Lie n-algebra models and Chern-Simons theory

Patricia Ritter

Napoli, November 19, 2015

Based on:
PR and Christian Sämann: arXiv:1308.4892, arXiv:1507.00972,
and work in progress

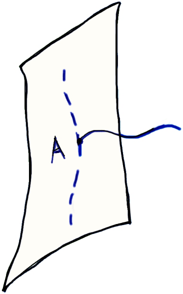
Outline

- Dealing with **extended objects** $\rightarrow$ World-surfaces:
    - Flash-intro to category theory;
    - Surface holonomy and Strict Lie 2-algebras;
    - Definition of Lie n-algebra;
    - Example of n-algebras on n-plectic spaces.
- **0-dimensional models** in quantum gravity:
    - IKKT model and its non-commutative emergent geometries;
    - Generalized AKSZ-construction for higher Chern-Simons theory;
    - Non-commutative n-plectic geometries solving 0-dim. models.

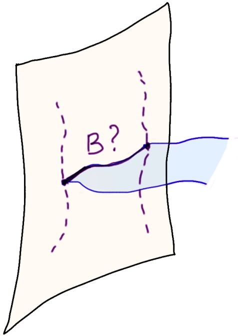
This Exists...

Why do we need higher gauge theory?

- "Everyone" agrees that higher-dimensional objects are needed
  $\rightsquigarrow$ how to implement higher dimensional gauge fields?
- In string- and M-theory:



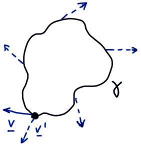
Gauge theory



¿2-Gauge theory?

What is the problem?

Reminder:



Map $\gamma \xrightarrow{F} e^{\mathcal{P}\int A} \in G$, $A \in \Omega^1(M, \mathfrak{g})$, $\mathfrak{g}$: Lie alg. of G

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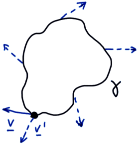

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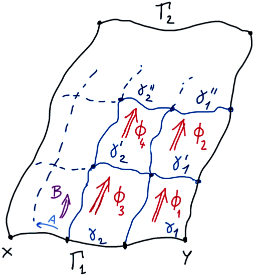
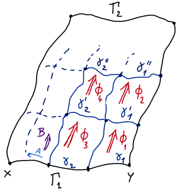
Problem with surface holonomy



- Need $B_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, w. $B \in \mathfrak{h}$, $\mathfrak{h}$: Lie alg. of H
- Problem: $\Phi_4 \cdot \Phi_3 \cdot \Phi_2 \cdot \Phi_1 = \Phi_4 \cdot \Phi_2 \cdot \Phi_3 \cdot \Phi_1$ iff G, H abelian
  $\Rightarrow$ Holonomy ill-defined for non-ablian str. gr. (path-ordering ill defined)!

Solution



Category theory to the rescue!

(Baez, '10)

In particular: **Categories** consist of the following:

Objects	*Morphisms*	2-Morphisms
$\parallel$	$\parallel$	$\parallel$
Points on M	Paths γ_i joining pts.	Surfaces Φ_i joining paths
	$\downarrow\ F$	$\downarrow\ F$
	elements $g \in G$	elements in $G \ltimes H$, $G \neq H$

- associative composition for morphisms
- horizontal ($\cdot$) and vertical ($\circ$) composition for 2-morphisms
- "compatibility" for compositions $\Rightarrow$ well-defined surface ordering

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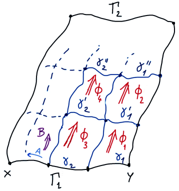

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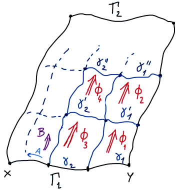

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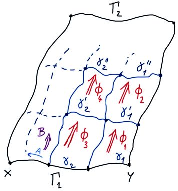

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(Strict) 2-connection

Explicitly: given a **strict** Lie 2-algebra

$$(\mathfrak{g}, \mathfrak{h}, t, \alpha), \quad t : \mathfrak{h} \to \mathfrak{g}, \quad \alpha : \mathfrak{g} \mapsto \mathfrak{aut}(H)$$

where t and α satisfy "composition-compatibility":

$$[g, t(h)] = t(\alpha(g)h), \qquad \alpha(t(h))h' = [h, h'], \quad g \in \mathfrak{g}, \quad h, h' \in \mathfrak{h}$$

that the holonomy functor F has to preserve:

A 2-connection consists of

a 1-form field $A \in \mathfrak{g}$ and a 2-form field $B \in \mathfrak{h}$ with

$$t(B) = F_A \qquad \text{for well-defined holonomy}$$

- **2-curvature** can be introduced: $Z = \mathrm{d}B + \alpha(A) \wedge B$
- **2-gauge transformations** are given by a function $g : M \to G$ AND $a \in \Omega^1(M, \mathfrak{h})$

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Truncated L_∞ algebras

Definition

An L_∞ algebra consists of a graded vector space $L = \oplus_i L_i$ endowed with k-ary graded multilinear totally antisymmetric products μ_k, $k \in \mathbb{N}^*$, satisfying

$$\sum_{i+j=k} \sum_\sigma (\pm) \mu_{j+1}(\mu_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(i+j)}) = 0, \quad (1)$$

for σ denoting the (i,j)-unshuffles. Grading $[\mu_k] = 2 - k$.

Or **dual picture**: graded vec.sp. w. grade 1 vec. field Q, s.t. $Q^2 = 0$:

$$\begin{aligned} Q &= -(v^i t_i^a + \tfrac{1}{2} w^b w^c f_{bc}^a) \frac{\partial}{\partial w^a} + (w^a w^b w^c h_{abc}^i - w^a v^j g_{aj}^i) \frac{\partial}{\partial v^i} \pm \dots \\ &\sim -\mu_1(v) - \mu_2(w,w) + \mu_3(w,w,w) - \mu_2(w,v) \pm \dots \end{aligned}$$

where e.g. $[w^a] = 1$, $[v^i] = 2, \dots$
Familiar from: BV-quantization, string field theory, higher spin algebras, ...

Truncated L_∞ algebras

Truncation

A **semi-strict Lie 2-algebra** is a truncated L_∞ algebra, for which $\mu_{n>3} = 0$. That is, it is a 2-term complex of real vector spaces $L_{-1} \equiv V$ and $L_0 \equiv W$

$$V \xrightarrow{\mu_1} W \xrightarrow{\mu_1} 0 \ , \tag{1}$$

equipped with products μ_1, μ_2, μ_3 satisfying

$$\mu_1^2(v) = 0 \ , \quad \mu_1(w) = 0 \ , \quad \mu_2(v_1, v_2) = 0 \ ,$$
$$\mu_1(\mu_2(w, v)) = \mu_2(w, \mu_1(v)) \ , \quad \mu_2(\mu_1(v_1), v_2) = \mu_2(v_1, \mu_1(v_2)) \ ,$$
$$\mu_3(v_1, v_2, v_3) = \mu_3(v_1, v_2, w) = \mu_3(v_1, w_1, w_2) = 0 \ ,$$

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$$\mu_1(\mu_3(w_1, w_2, w_3)) = -\mu_2(\mu_2(w_1, w_2), w_3) - \mathrm{cyclic}(1, 2, 3) \ ,$$
$$\mu_3(\mu_1(v), w_1, w_2) = -\mu_2(\mu_2(w_1, w_2), v) - \mathrm{cyclic}(w_1, w_2, v)$$

n- and 2-plectic manifolds

Definition

A manifold M that has a closed $(n+1)$-form ϖ that is non-degenerate, i.e. s.t. $\iota_X \varpi = 0 \Leftrightarrow X = 0$, is called a **$n$-plectic manifold**. ϖ is a **n-plectic form**. This is the categorification of a symplectic structure.

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The "*Poisson 2-algebra*"

Hamiltonian 1-forms $\mathfrak{H}(M) \equiv \{\alpha \in \Omega^1(M) \big| \exists X_\alpha \in TM \text{ s.t. } d\alpha = \iota_{X_\alpha}\varpi\}$.

Semistrict Lie 2-algebra $\Pi_{M,\varpi}$: $V \oplus W := \mathcal{C}^\infty(M) \oplus \mathfrak{H}(M)$ with

$$\pi_1(f) = \mathrm{d}f \ , \quad \pi_2(\alpha,\beta) = \iota_{X_\alpha}\iota_{X_\beta}\varpi \ , \quad \pi_3(\alpha,\beta,\gamma) = \iota_{X_\alpha}\iota_{X_\beta}\iota_{X_\gamma}\varpi \ ,$$

for $f \in \mathcal{C}^\infty(M)$ and $\alpha, \beta, \gamma \in \Omega^1(M)$.

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for $f \in \mathcal{C}^\infty(M)$ and $\alpha,\beta,\gamma \in \Omega^1(M)$.

Examples: $\mathbb{R}^3$ and S^3 with respective volume forms, $\varpi = \mathrm{dvol}$.

- $\mathbb{R}^3$ with basis ∂_i, Hamiltonian 1-forms $\xi_i = \frac{1}{2}\epsilon_{ijk}x^j\mathrm{d}x^k$:

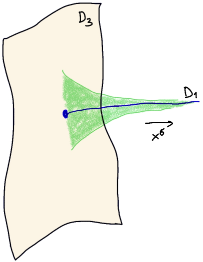
  $$\pi_2(\xi_i,\xi_j) = -\epsilon_{ijk}\mathrm{d}x^k \ , \quad \pi_3(\xi_i,\xi_j,\xi_k) = -\epsilon_{ijk}, \qquad \big(\pi_2(f,\xi_i) = \partial_i f\big) \ ,$$

  kind of **Heisenberg 2-algebra**.

- S^3 with $\varpi = \sin\eta_1\cos\eta_1\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3$ (Hopf coord.s), $\xi_i = \frac{1}{2}\epsilon_{ijk}\eta^j\mathrm{d}\eta^k$,

  $$X_{\xi_i} = \frac{1}{|\varpi|}\partial_i \ , \quad \pi_2(\xi_i,\xi_j) = \frac{\epsilon_{ijk}\mathrm{d}\eta^k}{|\varpi|} \ , \quad \pi_3(\xi_i,\xi_j,\xi_k) = -\frac{\epsilon_{ijk}}{|\varpi|^2}$$

Non-commutative spaces as solutions to brane models

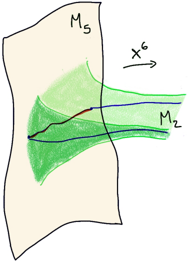


Nahm eq. for $D1 - D3$ system

$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^i + \varepsilon^i{}_{jk}[X^j, X^k] = 0$$

Sol.: $X^i = \frac{1}{x^6}G^i$, with $G^i = \varepsilon^i{}_{jk}[G^j, G^k]$

fuzzy S^2 funnel



Basu-Harvey eq. for $M2 - M5$

$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^\mu + \varepsilon^\mu{}_{\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

Sol.: $X^\mu = \frac{1}{\sqrt{x^6}}G^\mu$, with
$G^\mu = \varepsilon^\mu{}_{\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$

fuzzy S^3 funnel

Lie algebra matrix models

Candidate for non-perturbative actions (string- and M-theory):
E.g.: **IKKT model:** A_μ, ψ : $N \times N$ matrices, $[-,-]$: Lie br.
(Ishibashi, Kawai, Kitazawa, Tsuchiya, '96)

10-dimensional Super-Yang-Mills (SYM) theory

$\downarrow$ reduction to 0 dimensions

$$S_{\mathrm{IKKT}} = \alpha\,\mathrm{tr}\left(-\tfrac{1}{4}[A_\mu, A_\nu]^2 - \tfrac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu,\psi] + \beta\mathbb{1}\right)$$

$$\downarrow \quad \left.\begin{array}{l} [-,-] \rightarrow i\hbar\{-,-\} \\ \mathrm{tr}\,() \rightarrow \int \mathrm{d}\sigma^2\sqrt{g} \end{array}\right\} \text{ classical limit}$$

$$S_{\mathrm{Schild}} = \int_\Sigma \mathrm{d}^2\sigma\sqrt{g}\alpha\left(\tfrac{1}{4}\{X_\mu, X_\nu\}^2 - \tfrac{\mathrm{i}}{2}\bar{\psi}\Gamma^\mu\{X_\mu,\psi\}\right) + \beta\sqrt{g}$$

($X_\mu(\sigma^0,\sigma^1)$, σ^α: coords. on Σ, $\{-,-\}$: Poisson br. on Σ)

Non-commutative solutions

Postulate ***quantization***: $X^m \mapsto \hat{X}^m = A^m$,

$$\rightsquigarrow \{X^m, X^n\} \mapsto i\hbar\widehat{\{X^m, X^n\}} = [\hat{X}^m, \hat{X}^n] = [A^m, A^n]$$

Solution to IKKT	emergent geometry	Quant.^{n} of
Heisenberg: $[\hat{X}^m, \hat{X}^n] = i\theta^{mn}\mathbb{1}$	Moyal Plane $\mathbb{R}^n_\theta$	$\mathbb{R}^n$
$\begin{pmatrix}\text{deformed}\\ \text{IKKT}\end{pmatrix}$ $SU(2)$: $[\hat{X}^m, \hat{X}^n] \sim \epsilon^{mn}_k \hat{X}^k$	Fuzzy sphere	S^2

- Expanding IKKT around a solution $\rightsquigarrow$ YM-theory on non-commoutative space (Ishibashi, et al., '99)
- Same game is played for gravity (Steinacker, '11)

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Higher Chern-Simons models

Generalized AKSZ-construction

NQ-manifold: $(T[1]L[1], \mathrm{d}_{\mathsf{CE}})$;

Space-time: $(T[1]\Sigma, \mathrm{d})$

Coords.: $\xi^A = \{w^a,\ v^i,\ \ldots\}$
graded $(1, 2, \ldots)$;

Coords.: $(x^\mu, c^\mu(x) \sim \mathrm{d}x^\mu)$;

$\xrightarrow{a}$

Q-str.: de Rham d;

Q-str.: Chevalley-Eilenberg d_{CE};

$\xleftarrow{a^*}$

Fields: $a^*(w^a) := A^a_\mu \mathrm{d}x^\mu$,
$a^*(v^i) := B^i_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$,
etc.

Invariant polynomials:
$\omega = \mathrm{d}_{\mathsf{CE}}\xi^A \omega_{AB} \mathrm{d}_{\mathsf{CE}}\xi^B$

$\alpha = \xi^A \omega_{AB} \mathrm{d}_{\mathsf{CE}}\xi^B$

Higher Chern-Simons models

- General n-algebra C.S.-model:

$$S_{\mathrm{CS}} = \int_\Sigma \left((\langle \phi, \mathrm{d}_\Sigma \phi \rangle) + \sum_{k=1}^{n+1} \frac{(-1)^{k(k+1)/2}}{(k+1)!} \langle \mu_k(\phi, \ldots, \phi), \phi \rangle \right)$$

with equations of motion:

$$\frac{\overleftarrow{\partial} S_{\mathrm{CS}}}{\partial \phi^A} = (-2\mathrm{d}_\Sigma \phi^B \omega_{BA}) + \sum_{k=1}^{n+1} \frac{(-1)^{k(k+1)/2}}{k!} \mu_k(\phi, \ldots, \phi)^B \omega_{BA} = 0$$

Higher Chern-Simons models

- 4-dimensional, Lie 2-algebra C.S.-theory:

$$S_{\mathrm{AKSZ}} = \int_{\Sigma} \Big[\big(-B, \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) - \tfrac{1}{2}\mu_1(B) \big) - \tfrac{1}{4!}\big(A, \mu_3(A,A,A)\big) \Big]$$

with equations of motion:

$$0 = (\mathrm{d}A) + \frac{1}{2}\mu_2(A,A) - \mu_1(B) = \mathcal{F} \ ,$$
$$0 = (\mathrm{d}B) + \mu_2(A,B) - \frac{1}{6}\mu_3(A,A,A) = \mathcal{H} \ .$$

Solving Lie n-algebra C.S.-theory

To minimize 0-dimensional reduced higher C.S.-theory:

$$\sum_{k=1}^{n+1} \frac{(-1)^{k(k+1)/2}}{k!} \mu_k(\phi, \ldots, \phi)^B \omega_{BA} = 0$$

i.e. higher Maurer-Cartan equations $\equiv$ nill higher fake curvatures!

Use "n-Heisenberg algebra" on n-plectic $\mathbb{R}^{n+1}$:

$$\pi_1 \sim \mathrm{d} \ , \qquad \pi_k(\xi_{i_1}, \ldots, \xi_{i_k}) = (-1)^{k(k+1)/2} \iota_{X_{i_1}} \cdots \iota_{X_{i_k}} (\mathrm{dvol}) \ ,$$

for Hamiltonian $(n-1)$-forms $\xi_{i_0} = \frac{1}{n!} \varepsilon_{i_0 \cdots i_n} x^{i_1} \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_n}$
and $(n-k)$-forms $\phi_{i_1 \cdots i_k} \sim \epsilon_{i_1 \cdots i_{n+1}} \ x^{i_{k+1}} \mathrm{d}x^{i_{k+2}} \wedge \cdots \wedge \mathrm{d}x^{i_{n+1}}$

- Can obtain different spaces by adding deformations to the model;
- Expaning 0-dim. action around solutions, yields C.S.-theory on non-commutative background.

Thank you!