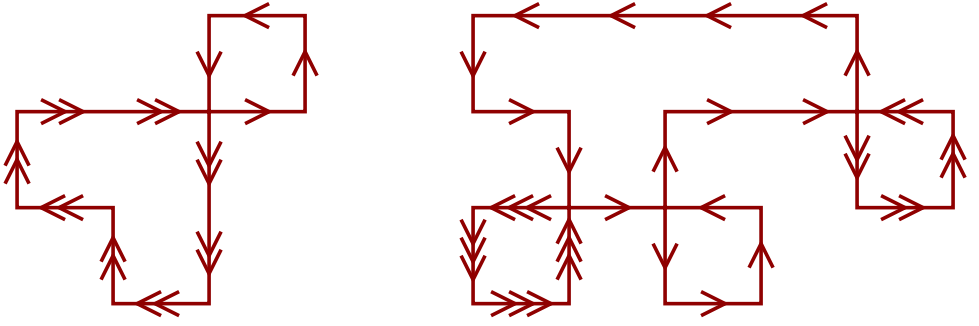
$$\mathcal{C}[\,d\,] \;=\; \prod_x \delta\Big(\sum_\nu [\,d_{x,\nu} - d_{x-\widehat{\nu},\nu}\,] \Big)$$

Admissible configurations are loops:

- Constraint from integrating over the symmetry group:

$$\forall\, x\ :\quad \sum_{\nu} \left[\, d_{x,\nu} - d_{x-\widehat{\nu},\nu} \,\right] \;=\; 0 \qquad\qquad \left(\, \vec{\nabla}\vec{d} \;=\; 0 \,\right)$$

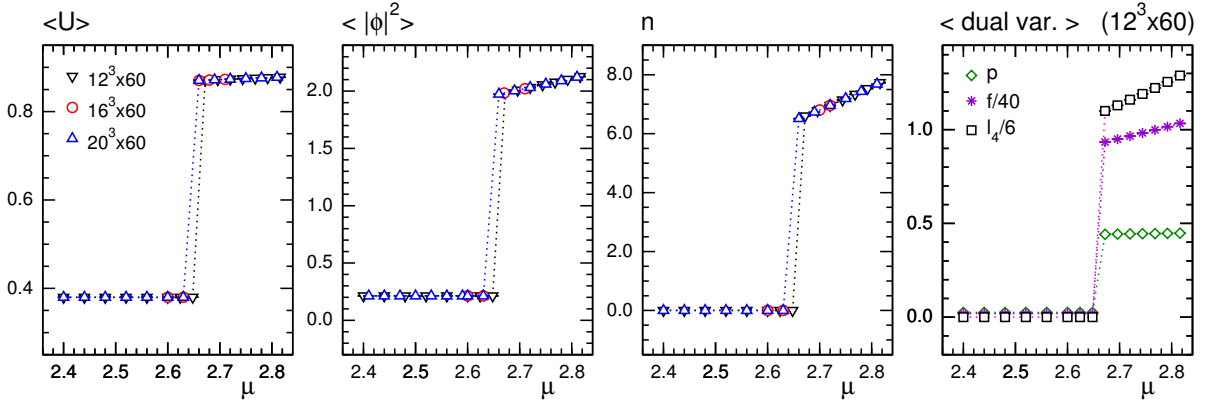
- Admissible configurations of dual variables are oriented loops of flux:



- Chemical potential μ couples to the temporal winding number of $d_{x,\nu}$-flux.

- MC simulations directly in terms of world lines overcome the complex action problem.

Example: The confining phase of U(1) gauge Higgs model at low T



In the confining phase the dependence on the chemical potential μ sets in only when μ reaches the mass of the lowest excitation. "Silver Blaze behaviour"

The corresponding BEC is accompanied by a condensation of dual variables.

Y. Delgado Mercado, C. Gattringer, A. Schmidt, PRL 111, 2013

Solving the complex action problem with dual variables

Dual representations were found for several interesting classes of models and dual simulations have helped to understand interesting finite density physics in quantum field theory.

The Monte Carlo simulation then is done in terms of the dual variables which are world lines for matter fields and world sheets for gauge fields.

- $\mathbb{Z}_3$ spin model with chemical potential. (PRL 2011, CPC 2012)
- Effective Polyakov loop model with chemical potential. (NPB 2011, NPB 2012)
- $\mathbb{Z}_3$ gauge-Higgs model with chemical potential. (PRD 2012, CPC 2013)
- Charged ϕ^4 field at finite density (relativistic Bose gas). (NPB 2013, PLB 2013)
- U_1 gauge-Higgs model with chemical potential. (PRL 2013, CPC 2013)
- Scalar QED_2 with topological term. (PRD 2015)
- Massless Schwinger model with topological term and chemical potential. (NPB 2015)
- Nanowires interacting with electromagnetic fields. (PRD 2015)
- O(N) and CP(N-1) models with chemical potential. (PLB 2015, PRL 2015)

Related work: S. Chandrasekharan, P. de Forcrand, T. Korzec, O. Philipsen, U. Wolff ...

Towards non-abelian gauge fields

Work with Carlotta Marchis

Where is the problem?

- Before integrating the scalar fields with $\int D[\phi] = \int \prod_x d\phi_x$ we had to reorder them:

$$\prod_{x,\nu} (\phi_x^\star \; \phi_{x+\widehat{\nu}})^{j_{x,\nu}} \; (\phi_x \; \phi_{x+\widehat{\nu}}^\star)^{\bar{j}_{x,\nu}} \;=\; \prod_x \phi_x^{\sum_\nu (\bar{j}_{x,\nu} + j_{x-\hat{\nu},\nu})} \phi_x^{\star \sum_\nu (j_{x,\nu} + \bar{j}_{x-\hat{\nu},\nu})}$$

(works also for abelian gauge fields)

- When reordering fermionic d.o.f. one picks up minus signs (Grassmann numbers).

- Reordering non-abelian gauge fields?

.... one does not even know how to do it!

- We currently explore decomposing the action into smaller building blocks:

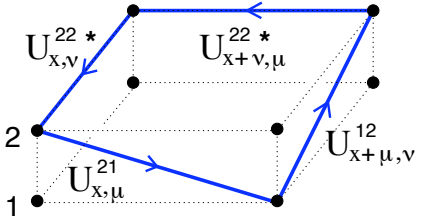
Abelian color cycles (ACC)

Decomposition of the non-abelian action into abelian color cycles:

- Action for SU(2) lattice gauge theory ($U_{x,\mu} \in \mathrm{SU}(2)$) :

$$S \;=\; -\frac{\beta}{2}\sum_{x,\mu<\nu} \mathrm{Tr}\; U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{\dagger}_{x+\hat{\nu},\mu}\, U^{\dagger}_{x,\nu} \;=\; -\frac{\beta}{2}\sum_{x,\mu<\nu} \sum_{a,b,c,d=1}^{2} U^{ab}_{x,\mu}\, U^{bc}_{x+\hat{\mu},\nu}\, U^{dc\,\star}_{x+\hat{\nu},\mu}\, U^{ad\,\star}_{x,\nu}$$

- The products $U^{ab}_{x,\mu}\, U^{bc}_{x+\hat{\mu},\nu}\, U^{dc\,\star}_{x+\hat{\nu},\mu}\, U^{ad\,\star}_{x,\nu}$ are the abelian color cycles (ACC) (= paths through color space along plaquettes) we use for expanding the Boltzmann factor. Example:



$$\longrightarrow \qquad p^{2122}_{x,\mu} \;\in\; \boldsymbol{N_0}$$

- Suitable parameterization:

$$U_{x,\mu} \;=\; \begin{bmatrix} \cos\theta_{x,\mu}\; e^{i\alpha_{x,\mu}} & \sin\theta_{x,\mu}\; e^{i\beta_{x,\mu}} \\ -\sin\theta_{x,\mu}\; e^{-i\beta_{x,\mu}} & \cos\theta_{x,\mu}\; e^{-i\alpha_{x,\mu}} \end{bmatrix} \qquad \theta_{x,\mu} \in [0,\pi/2]\;,\quad \alpha_{x,\mu},\beta_{x,\mu} \in [-\pi,\pi]$$

Expansion in ACCs

- Partition sum:

$$Z = \int D[U] \, \exp\left(\frac{\beta}{2} \sum_{x,\mu<\nu} \sum_{a,b,c,d} U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc \, \star}_{x+\hat{\nu},\mu} \, U^{ad \, \star}_{x,\nu} \right) , \quad \int D[U] = \prod_{x,\mu} \int_{SU(2)} dU_{x,\mu}$$

- Expansion of the Boltzmann factor:

$$Z = \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} e^{\frac{\beta}{2} \, U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc \, \star}_{x+\hat{\nu},\mu} \, U^{ad \, \star}_{x,\nu}}$$

$$= \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} \sum_{p^{abcd}_{x,\mu\nu}=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu} \, !} \left(U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc \, \star}_{x+\hat{\nu},\mu} \, U^{ad \, \star}_{x,\nu} \right)^{p^{abcd}_{x,\mu\nu}}$$

- Reordering the terms:

$$Z = \sum_{\{p\}} \prod_{x,\mu<\nu} \prod_{a,b,c,d} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu} \, !} \prod_{x,\mu} \int d_H[\theta_{x,\mu}, \alpha_{x,\mu}, \beta_{x,\mu}] \prod_{ab} \left(U^{ab}_{x,\mu} \right)^{N^{ab}_{x,\mu}[p]} \left(U^{ab \, \star}_{x,\mu} \right)^{\overline{N}^{ab}_{x,\mu}[p]}$$

Remaining link integrals can be solved and give constraints and weights for the configurations $\{p\}$ of the cycle occupation numbers $p^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$.

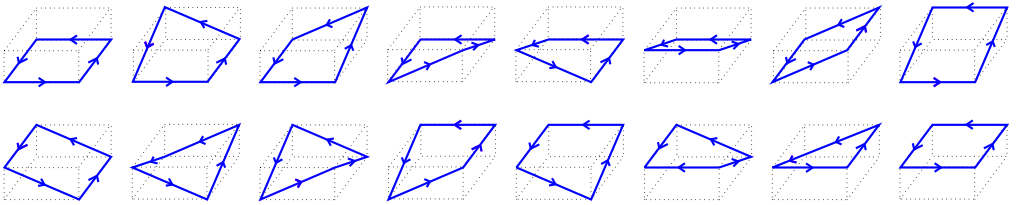
Partition function as sum over occupation numbers of ACCs

- Dual partition sum:

$$Z = \sum_{\{p\}} W_\beta[p]\ (-1)^{\sum_{x,\mu} J^{21}_{x,\mu}} \prod_{x,\mu<\nu} \delta\left(J^{11}_{x,\mu} - J^{22}_{x,\mu}\right)\ \delta\left(J^{12}_{x,\mu} - J^{21}_{x,\mu}\right)$$

$$J^{ab}_{x,\mu} = \text{total flux from } a \text{ to } b \text{ along the link } x,\mu$$

- 16 possible ACCs that can be occupied (i.e., $p^{abcd}_{x,\mu\nu} > 0$):



- Constraints at each link:

Adding matter

Staggered fermions in an SU(2) background

- Fermionic partition sum:

$$Z_F[U] = \int D[\overline{\psi},\psi]\; e^{-S_F[\overline{\psi},\psi,U]}$$

$$\overline{\psi}_x = (\,\overline{\psi}^1_x, \overline{\psi}^2_x\,)\;, \quad \psi_x = \begin{pmatrix} \psi^1_x \\ \\ \psi^2_x \end{pmatrix}$$

- Action and its decomposition into color bilinears:

$$S_F[\overline{\psi},\psi,U] = \sum_x \Big[m\,\overline{\psi}_x\psi_x + \sum_\mu \frac{\gamma_{x,\mu}}{2}\Big(\overline{\psi}_x U_{x,\mu}\psi_{x+\hat{\mu}} - \overline{\psi}_{x+\hat{\mu}} U^\dagger_{x,\mu}\psi_x\Big)\Big]$$

$$= \sum_x \Big[m\sum_a \overline{\psi}^a_x\psi^a_x + \sum_\mu \frac{\gamma_{x,\mu}}{2}\sum_{a,b}\Big(\overline{\psi}^a_x U^{ab}_{x,\mu}\psi^b_{x+\hat{\mu}} - \overline{\psi}^b_{x+\hat{\mu}} U^{ab\star}_{x,\mu}\psi^a_x\Big)\Big]$$

Loop expansion

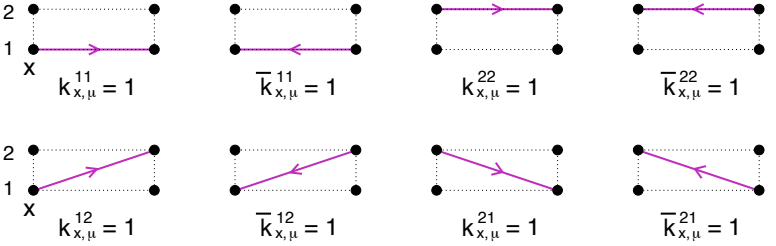
- Expanding the Boltzmann factors in the fermionic partition sum:

$$
\begin{aligned}
Z_F[U] &= \int D[\overline{\psi},\psi] \prod_x \prod_a e^{-m\overline{\psi}^a_x \psi^a_x} \prod_{x,\mu} \prod_{a,b} e^{-\frac{\gamma_{x,\mu}}{2}\overline{\psi}^a_x U^{ab}_{x,\mu} \psi^b_{x+\hat{\mu}}}\, e^{\frac{\gamma_{x,\mu}}{2}\overline{\psi}^b_{x+\hat{\mu}} U^{ab\star}_{x,\mu} \psi^a_x} \\
&= \int D[\overline{\psi},\psi] \prod_x \prod_a \sum_{s^a_x=0}^{1} (-m\overline{\psi}^a_x \psi^a_x)^{s^a_x} \\
&\qquad \times \prod_{x,\mu} \prod_{a,b} \sum_{k^{ab}_{x,\mu}=0}^{1} \left(-\frac{\gamma_{x,\mu}}{2}\, \overline{\psi}^a_x U^{ab}_{x,\mu} \psi^b_{x+\hat{\mu}}\right)^{k^{ab}_{x,\mu}} \sum_{\overline{k}^{ab}_{x,\mu}=0}^{1} \left(\frac{\gamma_{x,\mu}}{2}\, \overline{\psi}^b_{x+\hat{\mu}} U^{ab\star}_{x,\mu} \psi^a_x\right)^{\overline{k}^{ab}_{x,\mu}} \\
&= \frac{1}{2^{2V}} \sum_{\{s,k,\overline{k}\}} (2m)^{\Sigma_{x,a} s^a_x} \prod_{x,\mu} \prod_{a,b} (U^{ab}_{x,\mu})^{k^{ab}_{x,\mu}} \left(U^{ab\star}_{x,\mu}\right)^{\overline{k}^{ab}_{x,\mu}} \\
&\qquad \times \int D[\overline{\psi},\psi] \prod_x \prod_a (\overline{\psi}^a_x \psi^a_x)^{s^a_x} \prod_{x,\mu} \prod_{a,b} \left(-\gamma_{x,\mu} \overline{\psi}^a_x \psi^b_{x+\hat{\mu}}\right)^{k^{ab}_{x,\mu}} \left(\gamma_{x,\mu} \overline{\psi}^b_{x+\hat{\mu}} \psi^a_x\right)^{\overline{k}^{ab}_{x,\mu}}
\end{aligned}
$$

The Grassmann integral is saturated by monomers ($s^a_x = 1$), dimers ($k^{ab}_{x,\mu} = \overline{k}^{ab}_{x,\mu} = 1$) and loops of $k^{ab}_{x,\mu} = 1$ and $\overline{k}^{ab}_{x,\mu} = 1$. Only loops introduce signs!

Interaction with the gauge fields

- $k^{ab}_{x,\mu}$ and $\overline{k}^{ab}_{x,\mu}$ introduce color flux on the links:



- Full partition sum:

$$Z = \sum_{\{p,k,\overline{k},s\}} C_{MDL}[s,k,\overline{k}]\, W_\beta[p]\, W_m[s] \prod_{x,\mu} (-1)^{J^{21}_{x,\mu}+k^{21}_{x,\mu}+\overline{k}^{21}_{x,\mu}} \prod_L \text{sign}\,(L)$$

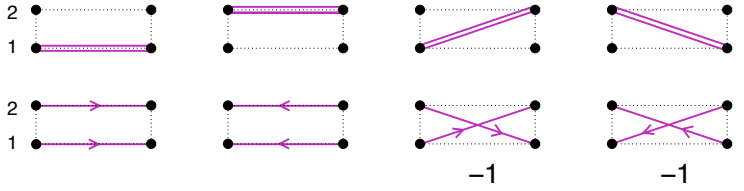
$$\times \prod_{x,\mu<\nu} \delta\left(J^{11}_{x,\mu}+k^{11}_{x,\mu}-\overline{k}^{11}_{x,\mu} - [J^{22}_{x,\mu}+k^{22}_{x,\mu}-\overline{k}^{22}_{x,\mu}]\right) \delta\left(J^{12}_{x,\mu}+k^{12}_{x,\mu}-\overline{k}^{12}_{x,\mu} - [J^{21}_{x,\mu}+k^{21}_{x,\mu}-\overline{k}^{21}_{x,\mu}]\right)$$

$$\text{sign}\,(L) = -(-1)^{\#\,plaquettes}\,(-1)^{length/2}\,(-1)^{temp.winding}$$

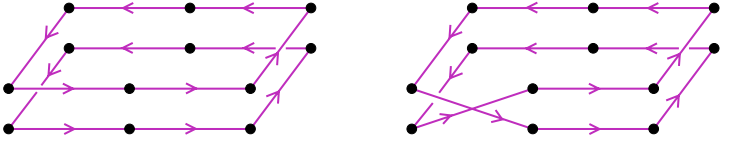
- Gauge constraints:

Strong coupling loops ($\beta = 0$)

- Strong coupling: $\beta = 0 \Rightarrow$ only fermion lines. The gauge constraints limit the number of admissible link elements:



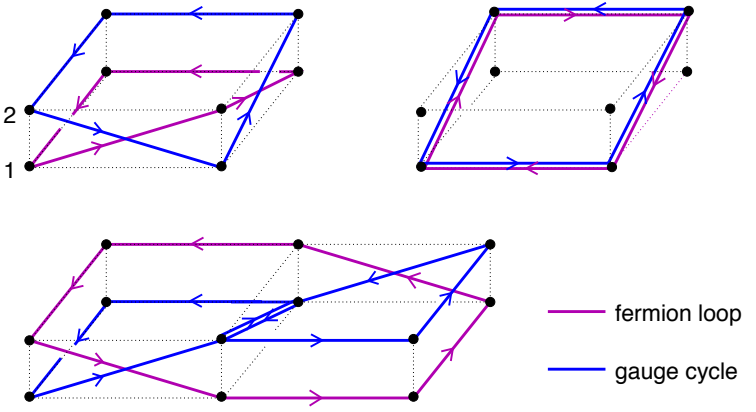
- Signs from color flips compensate fermion loop signs:



- In the strong coupling limit all contributions are positive.

Loops at $\beta > 0$

- At $\beta > 0$ one can activate gauge cycles to satisfy the gauge constraints for more general fermion loops. Examples of $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta^2)$:



- One can show that all contributions up to $\mathcal{O}(\beta^3)$ are positive. From $\mathcal{O}(\beta^4)$ on some configurations with negative signs appear. Chemical potential couples to total temporal winding number.

Summary

- Many field theories have a complex action problem at non-zero chemical potential.
- For some theories it is possible to exactly rewrite the lattice regularized partition sum in terms of dual variables.
- The dual variables are world lines for matter and world sheets for gauge fields.
- All contributions to the partition function are real and positive and the complex action problem is solved.
- To address the re-ordering problem of non-abelian gauge fields we introduce abelian color cycles, which are paths through color space along plaquettes.
- The ACC construction can be generalized by including matter fields.
- Weights for all terms of the strong coupling expansion are known in closed form.