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Some properties of Born-Infeld theories

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Abstract

In this presentation various n-field non-linear Abelian Born-Infeld type Lagrangians are obtained from a linear Lagrangian by integrating out the auxiliary fields. The potential of the linear Lagrangian is of the type Tr($\mathcal{M}(\phi)$), where $\mathcal{M}$ is a symmetric symplectic matrix depending on scalar fields ϕ, and if the scalar fields span a homogeneous symmetric space G/H with H a subgroup of a Lie group G and $\mathcal{M}$ defines an embedding $G/H \hookrightarrow Sp(2n)/U(n)$, then the linear Lagrangian exhibits H duality invariance.
In this way we are able to recover some previously known Lagrangians with $U(1)$ and $SU(2)$ on-shell symmetry. From a coset representative of $GL(n)/SO(n) \hookrightarrow Sp(2n)/U(n)$ we are able to construct a new Lagrangian with $SO(n)$ duality, which can be generalized to non-Abelian field strengths. This method allows to associate a non-linear Born-Infeld type Lagrangian to any supergravity theory based on $\mathcal{M}$.

Plan

1. The Born-Infeld Lagrangian in 4 dimensions
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The Born-Infeld Lagrangian in 4 dimensions

[M. Born, L. Infeld, Proc. Roy. Soc. Lond. A 144 (1934) 425;
M. Born, Proc. Roy. Soc. Lond. A 143 (1934) 410.]

$$\mathcal{L} = \frac{1}{\lambda}\left[1 - \sqrt{\left|\text{Det}\left(\eta_{\mu\nu} + \sqrt{\lambda} F_{\mu\nu}\right)\right|}\right] = \frac{1}{\lambda}\left[1 - \sqrt{1 + \frac{\lambda}{2} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda^2}{16}\left(F_{\mu\nu}{}^{*}F^{\mu\nu}\right)^2}\right].$$

Here λ is a real parameter, $\eta_{\mu\nu}$ the Minkowski metric, $F_{\mu\nu} = \partial A_\mu - \partial A_\nu$ an Abelian field strength, ${}^{*}F^{\mu\nu}$ its Hodge dual.

• It was found as a generalization of electromagnetism in an effort to impose an upper bound on the electric field of a point charge.

• Its main feature is that it is a non-linear Lagrangian, which realizes electric-magnetic duality in an interacting system and is self-dual [E. Schrödinger, Proc. Roy. Soc. Lond. A 150 (1935) 465].

• In string theory, it is related to the dynamics of open strings in a constant electromagnetic background and to D-branes.[E. S. Fradkin, A. A. Tseytlin, Phys. Lett. B 163 (1985) 123].

Linear Description of Born-Infeld theories with auxiliary fields

[M. Roček, A. A. Tseytlin, hep-th/9811232;
P. Aschieri, D. Brace, B. Morariu, B. Zumino, hep-th/0003228;
L. Andrianopoli, R. D'Auria, M. Trigiante, arXiv:1412.6786 [hep-th]]

Linear Lagrangian:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^T_{\mu\nu}\,g\,F^{\mu\nu} + \frac{1}{4}F^T_{\mu\nu}\,\theta\,{}^*F^{\mu\nu} - \frac{1}{2\lambda}\mathrm{Tr}(\mathcal{M}) + const.\\ &= -\frac{1}{4}\mathrm{Tr}(\mathbb{F}g) + \frac{1}{4}\mathrm{Tr}\left({}^*\mathbb{F}\theta\right) - \frac{1}{2\lambda}\mathrm{Tr}\left(g + g^{-1}(\mathbb{1}+\theta^2)\right) + const.\end{aligned}$$

Here:

λ is a small real parameter;

$F^{\mu\nu}$ abelian field-strengths; ${}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$ their Hodge duals;

$\mathbb{F} = F^{\mu\nu}\,F^T_{\mu\nu}$, ${}^*\mathbb{F} = F^{\mu\nu}\,{}^*F^T_{\mu\nu}$;

the auxiliary fields g and θ are $n\times n$ symmetric matrices, which depend on a set of scalar fields $\{\phi\}$;

$\mathcal{M}$ is a symplectic, symmetric matrix:

$$\mathcal{M}[g(\phi),\theta(\phi)] = \begin{pmatrix} g+\theta\cdot g^{-1}\cdot\theta & -\theta\cdot g^{-1} \\ -g^{-1}\cdot\theta & g^{-1}\end{pmatrix}. \qquad (1)$$

Observations

- Integrating out the non-dynamical scalar (auxiliary) sector through its algebraic equations of motion yields a non-linear n-vector Lagrangian of Born-Infeld type.
  In the case of a single vector field, the original non-linear Born-Infeld Lagrangian is recovered.

- If the scalar fields span a homogenous symmetric space $\{\phi\} \in \frac{G}{H}$, and L is a symplectic coset representative, then $\mathcal{M} := L \cdot L^T$ is a symmetric symplectic matrix:
  $\mathcal{M} = \mathcal{M}^T$ and $\mathcal{M}^T \mathbb{C} \mathcal{M} - \mathbb{C} = \mathbb{0}$, with $\mathbb{C} = \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ -\mathbb{1} & \mathbb{0} \end{pmatrix}$.

$$\Downarrow$$

- $\mathcal{M}(\Phi)$ defines an embedding:

$$\frac{G}{H} \hookrightarrow \frac{Sp(2n)}{U(n)}$$

  and this Born-Infeld like theory has duality symmetry described by H by construction $\Longrightarrow$ Self-duality is built-in.

2-field Lagrangian with electric $U(1)$ duality

Start with the spin $\frac{1}{2}$ coset representative of $\frac{SL(2)}{U(1)}$ defined in the solvable Iwasawa decomposition by:

$$L = \begin{pmatrix} 1 & 0 \\ -y & 0 \end{pmatrix} \begin{pmatrix} e^{-\phi/2} & 0 \\ 0 & e^{\phi/2} \end{pmatrix} = \begin{pmatrix} e^{-\phi/2} & 0 \\ -ye^{-\phi/2} & e^{\phi/2} \end{pmatrix}$$

where ϕ corresponds to the Cartan subalgebra and y to the nilpotent generator.

Consider its diagonal embedding in $\frac{Sp(4)}{U(2)}$:

$$\mathcal{M} = \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} & 0 & 0 \\ -\frac{y}{x} & \frac{x^2+y^2}{x} & 0 & 0 \\ 0 & 0 & \frac{x^2+y^2}{x} & \frac{y}{x} \\ 0 & 0 & \frac{y}{x} & \frac{1}{x} \end{pmatrix}; \; g = \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} \\ -\frac{y}{x} & \frac{x^2+y^2}{x} \end{pmatrix}; \quad \theta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

where $x = e^{\phi}$.

By means of their algebraic equations of motion, which are polynomial of degree 2 in x and y, integrating out the auxiliary fields g and θ in the linear Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\, F^T_{\mu\nu}\, g\, F^{\mu\nu} - \frac{1}{2\lambda}\,\mathrm{Tr}(\mathcal{M}) + \frac{2}{\lambda}$$

allows to recover the non-linear Lagrangian:

[S. Ferrara, A. Sagnotti, A. Yeranyan, arXiv:1503.04731 [hep-th]]

$$\mathcal{L} = \frac{2}{\lambda}\left[1 - \sqrt{1 + \frac{\lambda}{4}\left(F^1_{\mu\nu}F^{1\mu\nu} + F^2_{\mu\nu}F^{2\mu\nu}\right) + \frac{\lambda^2}{16}\left(F^1_{\mu\nu}F^{1\mu\nu}\,F^2_{\alpha\beta}F^{2\alpha\beta} - (F^1_{\mu\nu}F^{2\mu\nu})^2\right)}\,\right].$$

Since θ vanishes and the embedding is diagonal, this Lagrangian has manifest (electric) $U(1)$ duality. It is doubly self-dual under a Legendre tranform in both vectors.

Legendre transform to electric-magnetic $U(1)$ duality

A rotation $R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ of the $U(1)$ subgroup, transforming $\mathcal{M} \to R^{-1}\mathcal{M}R$, yields the Lagrangian:

$$\mathcal{L} = \frac{2}{\lambda}\left[1 - \sqrt{1 + \frac{\lambda}{4}\left(F^1_{\mu\nu}F^{1\mu\nu} + F^2_{\mu\nu}F^{2\mu\nu}\right) - \frac{\lambda^2}{16}\left(F^1_{\mu\nu}{}^*F^{2\mu\nu}\right)^2}\,\right].$$

• The same Lagrangian [S. Ferrara, A. Sagnotti, A. Yeranyan, arXiv:1503.04731 [hep-th]] can be obtained by performing a Legendre transform on one field, i.e. adding to the Lagrangian a term $-\frac{1}{4}\varepsilon^{\mu\nu\lambda\delta}F^2_{\mu\nu}G^2_{\lambda\delta}$ and then renaming the field $G^2 \to F^2$, ${}^*G^2 \to {}^*F^2$.

• The $U(1)$ subgroup is not embedded diagonally any more $\Longrightarrow$ The $U(1)$ duality becomes electric-magnetic and holds only on-shell on the equations of motion.

• For the consistent truncation $F^1 = F^2$ it reduces to the Born-Infeld theory for one field.

2-field theory with $SU(2)$ duality

The embedding $\frac{SO(1,3)}{SO(3)} \hookrightarrow \frac{Sp(4)}{U(2)}$:

$$\mathcal{M} = \begin{pmatrix} \frac{1}{w} + 16w(x^2+y^2) & 0 & 4wy & 4wx \\ 0 & \frac{1}{w} + 16w(x^2+y^2) & 4wx & -4wy \\ 4wy & 4wx & w & 0 \\ 4wx & -4wy & 0 & w \end{pmatrix}$$

yields the Lagrangian:

[S. Ferrara, A. Sagnotti, A. Yeranyan, arXiv:1602.04566 [hep-th]]

$$\mathcal{L} = \frac{2}{\lambda}\left[1 - \sqrt{1 + \frac{\lambda}{4}\left(F^1_{\mu\nu}F^{1\mu\nu} + F^2_{\mu\nu}F^{2\mu\nu}\right) - \frac{\lambda^2}{64}A}\,\right]$$

where

$$A = \left(F^1_{\mu\nu}{}^*F^{1\mu\nu}\right)^2 + \left(F^2_{\mu\nu}{}^*F^{2\mu\nu}\right)^2 + 4\left(F^1_{\mu\nu}{}^*F^{2\mu\nu}\right)^2 - 2F^1_{\mu\nu}{}^*F^{1\mu\nu}\,F^2_{\alpha\beta}{}^*F^{2\alpha\beta}.$$

This theory has SU(2) duality symmetry.

n-field theory with $SO(n)$ symmetry

Start with the diagonal embedding ($\theta = \mathbb{0}$):

$$\frac{GL(n)}{SO(n)} \hookrightarrow \frac{Sp(2n,\mathbb{R})}{U(n)}$$

and the linear Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(\mathbb{F}g) - \frac{1}{2\lambda}\mathrm{Tr}\left(g + g^{-1}\right) + \frac{n}{\lambda}.$$

Since $GL(n)$ does not have the constraint of unit determinant, it is possible to explicitly solve the equations of motion of the auxiliary field g:

$$\frac{\delta\mathcal{L}}{\delta g} = -\frac{1}{4}\mathbb{F} + \frac{1}{2\lambda}(g^{-2} - \mathbb{1}) = \mathbb{0}.$$

This yields the new non-linear Born-Infeld type Lagrangian:

$$\mathcal{L} = -\frac{1}{\lambda}\mathrm{Tr}\sqrt{\mathbb{1} + \frac{\lambda}{2}\mathbb{F}} + \frac{n}{\lambda}.$$

Observations

- By construction this Lagrangian has manifest $SO(n)$ duality. The $SO(n)$ subgroup is embedded diagonally.

$$\Downarrow$$

- The theory is self-dual under a Legendre transform on all the fields, i.e. under the addition of a term $-\frac{1}{4}\varepsilon^{\mu\nu\lambda\delta}F^I_{\mu\nu}G^I_{\lambda\delta}$.
- It is particularly interesting, because it can be generalized to non-Abelian field-strenghts: $F_{\mu\nu} = \partial A_\mu - \partial A_\nu - g[A_\mu, A_\nu]$.
- It has a simple form, because it does not contain the λ^2 term inside the square root, due to the fact that $GL(n)$ does not impose the constraint of unit determinant.

Conclusions and outlook

- This method [L. Andrianopoli, R. D'Auria, M. Trigiante, arXiv:1412.6786 [hep-th]] allows to associate a Born-Infeld like theory to any extended supergravity theory, as the symplectic matrix $\mathcal{M}$ entering in the potential Tr($\mathcal{M}$) is the same which encodes the scalar couplings to the gauge field-strengths in extended supergravity theories.

$$\Downarrow$$

- It is possible to study the Born-Infeld theory associated to the t^3 model. The corresponding coset representative is in the spin $\frac{3}{2}$ representation of $\frac{SL(2)}{U(1)}$. The algebraic equations of motion to eliminate the auxiliary fields are of higher degree, but they can be solved for the consistent truncation of vanishing axion where they are of degree 3.

- If the scalar fields span a coset $\frac{G}{H}$, the corresponding Born-Infeld like theories constructed with this method feature H duality symmetry. They are self-dual.

- There is a coset parametrization, which is a generalization of the Euler angles for $SU(2)$ [BLC, S.L. Cacciatori, arXiv:0906.0121 [math-ph]], where the expression of $\text{Tr}(\mathcal{M})$ simplifies considerably with respect to the Iwasawa decomposition, because the H-invariance of the potential explicitly isolates the dependence on the Cartan subalgebra. As a result in this frame the expression of the linear Lagrangian only depends on exponentials of the Cartan generators, reminiscent of a Toda system, and simplifying the equations of motion of the auxiliary fields.

- The original Born-Infeld theory admits a supersymmetric extension and it turns out there is a second hidden non-linearly realized supersymmetry. $\Longrightarrow$ The supersymmetric version represents the invariant action of the Goldstone multiplet in a $\mathcal{N} = 2$ supersymmetric theory spontaneously broken to $\mathcal{N} = 1$, with λ determining the supersymmetry breaking scale [J. Bagger, A. Galperin, hep-th/9608177]; S. Ferrara, M. Porrati, A. Sagnotti, arXiv:1411.4954 [hep-th]].
$\Longrightarrow$ Interesting to investigate possible supersymmetric extensions.

- We have been able to reproduce various previously known theories with $U(1)$ and $SU(2)$ duality [S. Ferrara, A. Sagnotti, A. Yeranyan, arXiv: 1503.04731 [hep-th], arXiv:1602.04566[hep-th]], understanding systematically how they are related and why they are self-dual.

- We have found a new theory with $SO(n)$ duality obtained from the embedding $GL(n)/SO(n) \hookrightarrow Sp(2n)/U(n)$, which can be extended to non-Abelian field strengths.