Recent advances in computation of twist fields correlators

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Foreword

- This talk builds over a vast literature but is mainly based on these papers

  - I.P., "Strings in an arbitrary constant magnetic field with arbitrary constant metric and stringy form factors," JHEP **1106** (2011) 138 [arXiv:1101.5898 [hep-th]].
  - I.P., "Green functions and twist correlators for N branes at angles," Nucl. Phys. B **866** (2013) 87 [arXiv:1206.1431 [hep-th]].
  - I.P, "Correlators of arbitrary untwisted operators and excited twist operators for N branes at angles," Nucl. Phys. B **886** (2014) 243 [arXiv:1401.6797 [hep-th]].
  - I.P., "Canonical quantization of a string describing N branes at angle," Nucl. Phys. B **889** (2014) 120 [arXiv:1407.4627 [hep-th]].
  - I.P. "Towards a fully stringy computation of Yukawa couplings on non factorized tori and non abelian twist correlators" to appear

(Some) Credits for abelian twist fields

L. J. Dixon, D. Friedan, E. J. Martinec, S. H. Shenker, 1987
T. T. Burwick, R. K. Kaiser and H. F. Muller, 1991
S. Stieberger, D. Jungnickel, J. Lauer and M. Spalinski,, 1992
J. Erler, D. Jungnickel, M. Spalinski and S. Stieberger, 1993
P. Anastasopoulos, M. D. Goodsell and R. Richter, 2013

and

J. J. Atick, L. J. Dixon, P. A. Griffin, D. Nemeschansky, 1988
M. Bershadsky, A. Radul, 1987
E. Corrigan, D. B. Fairlie, 1975
J. H. Schwarz, C. C. Wu, 1974
P. Hermansson, B. E. W. Nilsson, A. K. Tollsten, A. Watterstam, 1990
N. Di Bartolomeo, P. Di Vecchia, R. Guatieri, 1990
M. Bianchi, G. Pradisi and A. Sagnotti, 1991
M. Bianchi and E. Trevigne, 2005
P. Anastasopoulos, M. Bianchi and R. Richter, 2011
E. Kiritsis and C. Kounnas, 1994
G. D'Appollonio and E. Kiritsis, 2003
I. Antoniadis and K. Benakli, 1994
E. Gava, K. S. Narain and M. H. Sarmadi, 1997
J. R. David, 2000
S. A. Abel and A. W. Owen, 2003
S. A. Abel and M. D. Goodsell, 2006
M. Bertolini, M. Billo, A. Lerda, J. F. Morales and R. Russo, 2006
A. Lawrence and A. Sever, 2007
D. Duo, R. Russo, S. Sciuto, 2007
J. P. Conlon and L. T. Witkowski, 2011

(Some) Credits for the non abelian twist fields

and for non abelian twist fields

S. Thomas, 1987
K. Inoue, M. Sakamoto and H. Takano, 1987
B. Gato, 1988
K. Inoue and S. Nima, 1990
B. Gato, 1990

S. Förste and C. Liyanage, 2015

Plan of the talk

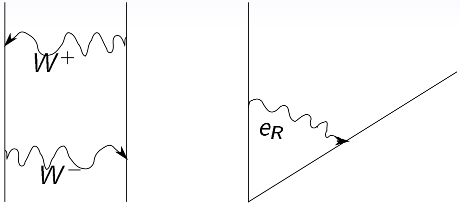
1. Start
2. Introduction and motivation
3. Branes at angles
   - The setup
   - Result 1: different sectors
4. Excited twist fields
5. Result 2: main result
   - Example of the main result for $\mathbb{R}^2$
   - Example with excited twist fields
   - Reggeon vertex
6. Conclusions
7. Future directions

Introduction and motivation

The big picture: why twist fields? (1)

- We would like to do "phenomenology" from string in a humble way start from the observed gauge group and matter:
  - consider D-brane worlds $\longrightarrow$ but $G_{GUT} \leq SU(5)$!
  - add instantons in order to get some needed/wanted features (Majorana masses, Yukawa couplings)

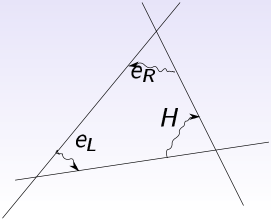
- Chiral matter appears in the twisted sector in the branes at angle setup.



Left: from displaced parallel branes we get $W^{\pm}$-like particles and their superpartners.
Right: from branes at angles we get chiral fermions, e.g. e_R and their superpartners.

The big picture: why twist fields? (2)

- Yukawa couplings are fundamental for generating masses



- They are computed as

$$Y \sim \langle V_{e_R} V_{e_L} V_H \rangle$$

- Since the matter vertex is like

$$V_e \sim e^{ik_\mu X^\mu} \times \sigma \qquad \Leftarrow \text{ associated with } X \text{ worldsheet fields}$$
$$\times S_\alpha \times S_{internal} \qquad \Leftarrow \text{ associated with } \psi \text{ worldsheet fields}$$

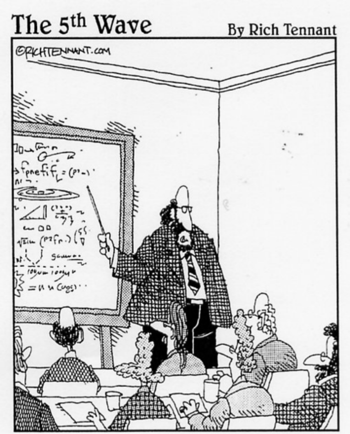
where S_α and $S_{internal}$ are spin fields and σ is a twist field.

The big picture: why twist fields? (3)

- We get
$$Y \sim \langle V_{e_R} V_{e_L} V_H \rangle \propto \langle \sigma_{e_r} \sigma_{e_L} \sigma_H \rangle \langle S_{e_r} S_{e_L} S_H \rangle$$
Therefore Yukawa computation requires computing correlators with (excited) twist fields and spin fields.
- Abelian spin field correlators can be easily computed using bosonization. Abelian vs non abelian is "only" due to global issues, e.g. light cone NSR vs light cone GS.
- Twist field correlators are harder, especially the non abelian one.

- Computations with twists appear also f.x.
  - stringy instantonic calculus
  - Melvin background and its T-dual versions
  - type II and heterotic compactifications on orbifolds (connected to algebraic solutions of Fuchsian eqs?)
- Therefore it is worth having a complete control over the correlators involving all kinds of excited twist fields.
- Also interesting arena for canonical quantization.

My true personal motivation



"After the discovery of 'antimatter' and 'dark matter', we have just confirmed the existence of 'doesn't matter', which does not have any influence on the Universe whatsoever."

Figure : I was bothered by not been able to deal with twist fields as one does with spin fields

The setup

The setup

The Euclidean action for the internal space string configuration is given by

$$S_E = \frac{1}{4\pi\alpha'} \int d\tau_E \int_0^\pi d\sigma \, (\partial_\alpha X^I)^2 = \frac{1}{4\pi\alpha'} \int_H d^2u \, (\partial_u Z^i \bar{\partial}_{\bar{u}} \bar{Z}^i + \bar{\partial}_{\bar{u}} Z^i \partial_u \bar{Z}^i)$$

$u = x + iy \in H$ (upper half plane)
$I = 1, \ldots 2N$, $i = 1, \ldots N$ (internal space has even dimension like $\mathbb{R}^{2N}$)

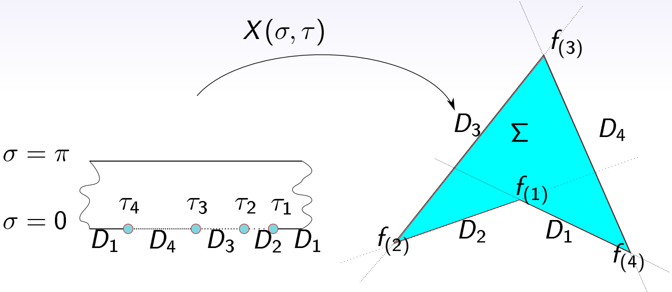
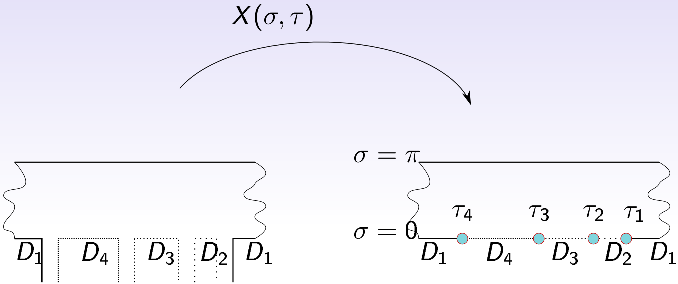


Figure : Map from the worldsheet to the target polygon $\Sigma \subset \mathbb{R}^2$ with a plain in and out string. The map $X(\sigma, \tau)$ folds the $\sigma = 0$ starting from $\tau = -\infty$ in a counterclockwise direction.

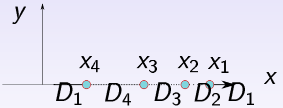
From interactions to boundary conditions

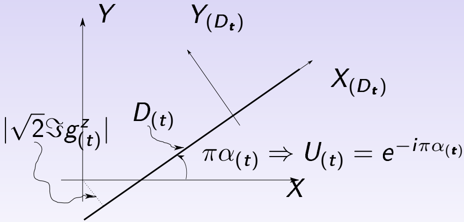
Pictorially



- We are mapping interactions to boundary conditions.
- This can be done both in path integral approach and in canonical quantization. ▸ Go to details on can.quan.
- Surely it works for ground states which are "pointlike".

Abelian vs non abelian twists (1)





- Locally all twist fields are abelian: the issue is global.
- Brane $D_{(t)} \equiv \mathbb{R}^N \subset \mathbb{C}^N \equiv \mathbb{R}^{2N}$ is on the segment $x_t < x < x_{t-1}$. In good local coordinates the brane D_t is described by

$$\Im(Z^i_{(D_t)}) = \Im(\frac{X^i + i\, X^{i+N}}{\sqrt{2}}) = 0, \qquad i = 1 \dots N$$

  which implies the string b.c.

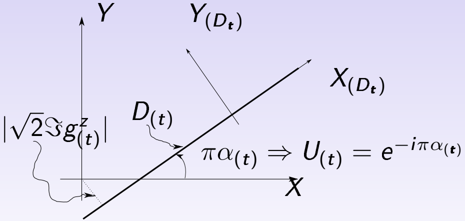
$$\Im(Z^i_{(D_t)}(u, \bar{u}))|_{y=0} = 0, \qquad \Re(\partial_y Z^i_{(D_t)}(u, \bar{u}))|_{y=0} = 0, \qquad x_t < x < x_{t-1}$$

- Good local coordinates $Z^i_{(D_t)}$ are connected with global coordinates Z^i as

$$Z^i_{(D_t)} = U^i_{(t)\ j} Z^j - g^i_{(t)} \qquad U_{(t)} \in U(N)$$

  This is not general since usually the relation is in real coordinates and we

Abelian vs non abelian twists (2)



- Finally the string bc for each boundary segment are

$$\Im(U_{(t)}Z(u,\bar{u}))|_{y=0} = \Im(g_{(t)}), \quad \Re(U_{(t)}\partial_y Z(u,\bar{u}))|_{y=0} = 0, \quad x_t < x < x_{t-1}$$

- They imply the monodromies

$$\partial_u Z(x_t + e^{i2\pi}(\epsilon + i0^+)) = M_{(t)}\partial_u Z(x_t + (\epsilon + i0^+)),$$
$$M_{(t)} = (U^T_{(t+1)}U_{(t+1)})^{-1}(U^T_{(t)}U_{(t)})$$

(Actually we should define the doubled $\partial_u Z$ defined on the whole $\mathbb{C}$)
(The monodromies starting in the lower half plane are different)

- The twist fields are abelian iff all the monodromies $M_{(t)}$ commute.
It is a global issue.
In $\mathbb{R}^2$ all abelian since $U_{(t)} = e^{-i\pi\alpha_{(t)}} \in U(1)$.

Result 1: Different sectors already for $\mathbb{R}^2$

- At given number of branes N_B there are different inequivalent sectors (i.e. not the analytic continuation of each other) already for the $\mathbb{R}^2$ case.
- For $\mathbb{R}^2$ there are different sectors only for $N_B \geq 4$.
- Labeled by M no. of convex angles (interior angles $< \pi$) minus 2.

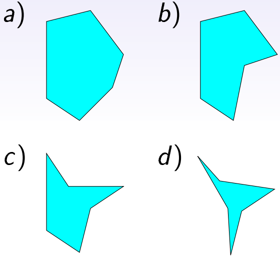


Figure : The four different cases with $N_B = 6$. $a)$ $M = 4$. $b)$ $M = 3$. $c)$ $M = 2$. $d)$ $M = 1$.

The intuitive reason: we need go through the straight line, i.e. no twist, if we want to go from a reflex angle ($> \pi$) to a more usual convex one ($< \pi$).

Result 1: Different sectors (2)

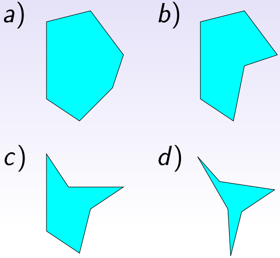


Figure : d) $M = 1$.

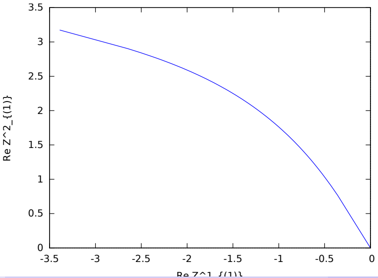
- One sector is more equal than the others: $M = 1$!
  It has holomorphic classical solution $(Z_{cl}(u), \bar{Z}_{cl}(\bar{u}))$ hence its classical action is the area but this does not happen for the other sectors.

Result 1: Different sectors and non holomorphic solutions

- Also the non abelian $SU(2)$ sector for $N_B = 3$ in $\mathbb{R}^4$ is NOT holomorphic and the classical action is not the area.
- The Yukawa couplings are smaller than in the abelian case since the area is bigger than the area of the triangle determined by the three space time interaction points.
- The picture shows the line traced by the endpoint of the classical string in good local coordinates for brane $D2_{(1)} \subset \mathbb{R}^4$. The naive path should is a segment. Data with numerical error $< 10^{-15}$. Embedding $\Im Z^i_{(D_t)} = 0$.

$$U_{(1)} = e^{i 2\pi\, 0.4\, \sigma_3}, \quad U_{(2)} = e^{i 2\pi\, (0.3\, \sigma_1 + 0.4\, \sigma_2 + 0.5\, \sigma_3)}, \quad U_{(3)} = e^{i 2\pi\, (0.5\, \sigma_1 + 0.6\, \sigma_2 + 0.7\, \sigma_3)}$$

$$M_{(1)} = \begin{pmatrix} 0.276341\, i - 0.00249408 & 0.914018 - 0.296982\, i \\ -0.296982\, i - 0.914018 & -0.276341\, i - 0.00249408 \end{pmatrix}$$



Excited twist fields

Excited twist fields: why?

They are needed for

- vertices in different pictures
- vertices for massive states (also KK states)

Zooming and usual twisted string

The local picture in $\mathbb{R}^2$

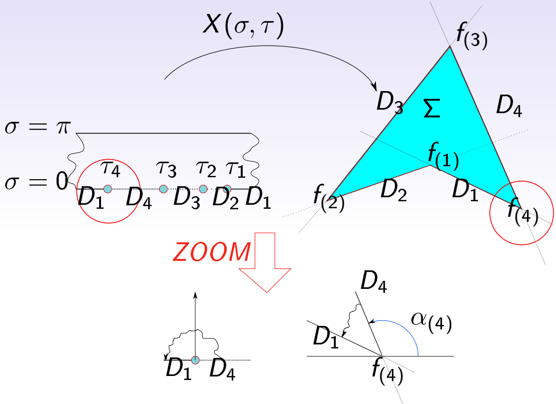


Figure : Zoom locally and get the usual twisted string.

where $f_{(t)}$ is the space time interaction point in space.
$U_{(t)} = e^{-i\pi\alpha_{(t)}}$ is the rotation between good local coords and global ones.

Usual twisted string

After zooming the expansion for the twisted string between brane D_t and D_{t+1} can be splitted into:

A classical part

$$Z_{cl} = f_{(t)},$$

A quantum part

$$Z_q(u, \bar{u}; \{x_t, \alpha_{(t)}\}) = + i\frac{1}{2}\sqrt{2\alpha'} e^{i\pi\alpha_{(1)}} \sum_{n=0}^{\infty} \left[\frac{\bar{\alpha}_{n+\bar{\epsilon}}}{n+\bar{\epsilon}} u^{-(n+\bar{\epsilon})} - \frac{\alpha^{\dagger}_{n+\epsilon}}{n+\epsilon} u^{n+\epsilon} \right]$$
$$+ i\frac{1}{2}\sqrt{2\alpha'} e^{i\pi\alpha_{1}} \sum_{n=0}^{\infty} \left[-\frac{\bar{\alpha}^{\dagger}_{n+\bar{\epsilon}}}{n+\bar{\epsilon}} \bar{u}^{n+\bar{\epsilon}} + \frac{\alpha_{n+\epsilon}}{n+\epsilon} \bar{u}^{-(n+\epsilon)} \right]$$

$(\epsilon = \alpha_{(t+1)} - \alpha_{(t)} + \theta(\alpha_{(t)} - \alpha_{(t+1)})$ the is the angle between the two branes; $\bar{\epsilon} = 1 - \epsilon)$

The splitting into a classical and quantum part is also needed for the existence of a conserved product between modes when performing the canonical quantization.

Go to details

Abstract excited twists and states in twisted Hilbert space (1)

In twisted Hilbert space there are the non normalized! states

$$\prod_{n=0}^{\infty} \left(n! \alpha^{\dagger}_{n+\epsilon} \right)^{N_n} \left(n! \bar{\alpha}^{\dagger}_{n+\bar{\epsilon}} \right)^{\bar{N}_n} |T\rangle$$

The vacuum $|T\rangle$ corresponds to the abstract plain twist $\sigma_{\epsilon}(x)$

$$|T\rangle = \lim_{x \to 0} \sigma_{\epsilon,f}(x) |0\rangle_{SL(2)}$$

All other states correspond the (generically **non primary**) abstract operators

$$\left[\prod_{n=0}^{\infty} \left(\partial_u^{n+1} Z \right)^{N_n} \left(\partial_u^{n+1} \bar{Z} \right)^{\bar{N}_n} \sigma_{\epsilon,f} \right] (x)$$

which are excited twists.
Notice that f.x. all $\bar{N}_n = 0$ are primary.

Abstract excited twists and states in twisted Hilbert space(2)

The notation

$$\left[\prod_{n=0}^{\infty} \left(\partial_u^{n+1} Z\right)^{N_n} \left(\partial_u^{n+1} \bar{Z}\right)^{\bar{N}_n} \sigma_{\epsilon,f}\right](x)$$

is non standard but better than the usual one since it does not use a symbol for each field

$$\left[\partial_u Z \sigma_{\epsilon,f}\right](x) \leftrightarrow \tau_\epsilon(x), \qquad \left[\partial_u \bar{Z} \sigma_{\epsilon,f}\right](x) \leftrightarrow \bar{\tau}_\epsilon(x),$$
$$\left[(\partial_u Z)^2 \sigma_{\epsilon,f}\right](x) \leftrightarrow \omega_\epsilon(x), \qquad \left[(\partial_u \bar{Z})^2 \sigma_{\epsilon,f}\right](x) \leftrightarrow \bar{\omega}_\epsilon(x),$$

However this notation can be partially misleading since it is *not* true that

$$\partial_u^2 Z(u,\bar{u})\ \sigma_{\epsilon,f}(x) \sim \frac{1}{(u-x)^{\#}} (\partial_u^2 Z \sigma_{\epsilon,f})(x) + \dots$$

but

$$\partial_u^2 Z(u,\bar{u}) \sigma_{\epsilon,f}(x) = (u-x)^{\epsilon-2}\ (\epsilon-1)\ (\partial_u Z \sigma_{\epsilon,f})(x) + (u-x)^{\epsilon-1}\ \epsilon\ (\partial_u^2 Z \sigma_{\epsilon,f})(x) + \dots$$

Result 2: main result

Result 2 in few words for $\mathbb{R}^2$

For branes at angle on $\mathbb{R}^2$ (T^2) the generic correlator

- with L untwisted operators
- and N (excited) twist fields

is given by a generalization of the Wick theorem.

For abelian twist given:

- x_t $(t = 1, \dots N)$ positions on ws of twists
- $f_{(t)}$ intersections in space of two consecutive branes
- $\pi\epsilon_{(t)}$ angles between two consecutive branes (the monodromies $M_{(t)} = e^{i2\pi\epsilon_{(t)}}$)

To compute any amplitude one needs

- classical solution $X^I_{cl}(u, \bar{u}; \{x_t, f_{(t)}, \epsilon_{(t)}\})$
- full Green function in presence of twist fields $G^{IJ}(u, \bar{u}; v, \bar{v}; \{x_t, \epsilon_{(t)}\})$ $(I, J = z, \bar{z})$
- correlator of the plain twist fields $\langle \prod_{t=1}^{N_B} \sigma_{\epsilon_{(t)}, f_{(t)}}(x_t) \rangle$ (full, i.e. quantum + classical contributions)
- a lot of patience

Result 2 in general

For branes at angle on $\mathbb{R}^{2N}$ the generic correlator

- with L untwisted operators
- and N (excited) twist fields

is still given by a generalization of the Wick theorem.
Mutatis mutandis.

Example of the main result (1)

- On $\mathbb{C} = \mathbb{R}^2$ with open string fields $Z(u, \bar{u}) = Z^1 = X^z \in \mathbb{C}$ and $\bar{Z}(u, \bar{u}) = Z^*(u, \bar{u}) = \bar{Z}^1 = X^{\bar{z}} \in \mathbb{C}$ with $u = x + iy \in H$ (the upper half plane)
- Untwisted sector: for comparison consider the boundary correlator in

$$\langle \partial_x \bar{Z}(x_1, x_1)\, \partial_x Z(x_2, x_2)\, (\partial_x^2 Z \partial_x \bar{Z})(x_3, x_3)\, \rangle$$

- it is given by

$$\begin{aligned} =&\partial_{x_1} \partial_{x_3}^2 G^{\bar{z}z}_{U,bou}(x_1, x_3)\, \partial_{x_2} \partial_{x_3} G^{z\bar{z}}_{U,bou}(x_2, x_3) \\ &+ \partial_{x_1} \partial_{x_3} G^{\bar{z}\bar{z}}_{U,bou}(x_1, x_3)\, \partial_{x_2} \partial_{x_3}^2 G^{zz}_{U,bou}(x_2, x_3) \end{aligned}$$

  where $G^{IJ}_{U,bou}(x_1, x_2)$ is the boundary Green function for Untwisted boundary conditions between two points $x_1, x_2 \in R$ on the boundary of the upper plane boundary ($G^{zz} \neq 0$ since brane breaks rotations)
- other possible terms like

$$\partial_{x_1} \partial_{x_2} G^{\bar{z}z}_{U,bou}(x_1, x_2)\, \partial_{x_3}^2 \partial_{x_3} [G^{z\bar{z}}_{U,bou}(x_2, x_3)]_{regularized}$$

  are absent because of normal ordering

Example of the main result (2)

- Twisted case: the boundary correlator in presence of N_B branes at angles

$$\langle \partial_x \bar{Z}(x_1,x_1)\, \partial_x Z(x_2,x_2)\, (\partial_x^2 Z \partial_x \bar{Z})(x_3,x_3) \prod_{t=1}^{N_B} \sigma_{\epsilon_t}(x_t)\rangle$$

$$= \langle \prod_{t=1}^{N_B} \sigma_{\epsilon_t}(x_t)\rangle \times \Big\{ \partial_{x_1}\partial_{x_3}^2 G_{bou}^{\bar{z}z}(x_1,x_3)\, \partial_{x_2}\partial_{x_3} G_{bou}^{z\bar{z}}(x_2,x_3)$$

$$+\partial_{x_1}\partial_{x_3} G_{bou}^{\bar{z}\bar{z}}(x_1,x_3)\, \partial_{x_2}\partial_{x_3}^2 G_{bou}^{zz}(x_2,x_3) \Leftarrow \text{ as before}$$

$$+\partial_{x_1}\partial_{x_2} G_{bou}^{\bar{z}z}(x_1,x_2)\, \partial_{x_3}^2 \partial_{y_3}|_{y_3=x_3} \Delta_{bou}^{z\bar{z}}(x_3,y_3) \Leftarrow \text{ left over from norm. ord.}$$

$$+\partial_{x_1}\partial_{x_2}^2 G_{bou}^{\bar{z}z}(x_1,x_2)\, \partial_{x_3}^2 Z_{cl}(x_2)\, \partial_{x_3}\bar{Z}_{cl}(x_3) + \partial_{x_1}\bar{Z}_{cl}(x_1)\, \partial_{x_2} Z_{cl}(x_2)\, \partial_{x_3}^2 \partial_{y_3}|_{y_3=x_3} \Delta_{bou}^{z\bar{z}}(x_3,y_3)$$

$$+\partial_{x_1}\partial_{x_3}^2 G_{bou}^{\bar{z}z}(x_1,x_3)\, \partial_{x_2} Z_{cl}(x_2)\, \partial_{x_3}\bar{Z}_{cl}(x_3) + \partial_{x_1}\bar{Z}_{cl}(x_1)\, \partial_{x_3}^2 Z_{cl}(x_3)\, \partial_{x_2}\partial_{x_3} G_{bou}^{z\bar{z}}(x_2,x_3)$$

$$+\partial_{x_1}\partial_{x_3} G_{bou}^{\bar{z}\bar{z}}(x_1,x_3)\, \partial_{x_2} Z_{cl}(x_2)\, \partial_{x_3}^2 Z_{cl}(x_3) + \partial_{x_1}\bar{Z}_{cl}(x_1)\, \partial_{x_3}\bar{Z}_{cl}(x_3)\, \partial_{x_2}\partial_{x_3}^2 G_{bou}^{zz}(x_2,x_3)$$

$$+\partial_x \bar{Z}_{cl}(x_1,x_1)\, \partial_x Z_{cl}(x_2,x_2)\, \partial_x^2 Z_{cl}(x_3,x_3)\, \partial_x \bar{Z}_{cl}(x_3,x_3) \Big\} \Leftarrow \text{ from classical solution } X_{cl}$$

where $G_{bou}^{IJ}(x,y)$ is the boundary Green function for twisted b.c.
and $\Delta_{bou}^{IJ}(x,y)$ its regularized version.
A NUMBER OF DETAILS HAVE BEEN OMITTED!

Example with excited twist fields

- Twisted case: the boundary correlator and excited twists in presence N_B branes at angles

$$\langle \partial_x \bar{Z}(\hat{x}_1,\hat{x}_1)\ (\partial_x Z\cdot\sigma_{\epsilon_1})(x_1,x_1)\ (\partial_x^2 Z \partial_x \bar{Z}\cdot\sigma_{\epsilon_2})(x_2,x_2)\ \prod_{t=3}^{N_B}\sigma_{\epsilon_t}(x_t)\rangle$$

where $(\partial_x^2 Z\partial_x\bar{Z}\sigma_{\epsilon_2})$ is the excited twist defined very roughly as $\lim_{u\to x_2}(\partial_x^2 Z\partial_x\bar{Z})(u,\bar{u})\sigma_{\epsilon_2}(x_2)$

$$= \langle\prod_{t=1}^{N_B}\sigma_{\epsilon_t}(x_t)\rangle\Big\{\partial_{v_2}[(v_2-x_2)^{\bar{\epsilon}_2}\partial_{\hat{x}_1}\partial_{v_2}G^{\bar{z}z}(\hat{x}_1,\hat{x}_1;v_2,\bar{v}_2)]\Big|_{v_2=x_2}$$

$$\times[(v_1-x_1)^{\bar{\epsilon}_1}(v_2-x_2)^{\epsilon_2}\partial_{v_1}\partial_{v_2}G^{z\bar{z}}(v_1,\bar{v}_1;v_2,\bar{v}_2)]|_{v_t=x_t}$$

$$+[(v_2-x_2)^{\epsilon_2}\partial_{\hat{x}_1}\partial_{v_2}G^{\bar{z}\bar{z}}(\hat{x}_1,\hat{x}_1;v_2,\bar{v}_2)]|_{v_2=x_2}$$

$$\times\partial_{v_2}[(v_1-x_1)^{\bar{\epsilon}_1}(v_2-x_2)^{\bar{\epsilon}_2}\partial_{v_1}\partial_{v_2}G^{zz}(v_1,\bar{v}_1;v_2,\bar{v}_2)]|_{v_t=x_t}\quad \text{“as before”}$$

$$+[(v_1-x_1)^{\bar{\epsilon}_1}\partial_{\hat{x}_1}\partial_{v_1}G^{\bar{z}z}(\hat{x}_1,\hat{x}_1;v_1,\bar{v}_1)]|_{v_1=x_1}$$

$$\times\partial_{v_2}[(u_2-x_2)^{\bar{\epsilon}_2}(v_2-x_2)^{\epsilon_2}\partial_{u_2}\partial_{v_2}\Delta^{z\bar{z}}(u_2,\bar{u}_2;v_2,\bar{v}_2)]|_{u_2=v_2=x_2}\quad \text{left over from norm. ord.}$$

$$+\ \text{terms with classical contributions}\Big\}$$

The Reggeon vertex

Is it possible to generate the previous correlators in a "mechanical" way? YES

For example the untwisted correlator

$$\langle \partial_x \bar{Z}(x_1, x_1)\ \partial_x Z(x_2, x_2)\ (\partial_x^2 Z \partial_x \bar{Z})(x_3, x_3)\ \rangle$$
$$= \frac{\partial}{\partial c_{(1)1}} \frac{\partial}{\partial \bar{c}_{(2)1}} \frac{\partial^2}{\partial \bar{c}_{(3)2} \partial c_{(3)1}} V(\{c_{(i)n}, \bar{c}_{(i)n}\})\Big|_{c=0}$$

where

- $V(\{c_{(i)n}, \bar{c}_{(i)n}\})$ is the Reggeon vertex
- $c_{(i)n}$ with i associated with x_i
- $c_{(i)n}$ with n associated with the number of derivatives $\partial_{x_i}^n$

Easy to derive for the untwisted correlators.

More complicated with the twisted ones. To Reggeon slides

Conclusions

We have shown that to compute any correlator involving excited twisted fields and untwisted vertices are needed three ingredients

- classical solution $X^I_{cl}(u, \bar{u}; \{x_t, U_{(t)}, f_{(t)}\})$
- correlator of the plain twist fields $\langle \prod_{t=1}^{N_B} \sigma_{M_{(t)}, f_{(t)}}(x_t) \rangle$
- full Green function in presence of twist fields $G^{IJ}(u, \bar{u}; v, \bar{v}; \{x_t, U_{(t)}\})$ $(I, J = 1, \dots 2N)$

and that the computation is more or less similar to the computation done using Wick theorem.
In this way they are not worse that correlators with spin fields
BUT
to get the amplitudes is almost impossible since the Green function is a product of generalization of hypergeometric functions.
Branes at angles Green function in NOT the same Green function for magnetized branes!

Future directions

As far the non abelian twists there are many issues open:

- only the classical solution $X^I_{cl}(u, \bar{u}; \{x_t, U_{(t)}, f_{(t)}\})$ and Green function for $N_B = 3$ for $\mathbb{R}^4$ with monodromies in $SU(2)$ are known;
- it should be easy to extend the result to $N_B = 3$ and monodromies in $SO(4)$;
- the normalization for $N_B = 3$ for $\mathbb{R}^4$, i.e. the correlator of the plain twist fields $\langle \prod_{t=1}^{3} \sigma_{M_{(t)}, f_{(t)}}(x_t) \rangle$ is unknown (but an educated guess can be done);
- the reason is that the $N_B \geq 4$ in $\mathbb{R}^4$ cannot be handled with the usual techniques because we cannot rely on the classical mathematics by Fuchs, Heun etc (the problem of the accessories parameters and global monodromies);
- use of factorization?
- even worse the case for higher dimensions like $\mathbb{R}^6$
- but it could be it is possible to use the researches by Schwarz, Fuchs, Klein, Gordan, Jordan... on algebraic solutions of Fuchsian equations.

Thanks for the attention!

Details

The Hermitian product for modes (1)

- Have a time dependent world-sheet since the boundary conditions vary with time.
- Need a proper way of defining an Hermitian product conserved in time.
- The solution: the Klein-Gordon metric used in QFT on curved spacetime. Note: not positive definite but is constant in time when solutions of KG equation are considered.
- Start from K-G current for any two 2-vectors $F_{1,2} = (f^z_{1,2}, f^{\bar z}_{1,2})$

$$j_\alpha(F_1, F_2) = i[(f^I_1)^* \partial_\alpha f^I_2 - (\partial_\alpha f^I_1)^* f^I_2]$$

- It is conserved on solutions.

Back bc
Back to Twisted string

The Hermitian product for modes (2)

- Consider on half an annulus $S(r_0, r_1)$ in the upper half plane

$$0 = \int_{S(r_0,r_1)} d * j = \int_{|u|=r_1} *j - \int_{|u|=r_0} *j + \int_{[r_0,r_1]} *j + \int_{[-r_1,-r_0]} *j$$

- "Metric" is at given time $r = |u|$, e.g. $\int_{|u|=r_0} *j$
  Term like $\int_{[r_0,r_1]} *j$ is not computed at constant time.
- We can write $\int_{[r_0,r_1]} *j = G(r_1) - G(r_0)$.
  Try to define a Hermitian product

$$(F_1, F_2) = (F_2, F_1)^* = \int_{|u|=r} *j + G(r) - G(-r)$$

- Good? Only if $G(r) - G(-r)$ does not depend on past bck values.
  This requires F to have quantum boundary conditions

$$e^{-i\pi\alpha_t} \partial_y f^z(u,\bar{u})|_{u=x+i0^+} + e^{i\pi\alpha_t} \partial_y f^{\bar{z}}(u,\bar{u})|_{u=x+i0^+} = 0 \qquad x_t < x < x_{t-1}$$

$$e^{-i\pi\alpha_t} f^z(u,\bar{u})|_{u=x+i0^+} - e^{i\pi\alpha_t} f^{\bar{z}}(u,\bar{u})|_{u=x+i0^+} = 0 \qquad x_t < x < x_{t-1}$$

Same for having a self-adjoint $\partial_u \bar{\partial}_{\bar{u}}$!

Back bc
Back to Twisted string

The Hermitian product for modes (3)

- Quantum boundary condition implies split

$$X^I(u,\bar u) = X^I_{cl}(u,\bar u;\{x_t, g_t, \alpha_t\}) + X^I_q(u,\bar u;\{x_t,\epsilon_t\})$$

with X_{cl} classical solution, X_q quantum fluctuation to be quantized

- The Hermitian form is then for quantum fluctuations

$$(F_1, F_2) = (F_2, F_1)^* = \int_{|u|=r} *j$$

- For the usual magnetic branes get the well known "weird" Hermitian form

$$(F_1, F_2) = \int_0^\pi i\, F_1^\dagger \overleftrightarrow{\partial_\tau} F_2 d\sigma \; + iF_1^\dagger \mathcal{F}_0 F_2|_{\sigma=0} - iF_1^\dagger \mathcal{F}_\pi F_2|_{\sigma=\pi}$$

where $\mathcal{F}_{IJs}$ are the magnetic fields.

Back bc
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Refined overlap condition

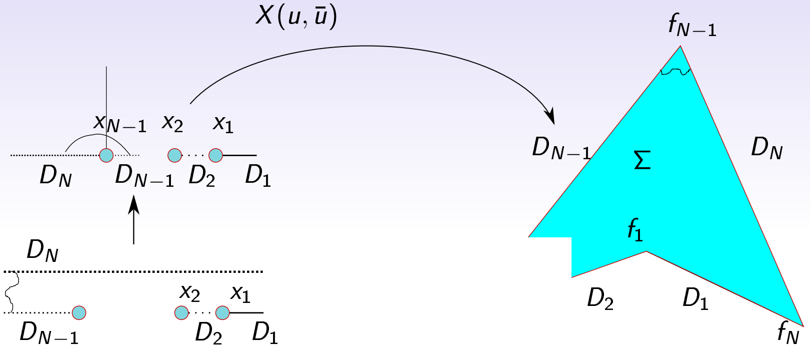
In principle it is possible to study the quantum modes of the $\partial_u \bar{\partial}_{\bar{u}}$ with quantum boundary conditions.

- BUT there are still some issues
- HENCE use the old overlap approach but improved
  - split
$$Z(u,\bar{u}) = Z_{cl}(u,\bar{u};\{x_t, f_t, \alpha_t\}) + Z_q(u,\bar{u};\{x_t, \alpha_t\})$$
  - compute the global classical solution $X_{cl}(u,\bar{u};\{x_t, f_t, \alpha_t\})$
  - when $x_t < |u| < x_{t-1}$ the string endpoints are f.x. on D_N and D_t use the appropriate quantum expansion as there were the appropriate twist at $u = 0$ and the corresponding antitwist at $u = \infty$
$$X_q(u,\bar{u};\{D_N, D_t\})$$
  - at transition "time" like $|u| = x_t$ require match of the two quantum expansions as
$$X_q(u,\bar{u};\{D_N, D_{t+1}\})\big|_{|u|=x_t^-} = X_q(u,\bar{u};\{D_N, D_t\})\big|_{|u|=x_t^+}$$

◂ Back to twisted string   ◂ Back to reggeon

In and out vacua in presence of N twist fields (1)

We consider the configuration



We use the improved overlap

Hence we take the in vacuum to be the twisted vacuum corresponding to the usual $N = 2$ twisted string

$$|0_{in}\rangle = |\, T_{D_{N-1} D_N}\rangle$$

Back to Reggeon

In and out vacua in presence of N twist fields (2)

What about $\langle 0_{out}|$?

- Compute Green function

$$G^{IJ}(u,\bar{u};v,\bar{v};\{x_t,\epsilon_t\})$$

  in the usual way
- consider the operatorial definition of the (derivative of) Green function

$$\partial_u\partial_v G^{IJ}(u,\bar{u};v,\bar{v};\{x_t,\epsilon_t\}) = \frac{\langle 0_{out}|\partial_u Z^I_q(u,\bar{u})\partial_v Z^J_q(v,\bar{v})|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle}$$

- take $|u|,|v| < x_{N-1}$ so we can write

$$\partial_u\partial_v G^{IJ} = \frac{\langle 0_{out}|\partial_u Z^I_{\{D_{N-1},D_N\},q}(u,\bar{u})\partial_v Z^J_{\{D_{N-1},D_N\},q}(v,\bar{v})|T_{D_{N-1}D_N}\rangle}{\langle 0_{out}|T_{D_{N-1}D_N}\rangle}$$

Back to Reggeon

In and out vacua in presence of N twist fields (3)

What about $\langle 0_{out}|$?

- Use mode expansion and normal order the result

$$\partial_u \partial_v \Delta^{IJ}_{(N,M)(N-1)} = \frac{\langle 0_{out}| : \partial_u Z^{I(-)}_{\{D_{N-1},D_N\},q}(u) \partial_v Z^{J(-)}_{\{D_{N-1},D_N\},q}(v) : |T_{D_{N-1}D_N}\rangle}{\langle 0_{out}|T_{D_{N-1}D_N}\rangle}$$

with

$$\Delta^{IJ}_{(N,M)(N-1)}(u,\bar{u};v,\bar{v};\{x_t,\epsilon_t\}) = G^{IJ}(u,\bar{u};v,\bar{v};\{x_t,\epsilon_t\}) - G^{IJ}_{N=2,\{D_{N-1},D_N\}}(u,\bar{u};v,\bar{v})$$

the regularized Green function.

- derive

$$\langle 0_{out}| \sim \langle T_{D_1 D_N}| e^{B_{\bar{z}\bar{z}}\alpha\alpha + B_{zz}\bar{\alpha}\bar{\alpha} + B_{\bar{z}z}\alpha\bar{\alpha}}$$

with $B \sim \Delta^{IJ}$

Back to Reggeon

The untwisted Reggeon vertex (2)

- Map untwisted abstract operator to a realization in an untwisted Hilbert space.
  E.g. in untwisted Hilbert space

$$(\partial_x^2 Z \partial_x \bar{Z})(x_3, x_3) = \frac{\partial^2}{\partial \bar{c}_{(3)2} \partial c_{(3)1}} \mathcal{S}(c_{(3)}, \bar{c}_{(3)})$$

$$+ \mathcal{S}(c_{(3)}, \bar{c}_{(3)}) = : e^{\sum_{n=0}^{\infty} \left[\bar{c}_{(3)n} \partial_x^n Z_{op}(x_3, x_3) c_{(3)n} \partial_x^n \bar{Z}_{op}(x_3, x_3) \right]} : = : e^{\sum_{n=0}^{\infty} c_{(3)nI} \partial_x^n \bar{Z}^I_{op}(x_3, x_3)} :$$

  The Sciuto-Della Selva-Saito vertex $\mathcal{S}$ is the generating function of this map.
- Compute the generating function of all correlators with L untwisted vertices in untwisted Hilbert space

$$V_L(\{c_{(i)n}, \bar{c}_{(i)n}\}) = \langle 0 | \mathcal{S}(c_{(1)}, \bar{c}_{(1)}) \dots \mathcal{S}(c_{(L)}, \bar{c}_{(L)}) | 0 \rangle$$

-

$$= \prod_{1 \le i < j \le L} e^{\sum_{n,m=0}^{\infty} c_{(i)nI} \; c_{(j)mJ} \; \partial_{x_i}^n \partial_{x_j}^m G_U^{IJ}(x_i, x_j)}$$

with $c_{(i)n} = c_{(i)n\bar{z}} = c^z_{(i)n}$ and $\bar{c}_{(i)n} = c_{(i)nz} = c^{\bar{z}}_{(i)n}$.

The Reggeon vertex (3)

The idea is to generalize the untwisted computation

$$V_L(\{c_{(i)n}, c_{(i)n}\}) = \langle 0|\mathcal{S}(c_{(1)}, \bar{c}_{(1)}) \dots \mathcal{S}(c_{(L)}, \bar{c}_{(L)})|0\rangle$$

to the twisted case

$$\begin{aligned} V_{N+L}(\{c_{(i)n}, d_{(t)n}\}) =& \langle 0_{out}|\mathcal{S}_{T(1)}(c_{(1)}, \bar{c}_{(1)}) \dots \mathcal{S}_{T(L)}(c_{(L)}, \bar{c}_{(L)})\times \\ & \times \mathcal{T}_{(1)}(d_{(1)}, \bar{d}_{(1)}) \dots \mathcal{T}_{(N)}(d_{(N)}, \bar{d}_{(N)})|0_{in}\rangle \end{aligned}$$

We need understanding

- the in vacuum $|0_{in}\rangle$
- the out vacuum $\langle 0_{out}|$ Details
- the Sciuto-Della Selva-Saito $\mathcal{S}_T(c_{(i)}, \bar{c}_{(i)})$ for the untwisted matter in the twisted sectors Details
- the Sciuto-Della Selva-Saito $\mathcal{T}(d_{(t)}, \bar{d}_{(t)})$ for the twisted matter, i.e. excited twist field Details

Details on computation

- It is also possible and more normal to perform the previous computation in path integral formalism

Sketch of computation

The Reggeon vertex (4)

The final result for L untwisted vertices and N twisted ones.

- Associate: space index $\leftrightarrow I$ with $I = z, \bar{z}$
  untwisted $\leftrightarrow c_{(i)nI}$ with $i = 1, \dots L$,
  twisted $\leftrightarrow d_{(t)nI}$ with $t = 1, \dots N$
- The generating function is

$$V_{N+L}(c,d) = \lim_{\{u_t\}\to\{x_t\}} \langle \sigma_{\epsilon_1,f_1}(x_1)\dots\sigma_{\epsilon_N,f_N}(x_N)\rangle \times V_{class} \times V_{self\ \ int} \times V_{int}$$

with

$$V_{class} = \prod_{t=1}^{N} e^{\sum_{n=1}^{\infty} d_{(t)nI}\,\partial_{u_t}^{n-1}[(u_t-x_t)^{\epsilon_{tI}}\,\partial_u Z_{cl}^I(u_t,\bar{u}_t)]} \times \prod_{i=1}^{L} e^{\sum_{n=0}^{\infty} c_{(i)nI}\,\partial_{x_i}^{n} Z_{cl}^I(x_i,x_i)}$$

where $Z_{cl}^I(u,\bar{u})$ is the classical solution.

$$V_{N+L}(c,d) = \lim_{\{u_t\}\to\{x_t\}} \langle \sigma_{\epsilon_1,f_1}(x_1)\dots\sigma_{\epsilon_N,f_N}(x_N)\rangle \times V_{clas} \times V_{self\ \ int} \times V_{int}$$

with

$$V_{self\ \ interaction} = \prod_{t=1}^{N} e^{\frac{1}{2}\sum_{n,m=1}^{\infty} d_{(t)nI}\, d_{(t)mJ}\,\partial_{u_t}^{n-1}\partial_{v_t}^{m-1}[(u_t-x_t)^{\epsilon_{tI}}(v_t-x_t)^{\epsilon_{tJ}}\,\partial_u\partial_v \Delta_{(t)}^{IJ}(u_t,\bar{u}_t;v_t,\bar{v}_t;\{x_{\bar{t}},\epsilon_{\bar{t}}\})]|_{v_t=u_t}} \times \prod_{i=1}^{L} e^{\frac{1}{2}\sum_{n=0}^{\infty} c_{(i)nI}\sum_{m=0}^{\infty} c_{(i)mJ}\,\partial_{x_i}^{n}\partial_{\hat{x}_i}^{m}\Delta_{(i)}^{IJ}(x_i,x_i;\hat{x}_i,\hat{x}_i;\{x_t,\epsilon_t\})|_{\hat{x}_i=x_i}}$$

where $\Delta_{(t)}^{IJ}$ is the Green function regularized at point x_t

$$V_{N+L}(c,d) = \lim_{\{u_t\}\to\{x_t\}} \langle \sigma_{\epsilon_1,f_1}(x_1)\dots\sigma_{\epsilon_N,f_N}(x_N)\rangle \times V_{classical} \times V_{self\ \ interaction} \times V_{interactions}$$

with

$$V_{interactions} = \prod_{1\le t<\hat{t}\le N} e^{\sum_{n,m=1}^{\infty} d_{(t)nI}\, d_{(\hat{t})mJ}\,\partial_{u_t}^{n-1}\partial_{v_{\hat{t}}}^{m-1}[(u_t-x_t)^{\epsilon_{tI}}(v_{\hat{t}}-x_{\hat{t}})^{\epsilon_{\hat{t}J}}\,\partial_u\partial_v G^{IJ}(u_t,\bar{u}_t;v_{\hat{t}},\bar{v}_{\hat{t}};\{x_{\bar{t}},\epsilon_{\bar{t}}\})]} \times \prod_{1\le i<j\le L} e^{\sum_{n=0}^{\infty} c_{(i)nI}\sum_{m=0}^{\infty} c_{(j)mJ}\,\partial_{x_i}^{n}\partial_{x_j}^{m} G^{IJ}(x_i,x_i;x_j,x_j;\{x_t,\epsilon_t\})} \times \prod_{1\le t\le N}\prod_{1\le j\le L} e^{\sum_{n=1}^{\infty} d_{(t)nI}\, c_{(j)mJ}\,\partial_{u_t}^{n-1}\partial_{x_j}^{m}[(u_t-x_t)^{\epsilon_{tI}}\,\partial_u G^{IJ}(u_t,\bar{u}_t;x_j,x_j;\{x_{\bar{t}},\epsilon_{\bar{t}}\})]}$$

The Reggeon vertex (4)

Our case L untwisted vertices and N twisted ones.
Putting all together the generating function is

$$
\begin{aligned}
V_{N+L}(c,d) = \lim_{\{u_t\}\to\{x_t\}} & \langle \sigma_{\epsilon_1,f_1}(x_1)\dots\sigma_{\epsilon_N,f_N}(x_N)\rangle \\
& \times \prod_{t=1}^{N} \Bigg\{ e^{\sum_{n=1}^\infty d_{(t)nI}\,\partial_{u_t}^{n-1}[(u_t-x_t)^{\epsilon_t I}\,\partial_u Z^I_{cl}(u_t,\bar u_t)]} \\
& \times e^{\frac12 \sum_{n,m=1}^\infty d_{(t)nI}\, d_{(t)mJ}\,\partial_{u_t}^{n-1}\partial_{v_t}^{m-1}[(u_t-x_t)^{\epsilon_t I}(v_t-x_t)^{\epsilon_t J}\,\partial_u\partial_v \Delta^{IJ}_{(N,M)(t)}(u_t,\bar u_t;v_t,\bar v_t;\{x_{\bar t},\epsilon_{\bar t}\})]|_{v_t=u_t}} \Bigg\} \\
& \times \prod_{i=1}^{L} \Bigg\{ e^{\sum_{n=0}^\infty c_{(i)nI}\,\partial_{x_i}^{n} Z^I_{cl}(x_i,x_i)} \\
& \times e^{\frac12 \sum_{n=0}^\infty c_{(i)nI} \sum_{m=0}^\infty c_{(i)mJ}\,\partial_{x_i}^{n}\partial_{\hat x_i}^{m} \Delta^{IJ}_{(N,M),bou(i)}(x_i,\hat x_i;\{x_t,\epsilon_t\})|_{\hat x_i=x_i}} \Bigg\} \\
& \times \prod_{1\le t<\hat t\le N} e^{\sum_{n,m=1}^\infty d_{(t)nI}\, d_{(\hat t)mJ}\,\partial_{u_t}^{n-1}\partial_{v_{\hat t}}^{m-1}[(u_t-x_t)^{\epsilon_t I}(v_{\hat t}-x_{\hat t})^{\epsilon_{\hat t} J}\,\partial_u\partial_v G^{IJ}_{(N,M)}(u_t,\bar u_t;v_{\hat t},\bar v_{\hat t};\{x_{\bar t},\epsilon_{\bar t}\})]} \\
& \times \prod_{1\le i<j\le L} e^{\sum_{n=0}^\infty c_{(i)nI} \sum_{m=0}^\infty c_{(j)mJ}\,\partial_{x_i}^{n}\partial_{x_j}^{m} G^{IJ}_{(N,M),bou}(x_i,x_j;\{x_t,\epsilon_t\})} \\
& \times \prod_{1\le t\le N}\prod_{1\le j\le L} e^{\sum_{n=1}^\infty d_{(t)nI}\, c_{(j)mJ}\,\partial_{u_t}^{n-1}\partial_{x_j}^{m}[(u_t-x_t)^{\epsilon_t I}\,\partial_u G^{IJ}_{(N,M)}(u_t,\bar u_t;x_j,x_j;\{x_{\bar t},\epsilon_{\bar t}\})]}
\end{aligned}
$$

SDS vertex for untwisted vertices (1)

The SDS vertex maps an abstract operator to an operatorial realization. The map for an untwisted abstract operator to its operatorial realization in twisted Hilbert space is

$$\mathcal{S}_T(c,\bar{c}) = :e^{\sum_{n=0}^{\infty}\left[\bar{c}_n \partial_x^n Z_{op\ T}(x+i0^+,x-i0^+)+c_n \partial_x^n \bar{Z}_{op\ T}(x+i0^+,x-i0^+)\right]}:$$
$$\exp\left\{\frac{1}{2}\sum_{n,m=0}^{\infty} \bar{c}_{nI}\ c_{mJ}\ \partial_{x_1}^n\ \partial_{x_2}^m \Delta_{bou\ T}^{IJ}(x_1;x_2)|_{x_1=x_2=x}\right\}$$

There is a new piece

$$\Delta_{bou\ T}^{IJ}(x_1;x_2) = G_{N=2\ T}^{IJ}(x_1+i0^+,x_1-i0^+;x_2+i0^+,x_2-i0^+) - G_U^{IJ}(x_1+i0^+,x_1-i0^+;x_2+i0^+,x_2-i0^+)$$

the left over of the “minimal subtraction”.

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SDS vertex for untwisted vertices (2)

Why is so?

- Consider "simplest" untwisted vertex in untwisted Hilbert space

$$: e^{ik^I Z^I_{op\ Untwisted}(x)} :$$

- can be derived from non normal ordered vertex by a point splitting procedure

$$: e^{ik^I Z^I_{op\ Untwisted}(x)} := \lim_{\eta \to 0} \mathcal{N}(\eta) e^{ik^I [X^{I(-)}_{op\ Untwisted}(xe^{-\eta}) + Z^{I(+)}_{op\ Untwisted}(x)]}$$

  with $\mathcal{N}(\eta)$ a regularization factor
- the vertex for the same state in twisted Hilbert space can be derived as

$$\lim_{\eta \to 0} \mathcal{N}(\eta) e^{ik^I [X^{I(-)}_{op\ Twisted}(xe^{-\eta}) + Z^{I(+)}_{op\ Twisted}(x)]}$$

  with the same regularization factor $\mathcal{N}(\eta)$, a kind of minimal subtraction.
- OK since realizations in twisted Hilbert reproduce the usual OPEs!

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SDS vertex for untwisted vertices (3)

Two examples

- to the boundary tachyonic vertex $e^{i\bar{k}Z(x,x)+ik\bar{Z}(x,x)}$ corresponds the operatorial realization

$$x^{-\alpha' k^2_{\parallel D}} e^{-\frac{1}{2}R^2(\epsilon)\alpha' k^2_{\parallel D}} \; : e^{i(\bar{k}Z(x,x)+k\bar{Z}(x,x))} :$$

with $R^2(\epsilon) = 2\psi(1) - \psi(\epsilon) - \psi(\bar{\epsilon}) > 0$ and $\psi(z) = \frac{d \ln \Gamma(z)}{dz}$ the digamma function and $k_{\parallel D}$ is the part of the momentum, parallel to the brane

- we can also compute the SDS for chiral operators: to the chiral operator $(\partial_u^2 Z\, \partial_u Z\, \partial_u \bar{Z})(u)$ corresponds

$$\begin{aligned} &: (\partial_u^2 Z \partial_u Z \partial_u \bar{Z})(u) : + \partial_u^2 \partial_v \Delta_c^{z\bar{z}}|_{v=u} \partial_u Z + \partial_u \partial_v \Delta_c^{z\bar{z}}|_{v=u} \partial_u^2 Z \\ =&: (\partial_u^2 Z \partial_u Z \partial_u \bar{Z})(u) : - \frac{k_\epsilon k_{\bar{\epsilon}} \epsilon(1-\epsilon)(2-\epsilon)}{2u^3} \partial_u Z + \frac{k_\epsilon k_{\bar{\epsilon}} \epsilon(1-\epsilon)}{2u^2} \partial_u^2 Z \end{aligned}$$

with $k_\epsilon = -i\frac{1}{2}\sqrt{2\alpha'}e^{i\pi\alpha_t}$ and $k_{\bar{\epsilon}} = -i\frac{1}{2}\sqrt{2\alpha'}e^{-i\pi\alpha_t}$

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SDS for excited twists (1)

- The main observation

$$\partial_u^{n-1}\left[u^{\bar\epsilon}\partial_u Z_{op}(u,\bar u)\right] = (n-1)!\ k_\epsilon \alpha^\dagger_{n-1+\epsilon} + O(u)$$

- therefore a normal ordered products of these operators gives directly an excited twist state, e.g.

$$\lim_{u\to 0} :\partial_u^{n-1}\left[u^{\bar\epsilon}\partial_u Z_{op}(u,\bar u)\right]\partial_u^{m-1}\left[u^{\epsilon}\partial_u \bar Z_{op}(u,\bar u)\right]: |T\rangle$$
$$= k_\epsilon k_{\bar\epsilon}(n-1)!(m-1)!\alpha^\dagger_{n-1+\epsilon}\bar\alpha^\dagger_{m-1+\epsilon}|T\rangle = \left(\partial^n Z \partial^m \bar Z \sigma_{\epsilon,f}\right)(0)|0\rangle_{SL(2)}$$

- then the SDS vertex is

$$\mathcal{T}_T(d,\bar d) =$$
$$\lim_{u\to 0} :\exp\left\{\sum_{n=1}^{\infty}\left[\bar d_n \partial_u^{n-1}\left[u^{\bar\epsilon}\partial_u Z_{op\ T}(u,\bar u)\right] + d_n\partial_u^{n-1}\left[u^{\epsilon}\partial_u \bar Z_{op\ T}(u,\bar u)\right]\right]\right\}:$$

since

$$\left[\prod_{n=1}^{\infty}\left(\partial_u^n Z\right)^{N_n}\left(\partial_u^n \bar Z\right)^{\bar N_n}\sigma_{\epsilon,f}\right](0)|0\rangle_{SL(2)} \leftrightarrow \lim_{u\to 0}\prod_{n=1}^{\infty}\frac{\partial^{N_n}}{\partial \bar d_n^{N_n}}\frac{\partial^{\bar N_n}}{\partial d_n^{\bar N_n}}\mathcal{T}(d,\bar d)\Big|_{d=0}|T\rangle$$

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SDS for excited twists (2)

What if the twist field is not located at $x = 0$? Translate the previous operator

$$\mathcal{T}(d, \bar{d}) =$$

$$\lim_{u \to x} : \exp\left\{ \sum_{n=1}^{\infty} \left[\bar{d}_n \partial_u^{n-1} \left[(u-x)^{\bar{\epsilon}} \partial_u Z_{op}(u, \bar{u}) \right] + d_n \partial_u^{n-1} \left[(u-x)^{\epsilon} \partial_u \bar{Z}_{op}(u, \bar{u}) \right] \right] \right\} :$$

This is what needed for exciting the other twist fields hidden in the boundary conditions discontinuities.

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Reggeon vertex for N excited twist fields and L untwisted states

We have now all ingredients to compute

$$V_{N+L}(\{c_{(i)n}, d_{(t)n}\}) = \langle 0_{out}| \mathcal{S}_{T(1)}(c_{(1)}, \bar{c}_{(1)}) \dots \mathcal{S}_{T(L)}(c_{(L)}, \bar{c}_{(L)}) \times \\ \times \mathcal{T}_{(1)}(d_{(1)}, \bar{d}_{(1)}) \dots \mathcal{T}_{(N)}(d_{(N)}, \bar{d}_{(N)}) |0_{in}\rangle$$

and get the stated result.
Notice that for the interactions not in the in Hilbert state we need to use the overlap condition to analytically continue them into the in Hilbert state.
In particular the relations are fundamental

$$[\mathcal{S}(c_{(i)}, \bar{c}_{(i)})|_{Hilbert(D_t D_N)}]_{analytically\ \ cont.} \sim e^{c^I c^J [G_{U(D_{N-1})} - G_{U(D_t)}]} \\ \mathcal{S}(c_{(i)}, \bar{c}_{(i)})|_{Hilbert(D_{N-1} D_N)}$$

and

$$[\mathcal{T}(d_{(t)}, \bar{d}_{(t)})|_{Hilbert(D_t D_N)}]_{analytically\ \ cont.} \sim e^{d^I d^J [G_{N=2,(D_{N-1} D_N)} - G_{N=2,(D_t D_N)}]} \\ \mathcal{T}(d_{(t)}, \bar{d}_{(t)})|_{Hilbert(D_{N-1} D_N)}$$

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The path integral approach

The path integral amounts to computing

$$V_{N+L}(\{c_{(i)}, d_{(t)}\}) = \int_{\mathcal{M}(\{x_t, \epsilon_t, f_t\})} \mathcal{D}Z \; e^{-S_E} \prod_{i=1}^{L} \mathcal{S}_{abs}(c_{(i)}, \bar{c}_{(i)}) \; \prod_{t=1}^{N} \mathcal{T}_{abs}(d_{(t)}, \bar{d}_{(t)})$$

where

- $\mathcal{M}(\{x_t, \epsilon_t, f_t\})$ is the space of string configurations satisfying the desired boundary conditions
- $\mathcal{S}_{abs}(c_{(i)}, \bar{c}_{(i)})$ is the abstract operator version of the SDS vertex
- $\mathcal{T}_{abs}(d_{(t)}, \bar{d}_{(t)})$ is the abstract operator version of the SDS vertex

Since the integral is quadratic can be easily done.

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