New supersymmetric localizations from topological gravity

Camillo Imbimbo

Napoli, Nov 19, 2015

Based on:
J. Bae, C.I., S.J Rey, D. Rosa, http://arxiv.org/1510.00006

Localization

- Localization is a long-known property of supersymmetric and topological theories, by virtue of which semi-classical approximation becomes, in certain cases, exact (Witten:1991).
- For instance, the topological quantum field theories (TQFTs) whose action is BRST-exact are semi-classically exact since their coupling constant is a gauge parameter which can be taken to be arbitrarily small.
- The traditional route for constructing TQFTs is by topologically twisting supersymmetric quantum field theories (SQFTs) by means of a conserved R-symmetry. Hence, the localization technique has frequently been associated to SQFTs since the early days.

Localization à la Pestun

- More recently (Pestun: 2007) a new paradigm for localization emerged for various SQFTs, which makes no explicit connection to TQFTs.
- Rather, localization is seen as a special property of SQFTs when defined on specific supersymmetry-preserving curved backgrounds.
- These external backgrounds may be identified (Festuccia et al:2011) with an off-shell, supersymmetric configuration of a supergravity (SUGRA) multiplet that the SQFT can couple to.

Generalized covariantly constant spinors

- In this case, there exists (at least) a generalized covariantly constant spinor that sets the supersymmetry variations of fermionic fields of the SUGRA multiplet to zero.
- The generalized covariantly constant spinor must satisfy integrability conditions which put stringent constraints on the bosonic fields of the SUGRA multiplet.
- These fields include the spacetime metric and also, in theories with extended supersymmetries, vector fields of gauged R-symmetries as well as off-shell auxiliary fields.

Generalized covariantly constant spinors

- A complete classification of generalized covariantly constant spinors is a complicated problem.
- Although explicit solutions have been obtained case by case in various spacetime dimensions, there is no general strategy for constructing the covariantly constant spinors and for classifying the background spacetime metrics and gauge fields which support them.
- I want to describe a different approach to this problem. Its application to 3d was discussed in C.I. and D.Rosa: 2014. Today I will discuss the 2d case.

A topological approach

Topological approach

Instead than coupling susy matter to supergravity, we will couple topological matter to (gauged)-topological gravity.

- The equations which are the counterpart of the equations for the covariantly constant spinors are obtained by equating to zero the BRST variations of the fermionic fields of (gauged)-topological gravity.
- These fixed-point equations for the BRST transformations determine the BRST-invariant backgrounds around which the topological computation localizes.

The BRST-invariant topological backgrounds

- The topological equations have several advantages with respect to the supergravity equations.
- One avoids all the complications of spinors and obtains much simpler (in a technical sense) equations.
- The equations are largely universal, i.e. independent on the space-time dimensions. In 2d they have a natural interpretation in terms of S^1-equivariant cohomology of the underlying space-time.
- One obtains a precise definition of topologically equivalent backgrounds. In 2d this permits the classification of all solutions up to topological equivalence.

Topological equations vs Supergravity equations

- But one obvious question is:

Do the topological eqs solve the same supersymmetric problem as the eqs for covariantly constant spinors?

- The answer is yes. In 2d we show this explicitly: every BRST invariant topological background is identifiable with every localizing SUGRA background.
- The map between the topological and the supersymmetric backgrounds, however, turns out to be quite non-trivial.

The map between topological and supergravity fields

- Most of topological background fields are not, in any sense, fields of some “topologically twisted” SUGRA.
- (Some of) the BRST invariant topological backgrounds are bilinears of the covariantly constant spinors of supergravity.
- For example, the ghost-for-ghost field of TG is identified with the spinorial bilinear which defines the Killing vector of the spacetime metric.
- Conversely, the localizing SUGRA fields are, in general, non-linear functionals of the BRST invariant topological backgrounds.

The topological background system

- The main problem in the topological approach is to identify the background topological gravity system which is required to describe localizations of a given supersymmetric matter system.
- In 3d it turns out that the background topological system which describes supersymmetric matter in the vector multiplet is pure topological gravity (C.I. D. Rosa, 2014).
- In 2d it turns out that to describe N=(2,2) supersymmetric matter, one needs topological gravity coupled to a background abelian topological gauge field.

The topological gravity and gauge background system

- The fields of topological gravity are

$$
\begin{array}{cccc}
 & g_{\mu\nu} & \psi_{\mu\nu} & \gamma^{\mu} \\
\text{ghost}\# & 0 & 1 & 2
\end{array}
$$

- The fields of the topological gauge multiplets are

$$
\begin{array}{cccc}
 & a^{(1)} & \psi^{(1)} & \gamma^{(0)} \\
\text{ghost}\# & 0 & 1 & 2
\end{array}
$$

The topological gravity and gauge background system

The nilpotent BRST operator s

$$s \equiv S + \mathcal{L}_\xi$$

acting as follows

$$\begin{array}{ccc} S\, g_{\mu\nu} = \psi_{\mu\nu} & S\,\psi_{\mu\nu} = \mathcal{L}_\gamma g_{\mu\nu} & S\,\gamma^\mu = 0 \\ S\, f^{(2)} = -d\,\psi^{(1)} & S\,\psi^{(1)} = -d\,\gamma^{(0)} + i_\gamma f^{(2)} & S\,\gamma^{(0)} = i_\gamma \psi^{(1)} \end{array}$$

where the BRST operator S satisfies

$$S^2 = \mathcal{L}_\gamma$$

Why do we need the background abelian topological gauge field?

- The standard (Witten:1992) “topological” reformulation of 2d YM matter is genuinely topological only in the zero-coupling limit. In the non-zero coupling limit it is not topological and it cannot be coupled to topological gravity.
- We need a reformulation of 2D YM which can be coupled to topological gravity.
- It turns out that this matter topological theory does exist if one introduces a background topological abelian multiplet.

A new formulation of topological 2D YM

$$
\begin{array}{cccc}
\mathcal{A} & \equiv c & A & \widetilde{\phi}\ , \\
\text{ghost\#} & 1 & 0 & -1 \\
\text{form degree} & 0 & 1 & 2 \\
\Phi & \equiv \phi & \widetilde{A} & \widetilde{c} \\
\text{ghost\#} & 0 & -1 & -2 \\
\text{form degree} & 0 & 1 & 2
\end{array}
$$

Topological 2D YM action

The BRST-invariant topological 2D YM action coupled to the topological backgrounds is

$$
\begin{aligned}
I_{matter} \quad &= \quad \int_\Sigma \left[f^{(2)} \, \frac{1}{2} \operatorname{Tr} \phi^2 + \psi^{(1)} \wedge \operatorname{Tr} \phi \, \widetilde{A} + \right. \\
&\qquad\qquad \left. + \gamma^{(0)} \operatorname{Tr} (\phi \, \tilde{c} + \frac{1}{2} \widetilde{A} \wedge \widetilde{A}) \right]
\end{aligned}
$$

Coupling Topological YM to topological gravity

$$
\begin{aligned}
S\,c &= -c^2 + \gamma^{(0)}\,\phi + i_\gamma A \\
S\,A &= -D\,c + \gamma^{(0)}\,\widetilde{A} + \psi^{(1)}\,\phi + i_\gamma \widetilde{\phi} \\
S\,\widetilde{\phi} &= -[c, \widetilde{\phi}] - F + \gamma^{(0)}\tilde{c} + \psi^{(1)}\widetilde{A} + f^{(2)}\phi \\
S\,\phi &= -[c, \phi] + i_\gamma \widetilde{A} \\
S\,\widetilde{A} &= -[c, \widetilde{A}] - D\,\phi + i_\gamma \widetilde{c} \\
S\,\widetilde{c} &= -[c, \widetilde{c}] - [\widetilde{\phi}, \phi] - D\widetilde{A}
\end{aligned}
$$

BRST invariant topological backgrounds

The topological equations

The topological analogues of covariantly constant spinor eqs of supergravity are obtained by setting to zero the BRST variations of the fermionic backgrounds $S\,\psi_{\mu\nu} = S\,\psi^{(1)} = 0$:

$$\mathcal{L}_\gamma g^{\mu\nu} \equiv D^\mu\,\gamma^\nu + D^\nu\,\gamma^\mu = 0$$
$$d\,\gamma^{(0)} - i_\gamma f^{(2)} = 0$$

- Our aim is to solve these equations and classify the solutions modulo BRST trivial ones.

Moduli

- The action depends on the topological backgrounds only through the BRST operator S.
- The BRST operator, in turn, depends on the ghost-for-ghost γ^μ of TG and on the $U(1)$ fields $\gamma^{(0)}$ and $f^{(2)}$ only.
- Therefore, the matter TQFT for BRST invariant backgrounds is automatically independent of any variation of the metric that preserve γ^μ, as well as of any topological variation of the $U(1)$ fields that preserve the class of $f^{(2)}$.

Killing vectors

- The first of the topological equations asserts that γ^{μ} has to be a Killing vector of the metric $g_{\mu\nu}$.
- In 2d this leads to consider the sphere $\mathbb{S}^2$ and the torus $\mathbb{T}^2$, equipped with metrics having a $U(1)$ isometry

S^1-equivariant cohomology

The second equation is

$$(d - \epsilon_\Omega\, i_V)\,(f^{(2)} + \epsilon_\Omega\, f^{(0)}) = 0$$
$$\gamma^{(0)} = \epsilon_\Omega\, f^{(0)} \qquad \text{and} \qquad \gamma^\mu = \epsilon_\Omega\, V^\mu$$

- $f^{(2)} + \epsilon_\Omega\, f^{(0)}$ is the equivariantly closed extension of the ordinary differential form $f^{(2)}$ and $D = d - \epsilon_\Omega\, i_V$ is the Cartan differential.
- V is the Killing vector associated with the S^1-equivariant action;
- ϵ_Ω is the degree-two generator of the ring of the S^1-equivariant cohomology.

S^1-equivariant cohomology of the 2-sphere

- There are two linearly independent equivariant classes x and y of degree-two

$$x = \epsilon_\Omega \qquad y = \widetilde{f}^{(2)} + \epsilon_\Omega \, \tilde{f}^{(0)}$$

$$\widetilde{f}^{(2)} = \sqrt{g}\, \frac{1}{2} \epsilon_{\mu\nu} dx^\mu \, dx^\nu \qquad D^2 \, \widetilde{f}^{(0)} = \sqrt{g}\, \epsilon_{\mu\nu} \, D^\mu \, V^\nu$$

- $\widetilde{f}^{(0)}$ is solved only up to an additive constant: given a choice of this constant, a shift to another value induces the change $y \to y + c\, x$
- We have $y^2 \sim x^2$: i.e. the S^1-equivariant cohomology at any degree is the polynomial ring generated by x and y modulo this relation.

Relation to Supergravity Backgrounds

- The generalized covariantly constant spinor equation for 2D N=(2,2) SUGRA is

$$(D_\mu - i\mathcal{A}_\mu)\,\zeta = -\frac{1}{2}\,H\,\Gamma_\mu\zeta + \frac{i}{2}\,G\,\Gamma_\mu\Gamma_3\zeta$$

- Introduce the spinorial bilinears

$$c_0(x) = \zeta^\dagger(x)\,\zeta(x) \quad c_\mu(x) = \zeta^\dagger(x)\Gamma_\mu\zeta(x) \quad \widetilde{c}_0(x) = \zeta^\dagger(x)\,\Gamma^3\,\zeta(x)$$

which are related by the Fierz identities

$$c^\mu c_\mu = c_0^2(x) - \widetilde{c}_0^2(x)$$

- The equation for the generalized covariantly constant spinor is equivalent to the equations for the bilinears

$$
\begin{aligned}
&D_\mu\, c_\nu + D_\nu\, c_\mu = 0\\
&D_\mu\, \widetilde{c}_0 = -i\, H \sqrt{g}\, \epsilon_{\mu\nu}\, c^\nu\\
&D_\mu\, c_\nu = \sqrt{g}\, \epsilon_{\mu\nu}\, (G\, c_0 + i\, H\, \widetilde{c}_0)\\
&D_\mu\, c_0 = G \sqrt{g}\, \epsilon_{\mu\nu}\, c^\nu
\end{aligned}
$$

- The first two equations are the same as topological equations with the identifications

$$
c^\mu = \gamma^\mu \qquad -i\, H = f \equiv *f^{(2)} \qquad \widetilde{c}_0 = \gamma^{(0)}
$$

"Composite" SUGRA backgrounds

What about the extra two SUGRA equations?

A solution of the first two equations, i.e. of the topological sub-system, identifies a unique solution of the full set of SUGRA equations via the relations

$$c_0 = \sqrt{\gamma^2 + (\gamma^{(0)})^2}$$
$$G = \frac{1}{c_0}\left[\frac{1}{2}\sqrt{g}\,\epsilon_{\mu\nu}\,D^{\mu}\,\gamma^{\nu} + f\,\gamma^{(0)}\right]$$

The topological equations play the role of "minimal subset" for the SUGRA equations.

Topological equivalence

- Topological invariance of the “costituent” topological fields

$$f^{(2)} \to f^{(2)} + \mathrm{d}\,\omega^{(1)} \quad \gamma^{(0)} \to \gamma^{(0)} + i_\gamma(\omega^{(1)})$$
$$\mathcal{L}_\gamma\,\omega^{(1)} = 0$$

induces topological transformations on the “composite” SUGRA fields

$$G^{(2)} \to G^{(2)} + \mathrm{d}\,\widetilde{\omega}^{(1)} \quad c_0 \to c_0 + i_\gamma(\tilde{\omega}^{(1)})$$

where

$$\widetilde{\omega}^{(1)} \equiv \frac{\gamma^{(0)}}{\sqrt{\gamma^2 + (\gamma^{(0)})^2}}\,\omega^{(1)} \qquad \text{and} \qquad \mathcal{L}_\gamma\,\widetilde{\omega}^{(1)} = 0\ .$$

$U(1)_R$ Field Strength Background

- Also the $U(1)_R$ background is obtained, via the integrability condition, from the "costituent" topological fields

$$\begin{aligned}\mathcal{F}_R &= \pm\frac{1}{2}\sqrt{(f^2 - G^2 - R)^2 + D_\mu\, f D^\mu f - D^\mu G\, D_\mu\, G} = \\ &= \frac{\epsilon^{\mu\nu}}{\sqrt{g}}\, D_\mu \left[\frac{(f\, c_0 - G\,\gamma^{(0)})\,\gamma_\nu}{2\,\gamma^2}\right] \equiv \frac{\epsilon^{\mu\nu}}{\sqrt{g}}\, D_\mu\, \mathcal{A}_\nu\end{aligned}$$

- Under a topological transformations of the topological "constituents", $\mathcal{F}_R$ changes by a globally defined total derivative:

$$\mathcal{A}^\mu \to \mathcal{A}^\mu + \left(\frac{\epsilon^{\rho\sigma}\,\partial_\rho\,\omega_\sigma}{2\,c_0\sqrt{g}} - \frac{\gamma^\rho\,\omega_\rho\, G}{2\,c_0^2}\right)\gamma^\mu$$

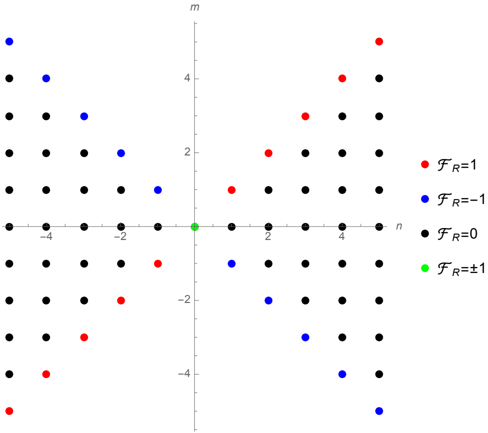
Classifying localizing SUGRA Backgrounds on $\mathbb{S}^2$

- For fixed isometry γ^μ, the general solution of the topological equations is an element of the equivariant S_1 cohomology of degree 2, determined by two parameters which can be identified with the two graviphoton fluxes

$$n \equiv \int_\Sigma \sqrt{g}\, f = \gamma^{(0)}(\pi) - \gamma^{(0)}(0) \qquad m \equiv \int_\Sigma \sqrt{g}\, G$$

- This are quantized if f and G are field strength of abelian gauge fields associated to compact gauge group.

Fluxes of localizing backgrounds on the 2-sphere: $n = \int_\Sigma \sqrt{g}\, f$, $m = \int_\Sigma \sqrt{g}\, G$.



The non-compact duality symmetry

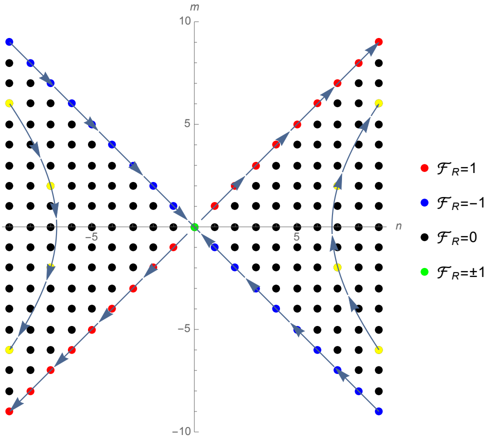
- The SUGRA eqs for the generalized covariantly constant spinor is invariant under a linear global $O(1,1;\mathbf{R})$ duality group

$$\begin{aligned}\begin{bmatrix} f \\ G \end{bmatrix} &\rightarrow \begin{bmatrix} f' \\ G' \end{bmatrix} = \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} f \\ G \end{bmatrix} \\ \zeta &\rightarrow \zeta' = e^{\frac{\alpha}{2}\Gamma_3}\zeta \\ \mathcal{A}_\mu &\rightarrow A'_\mu = \mathcal{A}_\mu \end{aligned}$$

- This duality symmetry is realized non-linearly on the topological "constituents" fields

$$\begin{aligned} f &\rightarrow f' = \cosh\alpha\, f + \sinh\alpha\, G[f,\gamma^{(0)},\gamma^\mu] \\ \gamma^{(0)} &\rightarrow (\gamma^{(0)})' = \sinh\alpha\, c_0[\gamma^{(0)},\gamma^\mu] + \cosh\alpha\ \gamma^{(0)} \end{aligned}$$

The action of the discrete non-compact duality symmetry on fluxes



Summary

- Supersymmetric localization *à la* Pestun can be understood in terms of background (gauged) topological gravity coupled to topological matter QFT's.
- The equations for generalized covariantly constant spinors are recast in terms of "classical" topological equations for equivariant cohomology.
- This appears to be a powerful point of view to classify inequivalent localizing backgrounds and their moduli spaces.
- In 2d a whole set of new localizing backgrounds arises: they are characterized by non-trivial fluxes for both graviphotons and are still to be explored.