Observations on Conformal Field Theories in $d > 2$

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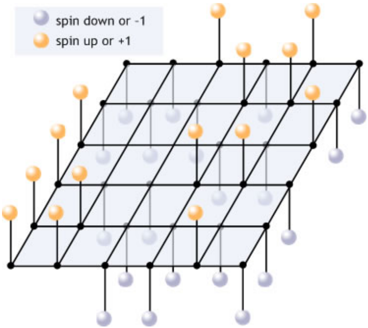
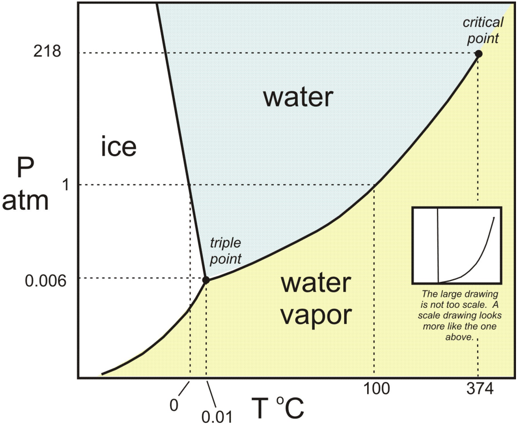


We have L^d spins with some nearest-neighbor interaction energy $J > 0$ if they are misaligned. So the spins want to be aligned at low temperatures. The magnetization M is the order parameter.

Alternatively, think of the 2nd order water-vapor transition diagram



At T_c there is a phase transition. Long range correlations develop. Lattice structure becomes irrelevant. Quantities depend non-analytically on the temperature and external field. Ginzburg-Landau theory:

$$H = \int d^d x \left(r(\nabla M)^2 + cM^2 + \lambda M^4 + ... \right)$$

The partition function

$$Z = \int [dM] e^{-H}$$

encodes all the thermodynamics properties at the phase transition.

Some experimentally interesting quantities are the usual $\alpha, \beta, \gamma, \delta, \eta, \nu$ exponents:

$$C \sim (T - T_c)^{-\alpha} \ , \qquad M \sim (T_c - T)^{\beta} \ , \qquad \chi \sim (T - T_c)^{-\gamma} \ ,$$

$$M \sim h^{1/\delta} \ , \qquad \langle M(\vec{n}) M(0) \rangle \sim \frac{1}{|\vec{n}|^{d-2+\eta}} \ , \qquad \xi \sim (T - T_c)^{-\nu} \ .$$

Amazingly, one discovers four relations between these 6 quantities:

$$\alpha + 2\beta + \gamma = 2 \ ,$$

$$\gamma = \beta(\delta - 1) \ ,$$

$$\gamma = \nu(2 - \eta) \ ,$$

$$\nu d = 2 - \alpha \ .$$

The explanation of this miracle is that at T_c the symmetry of the system is enhanced.

$$SO(d) \times \mathbb{R}^d \to SO(d) \times \mathbb{R}^d \times \Delta \ ,$$

with

$$\Delta : \ x \to \lambda x$$

and $\lambda \in \mathbb{R}^+$.

The dilation charge Δ can be diagonalized. If we have a local operator $\mathcal{O}$ in the theory, $\Delta(\mathcal{O})$ would uniquely determine its two-point correlator

$$\langle \mathcal{O}(n)\mathcal{O}(0)\rangle \sim \frac{1}{n^{2\Delta(\mathcal{O})}} \ .$$

Local operators could also have spin s, but we suppress it in the meantime.

In the Ising model, two of the infinitely many operators in the theory are $M(x)$ and $\epsilon(x)$, which are the magnetization and energy operators. They are the only relevant operators. This is why the phase diagram is two-dimensional.

The four miraculous relations among $\alpha, \beta, \gamma, \delta, \eta, \nu$ can be understood from scale invariance:

$$\alpha = \frac{d - 2\Delta_\epsilon}{d - \Delta_\epsilon} , \qquad \beta = \frac{\Delta_M}{d - \Delta_\epsilon} ,$$

$$\gamma = \frac{d - 2\Delta_M}{d - \Delta_\epsilon} , \qquad \delta = \frac{d - \Delta_M}{\Delta_M} ,$$

$$\eta = 2 - d + 2\Delta_M , \qquad \nu = \frac{1}{d - \Delta_\epsilon} .$$

This was essentially understood more than 70 years ago.

There has been a lot of recent progress based on the observation that the symmetry is actually bigger!!

$$SO(d) \times \mathbb{R}^d \to SO(d) \times \mathbb{R}^d \times \Delta \to SO(d+1,1)$$

Theories with this big symmetry are called **Conformal Field Theories**.

$SO(d+1,1)$ is the conformal group that acts on $\mathbb{R}^d$. This is the set of all transformations that preserve orthogonal lines.
It consists of

$$d(d-1)/2 \ \text{rotations}$$

$$d \ \text{translations}$$

$$1 \ \text{dilation}\ (\Delta)$$

$$d \ \text{special conformal transformations}$$

The last d symmetry generators are beyond Ginzburg-Landau theory.

Essentially all the examples that we know of second order phase transitions which have

$$SO(d) \times \mathbb{R}^d \times \Delta$$

have the full $SO(d+1,1)$. It has been even verified "experimentally" in some examples.

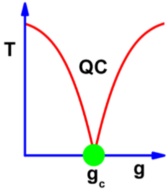
$d = 2$: An argument from the 80's by Joe Polchinski. The argument is a spin-off of Zamolodchikov's renormalization-group irreversibility theorem.

$d = 4$: With the advent of the irreversibility theorem in 2011-2012, it was possible to give an argument in $d = 4$ for this mysterious symmetry enhancement.

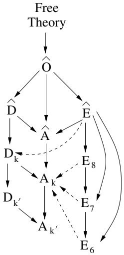
$d = 3$: Open question. Seems to hold in all known examples. Perhaps can be approached using tools of entanglement entropy?

Applications of Conformal Field Theories:

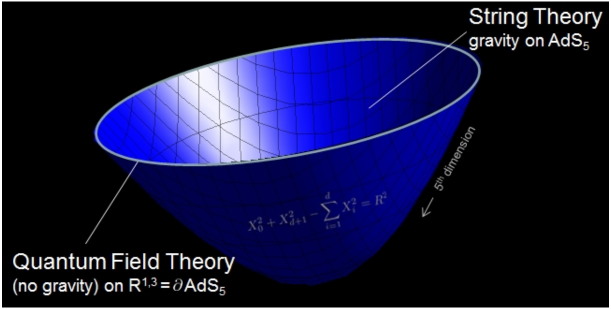
- Quantum phase transitions



- Fixed points of the Renormalization Group Flow.



Quantum gravity in AdS_{d+1} is described by a boundary Conformal Field Theory.



What are the applications and implication of this surprising symmetry enhancement?
The study has been somewhat haphazard. But we already know quite a bit. My goal here is to describe briefly some of the things we have learned.

The d bonus special conformal transformations act on space as

$$x^i \to \frac{x^i - b^i x^2}{1 - 2b\cdot x + b^2 x^2} ,$$

where b^i is any vector in $\mathbb{R}^d$.
It turns out that these are enough to fix three-point correlation functions as follows

$$\langle \mathcal{O}_1(n')\mathcal{O}_2(n)\mathcal{O}_3(0)\rangle = \frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{n^{\Delta_2+\Delta_3-\Delta_1}(n-n')^{\Delta_1+\Delta_2-\Delta_3}n'^{\Delta_1+\Delta_3-\Delta_2}}$$

Remember

$$\langle \mathcal{O}_i(n)\mathcal{O}_j(0)\rangle \sim \frac{\delta_{ij}}{n^{2\Delta_i(\mathcal{O})}} .$$

SO(d+1,1) symmetry therefore fixes the two- and three-point functions in terms of a collection of numbers

$$\Delta_i \qquad , \qquad C_{ijk}$$

which are called the "CFT data."

It turns out that ALL the correlation functions are fixed in terms of the CFT data.

How can we say anything useful about this collection of numbers ?

$$\{\Delta_i\} \qquad , \qquad \{C_{ijk}\}$$

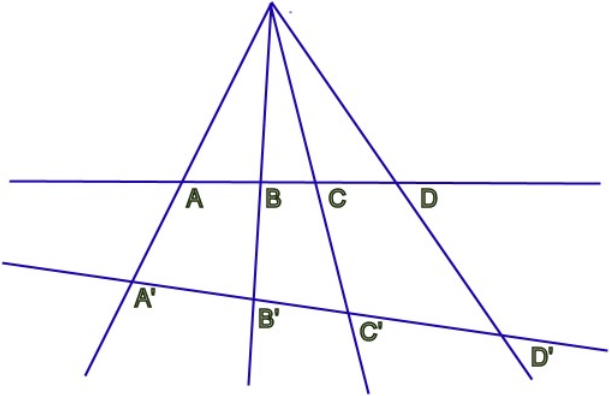
This collection is infinite because there are infinitely many operators in every Landau-Ginzburg theory. It is easiest to measure the relevant, low dimension, operators (such as M, ϵ), but also the others exist and are in principle measurable.

Consider a four-point function with operators at n_1, n_2, n_3, n_4. We can form conformally invariant ratios:

$$u = \frac{n_{12}^2 n_{34}^2}{n_{14}^2 n_{23}^2} \ , \qquad v = \frac{n_{13}^2 n_{24}^2}{n_{13}^2 n_{24}^2}$$

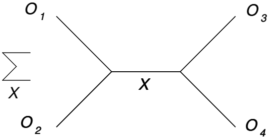
and the general four-point function is

$$\langle \mathcal{O}_1(n_1)\mathcal{O}_2(n_2)\mathcal{O}_3(n_3)\mathcal{O}_4(n_4)\rangle \sim F(u, v) \ .$$



$$\frac{(CA\,/\,CB)}{(DA\,/\,DB)} = \frac{(C'A'\,/\,C'B')}{(D'A'\,/\,D'B')}$$

We represent the four point function as a sum over infinitely many three point functions:



$$\sim \sum_X C_{12X} C_{X34} G(\Delta_{1,2,3,4,X}, u, v)$$

Therefore,

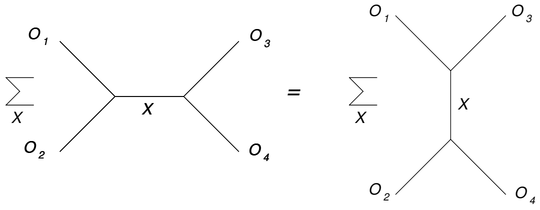
$$F(u,v) \sim \sum_X C_{12X} C_{X34} G(\Delta_{1,2,3,4,X}, u, v)$$

The functions G are partial waves (like Legendre polynomials). Thus, once we know the CFT data, the four-point function can be in principle computed.

The dynamics is in saying that we can make the decomposition in two different ways (**duality**). And we get

$$\sum_X C_{12X} C_{X34} G(\Delta_{1,2,3,4,X}, u, v) = \sum_X C_{13X} C_{X24} G(\Delta_{1,3,2,4,X}, v, u)$$

This equation is supposed to determine/constrain the allowed Δ_i and C_{ijk} that can furnish legal conformal theories.

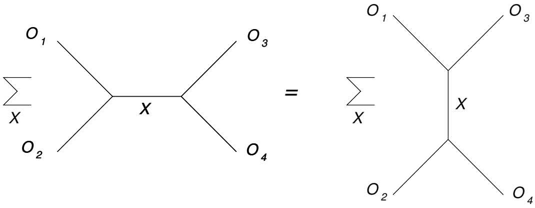


This is extremely surprising:

Maybe we could classify all the possible conformal theories by just solving self-consistency algebraic equations.

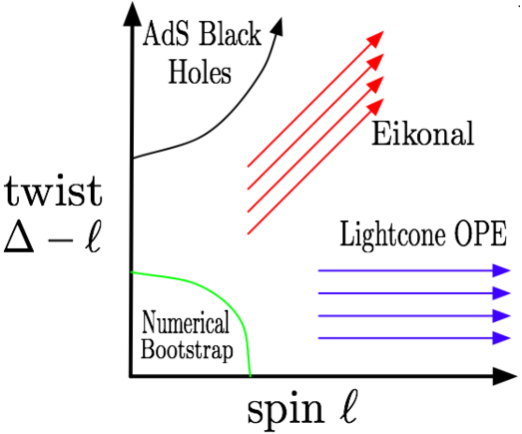
For the mathematically oriented: these equations are similar to the equations of associative rings. It is also very similar in sprit to the classification of Lie Algebras.

Therefore, if we had a better idea about what the equations



imply, that would be useful in many branches of physics. More ambitiously, we could hope to classify all the solutions!

Recently, there has been dramatic progress on this problem both from the analytic and numeric points of view.



The Lightcone, Eikonal, and AdS limits are universal and amenable to an analytic treatment. The region of small Δ is very model-dependent and can be studied by imposing various assumptions on the Conformal Theory one is looking for.

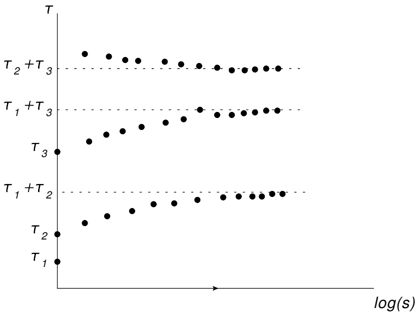
I will quote, without detailed proofs, four very general analytic results.

Result I: Additivity of the Spectrum

If we have (Δ_1, s_1) and (Δ_2, s_2) in the spectrum, then there are operators (Δ_i, s_i) which have $\Delta_i - s_i$ arbitrarily close to $\Delta_1 - s_1 + \Delta_2 - s_2$.

Typically, to find operators with $\Delta_i - s_i \sim \Delta_1 - s_1 + \Delta_2 - s_2$ we would need to take large Δ_i, s_i.

This leads to a rather peculiar spectrum, schematically as follows



(In the figure, $\tau \equiv \Delta - s$.)

Note that the spectrum of many famous 2d systems such as the Ising and Potts models is not as complicated. This is because in the proof of the additivity theorem it is **assumed** that $d > 2$.

Outline of the Proof of the Additivity Theorem

The key is to study the bootstrap equations in the light-cone regime. This is suggested by the fact that additivity is naturally phrased in terms of twists. We begin with the OPE in this regime

$$\lim_{x^+\to 0} \mathcal{O}_1(x^+, x^-)\mathcal{O}_2(0) \sim \sum_i (x^+)^{\frac{1}{2}\tau_i} \mathcal{F}_i(x^-)\mathcal{O}_i(0)$$

Hence, the light-cone expansion is not in dimensions but in twists $\tau_i = \Delta_i - s_i$. $\mathcal{F}_i$ are the $SL(2,\mathbb{R})$ blocks.

Outline of the Proof of the Additivity Theorem

The dimensionality of space-time drops out and we always have blocks of $SL(2,\mathbb{R})\times\mathbb{R}$. This allows to prove things for any d even though there are no nice analytic expressions for conformal blocks in, e.g., $d=3$. It is enough for our purposes to have the collinear conformal blocks.

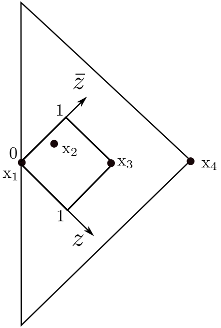
Outline of the Proof of the Additivity Theorem

The collinear conformal blocks have a slightly nontrivial analytic structure, with certain branch cuts. Those will be absolutely crucial.

Amusingly, also a lot of the work on the large-N AdS/CFT bootstrap relies on these non-analytic pieces.

Outline of the Proof of the Additivity Theorem

We consider the four point function $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_1 \rangle$ and put the four operators on a plane



when $z \to 0$ with generic z we get the light-cone OPE expansion in the (12)(34) channel. But when we start approaching $\bar{z} \to 1$ the separation $x_2 - x_3$ becomes light-like as well.

Outline of the Proof of the Additivity Theorem

So we essentially rewrite the bootstrap equations in light-cone form

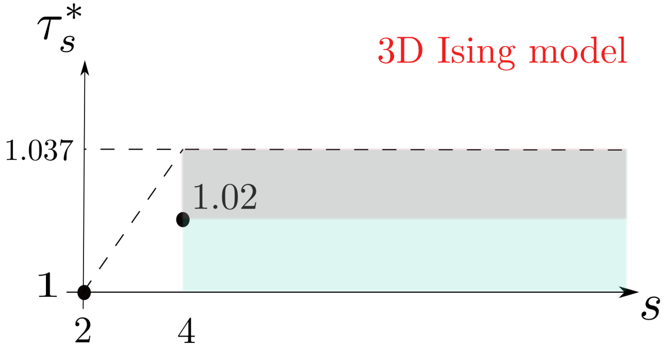
$$(12)(34)_{light-cone} = (23)(14)_{light-cone}$$

Now we take the limit $\bar{z} \to 1$ faster than $z \to 0$. Then the four-point function is dominated by the exchange of the unit operator in the (23)(14) channel. Writing this down in terms of the (12)(34) channel one finds that the spectral density of twists in any vicinity of $\tau_1 + \tau_2$ is nonzero.

Result II: Convexity

$\Delta_i - s_i$ approaches the limiting value $\Delta_1 - s_1 + \Delta_2 - s_2$ in a convex manner.

These already lead to nontrivial results for the 3d Ising model. Since the spin field has $\Delta(M) = 0.518...$, there needs to be a family of operators with $\Delta - s$ approaching 1.037... from below, in a convex fashion:



This is beautifully verified by the measurement of the spin-4 operator dimension. Recently verified for spin 6. The rest awaits confirmation.

The approach to the asymptote is happening at large spin, i.e. small angular resolution. It turns out that in an appropriate sense

Result III: Weak Coupling at Small Angles

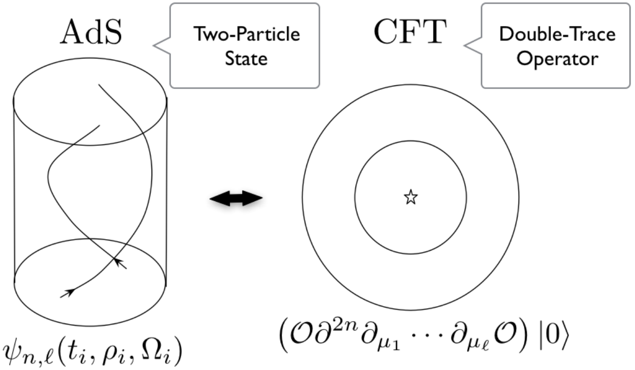
The approach to the asymptote is under analytic, perturbative control, even in strongly coupled CFTs!

For example, the leading deviation from the asymptote at large spin is

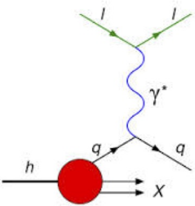
$$2\tau - \frac{d^2\Gamma(d+2)\Delta^2\Gamma^2(\Delta)}{2c_T(d-1)^2\Gamma^2(d/2+1)\Gamma^2(\Delta-d/2+1)}\frac{1}{s^2} + \cdots$$

This is like a one-loop calculation around the weakly coupled "$s = \infty$" point.

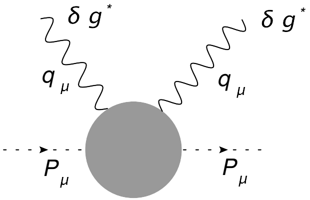
This asymptotic calculation for large spin has been generalized in many directions. One can view it as a proof of cluster decomposition for quantum gravity in AdS.



So far we have mostly discussed the spectrum Δ_i. What about the C_{ijk} (i.e. the three-point functions)? It turns out that considering deep-inelastic scattering gedanken experiments we find interesting constraints



Here we consider gravitational deep-inelastic scattering gedanken experiments. The total cross section is given by the imaginary part of



We use the Kramers-Kronig relation to relate the OPE regime with integrals of the imaginary part. This gives positivity constraints

$$\langle T_{\mu\nu} O_{\mu_1 \ldots \mu_s} O_{\nu_1 \ldots \nu_s} \rangle > 0$$

for many of the tensor structures that appear in these three-point functions.

A particular special case is the state that we get by acting with the energy-momentum tensor on the vacuum. Then we get

$$\langle T_{\mu\nu} T_{\lambda_1\lambda_2} T_{\delta_1\delta_2}\rangle > 0$$

which **exactly** coincide with the Hofman-Maldacena bounds

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3}$$

We get many other bounds for different choices of the operators O.

Conclusions

- Conformal Field Theories are abundant in physics.
- The are determined by an intricate self-consistency condition. We still don't know much about the general consequences of this self-consistency condition.
- There are some results though on monotonicity, convexity, and additivity of the spectrum of dimensions. Also results about three-point functions (mostly bounds).
- Many open questions (upper critical dimension, sharpening bulk locality, precision computations in AdS_3, RG flows...)

Thank You!

Sorry for not giving references – if you are interested come and ask...