Non-dissipative corrections to energy-momentum tensor for a relativistic fluid

Eduardo Grossi
University of Florence
in collaboration with F. Becattini and M. Buzzegoli

F. Becattini, L. Bucciantini , E. G., L. Tinti Eur.Phys.J. C75 (2015)
F. Becattini, E. G. Phys.Rev. D92 (2015) 045037

GGI2016, New Frontiers in Theoretical Physics18/05/2016

Outline

- Relativistic Hydrodynamics
- Generalized global equilibrium
- Non-Dissipative second order coefficient

Energy momentum tensor at equilibrium

The equation of relativistic hydrodynamics:

$$\partial_\mu T^{\mu\nu} = 0 \quad \partial_\mu j^\mu = 0$$

At global and homogeneous equilibrium the energy momentum tensor and charge current are:

$$T^{\mu\nu}(x) = (\rho + p) u^\mu u^\nu - g^{\mu\nu} p$$

$$j^\mu = n u^\mu$$

The basic assumption of hydrodynamics is the local equilibrium condition

$$\rho(x) = \rho_{eq}(T(x), \mu(x)) \quad p(x) = p_{eq}(T(x), \mu(x))$$
$$n(x) = n_{eq}(T(x), \mu(x))$$

Navier-Stokes equations

The general form of energy momentum tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu - p\Delta^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu + \Pi^{\mu\nu} - \Pi\Delta^{\mu\nu}$$
$$j^\mu = n u^\mu + v^\mu$$

Using the definition of entropy and the equation of motion :

$$T\partial \cdot S = -q \cdot (\frac{\partial T}{T} - Du) + \Pi^{\mu\nu}\partial_\mu u_\nu + \Pi\partial \cdot u - T\nu \cdot \partial \left(\frac{\mu}{T}\right) \geq 0$$

The new terms depend on the value of the transport coefficient

$$q^\mu \equiv \kappa T \Delta^{\mu\nu}(\partial_\nu \log T - Du_\nu)$$
$$\Pi^{\mu\nu} \equiv 2\eta \left[\tfrac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha) - \tfrac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] \partial^\alpha u^\beta$$
$$\Pi \equiv \zeta \partial \cdot u$$
$$\nu^\mu \equiv D\Delta^{\mu\nu}\partial_\nu \left(\tfrac{\mu}{T}\right)$$

Israel Stewart theory

The Navier-Stokes theory is unstable and a-causal.

Adding a second order term the equation became casual and stable

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_{\Pi} D\sigma^{\mu\nu}$$

$$\eta\sigma^{\mu\nu} \longrightarrow \Pi^{\mu\nu}$$

The shear tensor becomes a dynamical variabile that relax at its Navier-Stokes value

$$\tau_{\Pi} D\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \Pi^{\mu\nu} + \cdots$$

[Muller 1967, Israel Stewart 1976]

[Hiscock 1983]

Second order in gradients

Shear Tensor:

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\pi\left[\langle D\sigma^{\mu\nu}\rangle + \frac{\nabla\cdot u}{3}\sigma^{\mu\nu}\right] + \kappa\left[R^{<\mu\nu>} - 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta}\right]$$
$$+\lambda_1\sigma^{<\mu}{}_\lambda\sigma^{\nu>\lambda} + \lambda_2\sigma^{<\mu}{}_\lambda\Omega^{\nu>\lambda} + \lambda_3\Omega^{<\mu}{}_\lambda\Omega^{\nu>\lambda}$$
$$+\kappa^* 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} + \eta\tau_\pi^*\frac{\nabla\cdot u}{3}\sigma^{\mu\nu} + \lambda_4\nabla^{<\mu}\ln s\nabla^{\nu>}\ln s\,.$$

Bulk pressure:

$$\Pi = -\zeta\left(\nabla\cdot u\right) + \zeta\tau_\Pi D\left(\nabla\cdot u\right) + \xi_1\sigma^{\mu\nu}\sigma_{\mu\nu} + \xi_2\left(\nabla\cdot u\right)^2$$
$$+\xi_3\Omega^{\mu\nu}\Omega_{\mu\nu} + \xi_4\nabla^\perp_\mu\ln s\nabla^\mu_\perp\ln s + \xi_5 R + \xi_6 u^\alpha u^\beta R_{\alpha\beta}\,.$$

$$\sigma^{\mu\nu} = 2\nabla^{\langle\mu}u^{\nu\rangle} = (\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\nu_\alpha\Delta^\mu_\beta - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta})\nabla^\alpha u^\beta$$
$$\Omega^{\mu\nu} = \frac{1}{2}\Delta^\mu_\alpha\Delta^\nu_\beta(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha)$$

R. Baier et al. JHEP 0804 (2008)

[P. Romatschke Class.Quant.Grav. 27 (2010)]

Second order in gradients II

- The second order coefficients $\lambda_1, \lambda_2, \tau_\pi, \tau_\Pi, \zeta, \text{etc..}$ are dissipative, depend by the coupling of the theory

- Instead $\lambda_3, \lambda_4, \xi_3, \xi_4, \kappa, \text{etc..}$ depend, at leading order only by the temperature and are non zero also for a free theory

$$\lambda_3 = -\frac{T^2}{12}$$

← For a free massless boson field

[G.Moore et al. JHEP 1211 (2012)]

Generalized equilibrium density matrix

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

1. If the vector field β_μ is a Killing vector field

$$\beta^\mu = \frac{u^\mu}{T}$$

2. If ζ is a constant

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 \qquad \nabla_\nu \zeta = \nabla_\nu \left(\frac{\mu}{\sqrt{\beta^2}} \right) = 0$$

The density matrix is stationary, i.e. independent from the choice of the hyper-surface

$$\Sigma(\tau_1) = \Sigma(\tau_2) = \Sigma(\tau_3) = \cdots$$

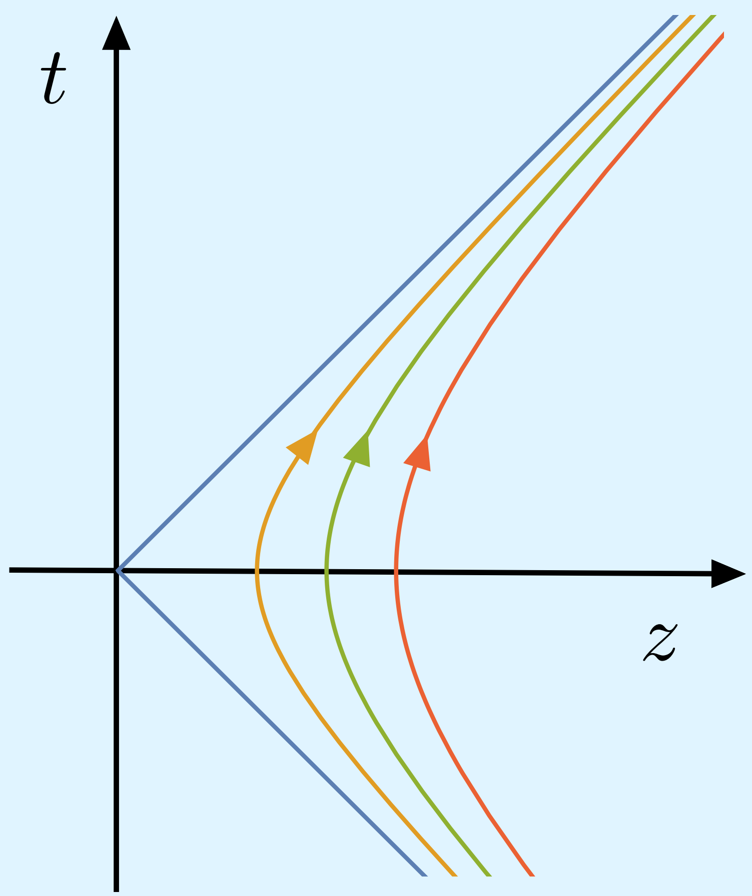
Equilibrium with rotation and acceleration

The general solution of Killing equation in Minkowsky space-time depends on 10 constant parameters:

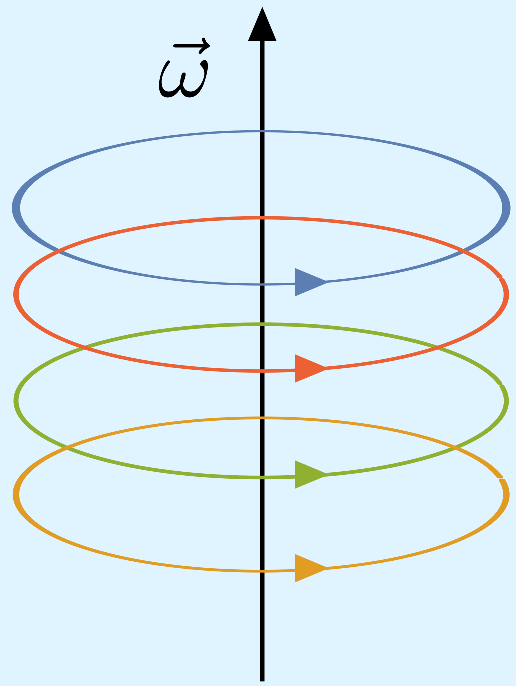
$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu} x_\nu \qquad \varpi^{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at) \qquad \beta^\mu = \frac{1}{T_0}(1, \boldsymbol{\omega} \times \mathbf{x})$$

acceleration



angular velocity



Density operator at generalized equilibrium

Inserting the the solution of the Killing equation the density operator can be written in terms of the element of the Poincaré algebra

$$\rho = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} \right]$$

$$J^{\mu\nu} = \int_\Sigma \mathrm{d}\Sigma_\lambda (x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu})$$

Depends only on conserved charges

$$\widehat{K}^\mu = u_\lambda \widehat{J}^{\lambda\mu}$$

$$\widehat{J}^\mu = \frac{1}{2} \epsilon^{\alpha\beta\gamma\mu} u_\alpha \widehat{J}_{\beta\gamma}$$

- Boost
- Angular momentum

Basis vectors

The thermal vorticity tensor can be decompose along the four velocity

$$\varpi^{\mu\nu} = \alpha^\mu u^\nu - \alpha^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} w_\rho u_\sigma$$

Four velocity

$$u_\mu = \beta_\mu / \sqrt{\beta^2}$$

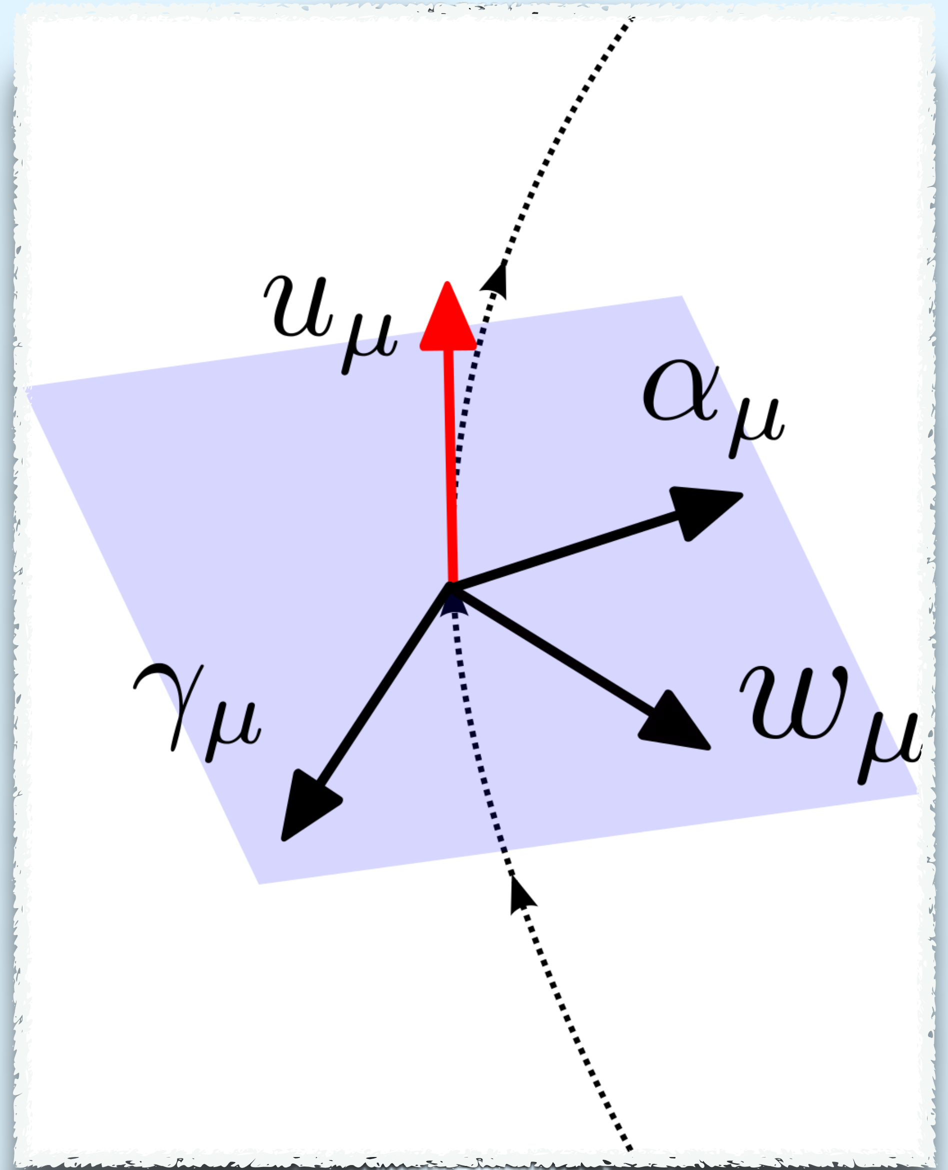
Acceleration over temperature

$$\alpha^\mu = \frac{1}{T} D u^\mu = \frac{a^\mu}{T}$$

Angular velocity over temperature

$$w^\mu = \frac{1}{2T} \epsilon^{\mu\nu\rho\sigma} u_\sigma \partial_\nu u_\rho = \frac{\omega^\mu}{T}$$

$$\gamma_\mu = w^\nu \alpha^\rho u^\sigma \epsilon_{\mu\nu\rho\sigma}$$



Expansion of the energy-momentum tensor

The mean value of the energy momentum tensor is

$$\langle \widehat{T}^{\alpha\beta}(x) \rangle = \langle \widehat{T}^{\alpha\beta}(0) \rangle_{\beta(x)} + \frac{1}{2}\varpi_{\mu\nu} \mathrm{Re}(\langle \widehat{J}^{\mu\nu}\, \widehat{T}^{\alpha\beta}(0) \rangle_{\beta(x)} - \langle \widehat{J}^{\mu\nu} \rangle_{\beta(x)} \langle \widehat{T}^{\alpha\beta}(0) \rangle_{\beta(x)})$$

For PT symmetry the first order is zero

Instead the second order is:

$$\begin{aligned} \langle \widehat{T}^{\alpha\beta}(x) \rangle &= \langle \widehat{T}^{\alpha\beta}(0) \rangle_{\beta(x)} \\ &+ \varpi_{\mu\nu}\varpi_{\rho\sigma} \Bigg[\frac{1}{8} \mathrm{Re}(\langle \widehat{J}^{\mu\nu}\, \widehat{J}^{\rho\sigma}\, \widehat{T}^{\alpha\beta}(0) \rangle_{\beta(x)} - \langle \widehat{J}^{\mu\nu}\, \widehat{J}^{\rho\sigma} \rangle_{\beta(x)} \langle \widehat{T}^{\alpha\beta}(0) \rangle_{\beta(x)}) \\ &\qquad + \frac{1}{8} \beta^{\mu}\beta^{\rho} \frac{\partial^2}{\partial\beta_\nu\, \partial\beta_\sigma} \langle \widehat{T}^{\alpha\beta}(0) \rangle_{\beta(x)} + \frac{1}{12} \beta^{\mu} g^{\nu\rho} \frac{\partial}{\partial\beta_\sigma} \langle \widehat{T}^{\alpha\beta}(0) \rangle_{\beta(x)} \Bigg] \\ &+ \mathcal{O}(\varpi^2) \end{aligned}$$

Mean value of the energy-momentum tensor

Because of the rotation invariance only 7 coefficient are different from zero

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu \gamma^\nu + \gamma^\mu u^\nu)$$

$$\partial_\mu T^{\mu\nu} = 0$$

The conservation of the energy momentum leads:

$$U_\alpha = -\beta \frac{\partial}{\partial \beta}(D_\alpha + A) - (D_\alpha + A)$$

$$U_w = -\beta \frac{\partial}{\partial \beta}(D_w + W) - D_w + 2A - 3W$$

$$2G = 2(D_\alpha + D_w) + A + \beta \frac{\partial}{\partial \beta} W + 3W$$

Free complex scalar field

We consider a free complex scalar field at finite temperature and chemical potential:

- The improved energy-momentum:

$$T_{\alpha\beta} = (1-2\xi)\left(\partial_\alpha\phi^\dagger\partial_\beta\phi + \partial_\beta\phi^\dagger\partial_\alpha\phi\right) - (1-4\xi)g_{\alpha\beta}\partial\phi^\dagger\cdot\partial\phi + m^2 g_{\alpha\beta}\phi^\dagger\phi + 2\xi(g_{\alpha\beta}\phi^\dagger\Box\phi + \Box\phi^\dagger\phi - \phi^\dagger\partial_\alpha\partial_\beta\phi - \partial_\alpha\partial_\beta\phi^\dagger\phi)$$

The coefficients can be extract from the three point euclidean Green function

$$\delta\langle T^{\mu\nu}\rangle = -\frac{\varpi_{\alpha\lambda}\varpi_{\gamma\sigma}}{2\beta^2}\frac{\partial}{\partial p^\lambda\partial q^\sigma}G_E^{0\alpha|0\gamma|\mu\nu}(p,q)$$

Acceleration and rotation

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$$U_w = \frac{(1-4\xi)}{12\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} p^4 \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu)\right)$$

$$U_\alpha = \frac{1}{48\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} (p^2+m^2)(m^2+4p^2(1-6\xi)) \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu)\right)$$

$$W = \frac{(2\xi-1)}{24\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} p^4 \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu)\right)$$

$$A = \frac{1}{48\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} \left(2p^4(1-6\xi) + p^2m^2(3-12\xi)\right) \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu)\right)$$

$$G = \frac{1}{92\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} \left(p^4(1+6\xi) + 3p^2m^2\right) \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu)\right)$$

$$D_\alpha = \frac{1}{144\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} \left(8p^4(6\xi-1) + 3m^2(24\xi-5)\right) \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu)\right)$$

$$D_w = \frac{\xi}{6\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} p^4 \left(n_B^{(2)}(E_p-\mu) + n_B^{(2)}(E_p+\mu)\right)$$

Massless case

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu \gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$\xi = \text{generic}$

$$U_w = \frac{(1-4\xi)}{12\beta^4}$$
$$U_\alpha = \frac{(1-6\xi)}{12\beta^4}$$
$$W = \frac{(2\xi-1)}{12\beta^4}$$
$$A = \frac{(1-6\xi)}{12\beta^4}$$
$$G = \frac{(1+6\xi)}{36\beta^4}$$
$$D_\alpha = \frac{(6\xi-1)}{18\beta^4}$$
$$D_w = \frac{\xi}{6\beta^4}$$

$\xi = 1/6$

$$U_w = \frac{1}{36\beta^4}$$
$$U_\alpha = 0$$
$$W = -\frac{1}{18\beta^4}$$
$$A = 0$$
$$G = \frac{1}{18\beta^4}$$
$$D_\alpha = 0$$
$$D_w = \frac{1}{36\beta^4}$$

$\xi = 0$

$$U_w = \frac{1}{12\beta^4}$$
$$U_\alpha = \frac{1}{12\beta^4}$$
$$W = -\frac{1}{12\beta^4}$$
$$A = \frac{1}{12\beta^4}$$
$$G = \frac{1}{36\beta^4}$$
$$D_\alpha = -\frac{1}{18\beta^4}$$
$$D_w = 0$$

Dirac Field

We consider a free Dirac field at finite temperature and chemical potential and we compute the coefficients using two different energy-momentum tensor

- The symmetric

$$T_{\alpha\beta} = \frac{\mathrm{i}}{4}\left[\bar{\psi}\gamma_\alpha\partial_\beta\psi - \partial_\beta\bar{\psi}\gamma_\alpha\psi\bar{\psi}\gamma_\beta\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma_\beta\psi\right]$$

- The canonical

$$T_{\alpha\beta} = \frac{\mathrm{i}}{2}\left[\bar{\psi}\gamma_\alpha\partial_\beta\psi - \partial_\beta\bar{\psi}\gamma_\alpha\psi\right]$$

- The currents

$$j_\alpha = \bar{\psi}\gamma_\alpha\psi \qquad\qquad j^5_\alpha = \bar{\psi}\gamma^5\gamma_\alpha\psi$$

Coefficients for the symmetric tensor

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$$U_w = -\frac{1}{8\pi^2\beta^2}\int_0^\infty \mathrm{d}p\,(3p^2+m^2)\left(n_F^{(1)}(E_p-\mu)+n_F^{(1)}(E_p+\mu)\right)$$

$$U_\alpha = \frac{1}{24\pi^2\beta^2}\int_0^\infty \frac{\mathrm{d}p}{\sqrt{p^2+m^2}}(p^2+m^2)^2\left(n_F^{(2)}(E_p-\mu)+n_F^{(2)}(E_p+\mu)\right)$$

$$W = 0$$

$$A = 0$$

$$G = -\frac{1}{24\pi^2\beta^2}\int_0^\infty \mathrm{d}p\,(4p^2+m^2)\left(n_F^{(1)}(E_p-\mu)+n_F^{(1)}(E_p+\mu)\right)$$

$$D_\alpha = -\frac{1}{24\pi^2\beta^2}\int_0^\infty \mathrm{d}p\,(p^2+m^2)\left(n_F^{(1)}(E_p-\mu)+n_F^{(1)}(E_p+\mu)\right)$$

$$D_w = -\frac{1}{8\pi^2\beta^2}\int_0^\infty \mathrm{d}p\,p^2\left(n_F^{(1)}(E_p-\mu)+n_F^{(1)}(E_p+\mu)\right)$$

Coefficients massless case

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

Symmetric

$$U_w = \frac{1}{8\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$U_\alpha = \frac{1}{24\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$W = 0$$
$$A = 0$$
$$G = \frac{1}{18\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$D_\alpha = \frac{1}{72\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$D_w = \frac{1}{24\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$

Canonical

$$U_w = \frac{1}{8\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$U_\alpha = \frac{1}{24\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$W = 0$$
$$A = 0$$
$$G_1 = \frac{2}{9\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2}) = -2G_2$$
$$D_\alpha = \frac{1}{72\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$
$$D_w = \frac{1}{24\beta^4}(1 + \frac{3\beta^2\mu^2}{\pi^2})$$

Coefficients canonical (T=0)

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

For T=0 the distribution function becomes a step function

$$U_\alpha = \frac{3E_{\rm F}^5 - 4m^2 E_{\rm F}^3}{12\pi^2\beta^2 p_{\rm F}^3}, \; D_\alpha = -\frac{E_{\rm F}^2}{12\pi^2\beta^2}, \; A = 0,$$

$$U_w = -\frac{E_{\rm F}^2 + 2p_{\rm F}^2}{4\pi^2\beta^2}, \; D_w = -\frac{p_{\rm F}^2}{4\pi^2\beta^2}, \; W = 0, \; G = -\frac{E_{\rm F}^2 + 3p_{\rm F}^2}{12\pi^2\beta^2}$$

Conclusions and Outlook

- ✦The energy momentum tensor gets extra correction at equilibrium due to vorticity and acceleration.
- ✦ The second order coefficients involving vorticity and acceleration are generally different from zero for a free field.
- ✦They also depend on the choice of the energy momentum tensor operator.
- ✦They seen to be relevant in heavy ion physics, but could be relevant also in other physical situation