Exact results for supersymmetric gauge theories on 4d (and 5d) compact manifolds

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Supersymmetry algebra

$\mathcal{N} = 2$ multiplet: bosons $\{\phi, \tilde{\phi}, A\}$ and fermions $\{\eta, \psi, \chi^+\}$

$$\phi, \tilde{\phi}, \eta \in \Omega^0(M), \quad A, \psi \in \Omega^1(M), \quad \chi^+ \in \Omega^{2+}(M)$$

Witten equivariant twisted algebra
(obtained twisting rotation group by R-symmetry)

$$\mathcal{Q} = \varepsilon Q + \varepsilon^\mu(V) Q_\mu$$

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generates

$$\mathcal{Q}A = \psi, \qquad \mathcal{Q}\psi = D\phi + i\iota_V F, \qquad \mathcal{Q}\phi = 0 + i\iota_V \psi$$

$$\mathcal{Q}\tilde{\phi} = \eta, \qquad \mathcal{Q}\eta = i[\phi, \tilde{\phi}] + i\,\iota_V D\tilde{\phi},$$

$$\mathcal{Q}\chi^+ = F^+,$$

Squares on bosonic symmetries $\mathcal{Q}^2 = \mathrm{Gauge}(\phi) + \mathcal{L}_V$

Toric 4-Manifold

$M\,(\mathbb{CP}^2)$ with affine coordinate patches

$$U_\ell \cong \mathbb{C}^2, \qquad \ell = 1, \dots, \chi(M)\,(=3)$$

The isometry generated by the killing vector V acts on local coordinates as

$$U_\ell \ni (z_1^{(\ell)}, z_2^{(\ell)}) \mapsto (e^{i\epsilon_1^{(\ell)}} z_1^{(\ell)}, e^{i\epsilon_2^{(\ell)}} z_2^{(\ell)})$$

with isolated fixed points

$$V(P_\ell) = 0, \quad P_\ell \in M, \quad \ell = 1, \dots, \chi(M)$$

Moreover $dim\, H^{2+}(M) = 1$ and its generated by the Kähler form ω_K and $dim\, H^{2-}(M) = \chi(M) - 3$

Localization of the path integral

Fields of twisted $\mathcal{N} = 2$ multiplet $\varphi = \{\phi, \tilde{\phi}, A, \eta, \psi, \chi\}$.
Evaluate the path integral

$$\mathcal{Z}[\not{s}] = \int \mathcal{D}\varphi e^{-S[\varphi] - s\mathcal{QV}[\varphi]}$$

Where $\mathcal{QV}[\varphi] \geq 0$

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In the limit $s \to \infty \Rightarrow$ localizes on $\hat{\varphi}$ s.t. $\mathcal{QV}[\hat{\varphi}] = \mathcal{QV}'[\hat{\varphi}] = 0$

$$\mathcal{Z} = \int_{\mathcal{M}} d\hat{\varphi}\, e^{-S[\hat{\varphi}]} \underbrace{\int \mathcal{D}\delta\varphi\, e^{-\frac{1}{2}\mathcal{QV}''[\hat{\varphi}]\delta\varphi^2}}_{\substack{\text{Gaussian integration} \\ \text{(index theorem)}}}$$

Topological theory

Topological term, with gauge group $G = U(N)$

$$S[\varphi] = \frac{i\tau}{4\pi}\int_M \mathrm{Tr} F \wedge F$$

Localizing action, $\lambda =$ const.

$$\mathcal{QV}\big|_{\text{bos}} = \int_M |\mathcal{Q}\eta|^2 + |\mathcal{Q}\psi|^2 + |\mathcal{Q}\chi^+ - \lambda\,\omega_K \mathbb{1}|^2$$

notation: $|\alpha|^2 = \alpha \wedge \star\alpha^\dagger$.

Decomposition of self dual 2-form

$$\chi^+ \simeq \chi^{(2,0)} + \chi^{(0,2)} + \chi^{(1,1)}_{\|\omega_K}$$

Supersymmetric minima

Look at the $\mathcal{Q}$-transformation of the fermions

$$\left.\begin{aligned} \mathcal{Q}\eta &= \iota_V D\tilde{\phi} + [\phi, \tilde{\phi}] = 0 \\ \mathcal{Q}\psi &= i\iota_V F + D\phi = 0 \end{aligned}\right\} \quad \begin{matrix} \phi^\dagger = -\tilde{\phi} \\ \Longrightarrow \end{matrix} \quad \begin{cases} [\phi, \tilde{\phi}] = \iota_V d\tilde{\phi} = 0 \\ [F, \phi] = [F, \tilde{\phi}] = 0 \\ \iota_V F = -id\phi \\ (\mathbf{3} \text{ equations}) \end{cases}$$

and

$$(\mathcal{Q}\chi^+ - \lambda\omega_K \mathbb{1}) = 0 \quad \Longrightarrow \quad \begin{cases} F^{(2,0)} = F^{(0,2)} = 0 \\ F^{(1,1)} \wedge \omega_K = \lambda\omega_K \wedge \omega_K \mathbb{1} \\ (\mathbf{3} \text{ eq, ''Hermitian-Yang-Mills''}) \end{cases}$$

Supersymmetric minima

- $(\iota_V - d)(F + \phi) = 0$
- $F = F^- + \lambda\omega_K \mathbb{1} \;\Rightarrow\; d^\star F = 0$

This implies that $F + \phi \in \Omega^0(M) \oplus \Omega^2(M)$ is an equivariant harmonic 2-form: $F + \phi \in H^2_V(M, \mathbb{Z})$

In *toric manifolds* $H^2_V(M, \mathbb{Z})$ has $\chi(M)$ generators $\omega^{(\ell)}(\epsilon_1, \epsilon_2)$

$$F + \phi = \sum_{\ell=1}^{\chi(M)} k^{(\ell)} \omega^{(\ell)} + a$$

with $k^{(\ell)} \in \mathbb{Z}$ and $a = \text{const.}$
Can be written as two different equations for the 2-form and the 0-form. 2-form: magnetic fluxes around 2-cycles

Equivariant weights

From 0-form we can read the expression ϕ at the fixed points P_ℓ

$$\phi^{(\ell)} := \phi(P_\ell) = a + k^{(\ell)}\epsilon_1^{(\ell)} + k^{(\ell+1)}\epsilon_2^{(\ell)}$$

Calculation of the partition function localizes into the fixed points P_ℓ of the isometry generated by V. Around each fixed points the coordinate patch looks like a copy of $\mathbb{C}^2$ with an isometry generated by $V(\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)})$ and gauge transformation $\phi^{(\ell)}$.

This is the so called "Ω-background". Nekrasov calculated the partition function in this case ($q = \exp 2\pi i\tau$.)

$$\mathcal{Z}^{\mathbb{C}^2}(\phi, \epsilon_1, \epsilon_2, q) = Z^{\mathbb{C}^2}_{\text{pert}}(\phi, \epsilon_1, \epsilon_2) Z^{\mathbb{C}^2}_{\text{inst}}(\phi, \epsilon_1, \epsilon_2, q)$$

$\Rightarrow$ $\mathcal{Z}^M$ will be given as product of copies of $\mathcal{Z}^{\mathbb{C}^2}$

Integration of zero modes

Zero modes remain to be integrated in the path integral. The zero mode of the scalar ϕ is a. But being complex in principle we have

$$\int da\, d\bar{a}\, d\eta_0\, d\chi_0^+$$

where η_0 and χ_0^+ are the fermionic zero modes that are always present in toric manifold because $dim\, H^{2+}(M) = 1$. These fermionic zero modes don't appear in the integrand and make the measure ill-defined.

We have to insert an observable

$$\int_D da\, d\bar{a}\, d\eta_0\, d\chi_0 e^{-\mathcal{Q}\int_M \mathrm{Tr}\bar{a}\chi_0\wedge\omega_K} = \cdots = \int_D da\, d\bar{a}\, \frac{\partial}{\partial\bar{a}} = \oint_{\partial D} da$$

Coulomb branch with gauge group maximally broken to $U(1)^N$ if the v.e.v. of $\alpha\phi \neq 0$, $\forall\alpha$ roots of G. Therefore $D = \mathbb{C}^N \setminus \bigcup_\alpha \{\alpha(a) = 0\}$.

The partition function

The resulting partition function on M is written as a contour integral. The integrand is the product of copies of the partition function on $\mathbb{C}^2$ with equivariant weights $\epsilon_1^{(\ell)}(\epsilon_1, \epsilon_2)$, $\epsilon_2^{(\ell)}(\epsilon_1, \epsilon_2)$, $\phi^{(\ell)}(a, \epsilon_1, \epsilon_2)$

$$\mathcal{Z}^M(a, \epsilon_1, \epsilon_2, q) = \sum_{\{k^{(\ell)}\}} \oint da \; q^{-S_0(\{k^{(\ell)}\})} \prod_{\ell=1}^{\chi(M)} \mathcal{Z}^{\mathbb{C}^2}(\phi^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, q)$$

with $\mathcal{Z}^{\mathbb{C}^2}(\phi, \epsilon_1, \epsilon_2, q) = Z^{\mathbb{C}^2}_{\text{pert}}(\phi, \epsilon_1, \epsilon_2) Z^{\mathbb{C}^2}_{\text{inst}}(\phi, \epsilon_1, \epsilon_2, q)$

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We have to study the analytic structure of the integrand. The perturbative part is

$$Z_{\text{pert}}^{\mathbb{C}^2}(\phi, \epsilon_1, \epsilon_2) \simeq \Gamma_2(\phi|\epsilon_1, \epsilon_2)^{-1} \simeq \prod_{m,n=0}^{\infty} (\phi + m\epsilon_1 + n\epsilon_2)$$

and...

...the non-perturbative (instanton) term

$$Z^{\mathbb{C}^2}_{\text{inst}}(\phi, \epsilon_1, \epsilon_2, q) = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}^k_{ASD}(\phi,\epsilon_1,\epsilon_2)} \star 1$$

Very complicated combinatorical expression in terms of Young diagrams.

But, if $G = U(2)$ we know

Nekrasov instanton partition function	AGT $\longleftrightarrow$	Virasoro algebra conformal blocks

Can use Zamolodchikov's recursion relation for Virasoro conformal blocks translated by the AGT dictionary [Poghossian]

$$Z^{\mathbb{C}^2}_{\text{inst}}(\phi, \epsilon_1, \epsilon_2, q) = 1 - \sum_{m,n=1}^{\infty} \frac{q^{mn} R^{(\epsilon_1,\epsilon_2)}_{m,n} Z^{\mathbb{C}^2}_{\text{inst}}(m\epsilon_1 - n\epsilon_2, \epsilon_1, \epsilon_2, q)}{\phi^2 - (m\epsilon_1 + n\epsilon_2)^2}$$

Simple poles at $\phi = \pm(m\epsilon_1 + n\epsilon_2)$ with $m, n \in \mathbb{Z}_{>0}$

Case of $\mathbb{CP}^2$: instantons

$$Z_{\text{inst}}^{\mathbb{CP}^2}(a, \epsilon_1, \epsilon_2, q) = \prod_{\ell=1}^{3} Z_{\text{inst}}^{\mathbb{C}^2}(\phi^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, q)$$

Case of $\mathbb{CP}^2$: instantons

$$Z^{\mathbb{CP}^2}_{\text{inst}}(a,\epsilon_1,\epsilon_2,q)=\prod_{\ell=1}^{3} Z^{\mathbb{C}^2}_{\text{inst}}(\phi^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},q)$$

Analytic structure of the instanton part:

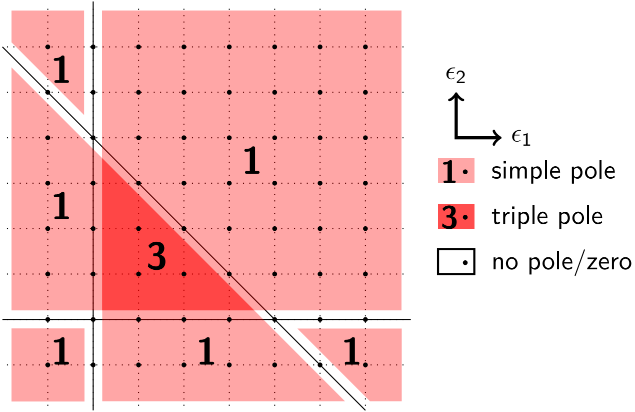

Figure: Poles of instanton partition function.

Case of $\mathbb{CP}^2$: perturbative term

$$Z^{\mathbb{CP}^2}_{\rm pert}(a,\epsilon_1,\epsilon_2) = \prod_{\ell=1}^{3} \Gamma_2(\phi^{(\ell)}|\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}) = \prod_{\rm finite}(a+m\epsilon_1+n\epsilon_2)$$

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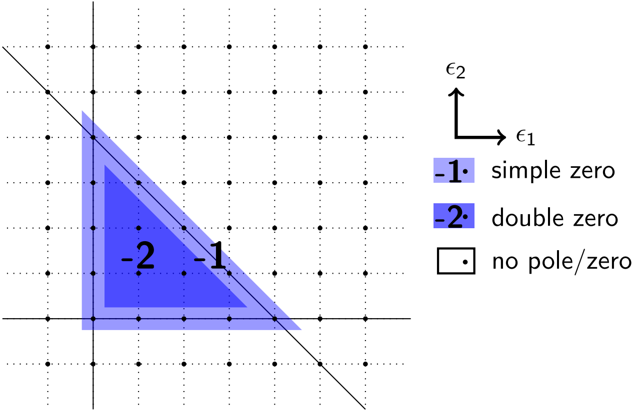

Figure: Zeros of one-loop partition function.

Case of $\mathbb{CP}^2$: analytic structure of the integrand

Put together $Z^{\mathbb{CP}^2}_{\text{inst}}(a,\epsilon_1,\epsilon_2,q)Z^{\mathbb{CP}^2}_{\text{pert}}(a,\epsilon_1,\epsilon_2)$

Analytic structure of the partition function for $\mathbb{CP}^2$
(balance between perturbative and non perturbative part)

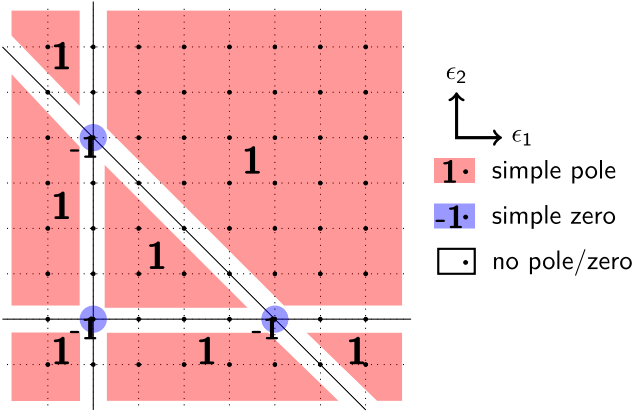

Figure: Poles of the full partition function.

(5d case)

$U(2)$ partition function on squashed S^5,

$$\mathcal{Z}^{S^5} = \int_{\mathbb{R}} d\sigma\, e^{-\Lambda\sigma^2} S_3(i\sigma) S_3(-i\sigma) \prod_{\ell=1}^{3} Z_{\text{inst}}^{\mathbb{C}^2\times S^1}(i\sigma, \vec{\omega}^{(\ell)}, \Lambda)$$

where $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ are the squasching parameters.

We can write a recursion relation for $Z_{\text{inst}}^{\mathbb{C}^2\times S^1}$ analogous to the $4d$ case. (Its CFT dual should give a recursion relation for q-deformed Virasoro conformal blocks)

(5d case)

Also in this case the perturbative and the non-perturbative term balance. Infact

$$Z^{S^5}_{\text{pert}} = S_3(i\sigma)S_3(-i\sigma) \quad \rightsquigarrow \quad \begin{array}{l}\text{double zeros in} \\ i\sigma = \vec{n}\cdot\vec{\omega}, \quad n_i > 0\end{array}$$

$$Z^{S^5}_{\text{inst}} = \prod_{\ell=1}^{3} Z^{\mathbb{C}^2\times S^1}_{\text{inst}}(\ell) \quad \rightsquigarrow \quad \begin{array}{l}\text{triple poles in} \\ i\sigma = \vec{n}\cdot\vec{\omega}, \quad n_i > 0 \\ \text{simple poles elsewhere}\end{array}$$

Therefore eventually only simple poles!

Case of $\mathbb{CP}^2$: result of the integration

The path integral with the insertion of the equivariant observables gives the generatig function for equivariant Donaldson invariant

$$\mathcal{Z}^{\mathbb{CP}^2}(\epsilon_1, \epsilon_1, q) = \sum_{\{k^{(\ell)}\}} q^{-\frac{1}{4}\left[\sum_{\ell=1}^{3}(k^{(\ell)})^2 - \sum_{\ell' \neq \ell} k^{(\ell)} k^{(\ell')}\right]} \prod_{\{(i,j)\} \in D(\{k^{(\ell)}\}) \cap \mathbb{Z}^2} \frac{1}{i\epsilon_1 + j\epsilon_2}$$

$$\prod_{\ell=1}^{3} Z_{\text{inst}}^{\mathbb{C}^2}\left(k^{(\ell)}\epsilon_1^{(\ell)} - k^{(\ell+1)}\epsilon_2^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, q\right)$$

Expanding in series of q and taking the appropriate limit $\epsilon_1, \epsilon_2 \to 0$ gives non equivariant Donaldson polynomials.

Link with SW geometry

In the non equivariant limit

$$\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2\log\mathcal{Z}^{\mathbb{C}^2}(a,\epsilon_1,\epsilon_2,q)=\mathcal{F}(a,q)+\text{higher terms}$$

$\mathcal{F}$ is the Seiberg-Witten prepotential of the $\mathcal{N}=2$ theory.
Therefore

$$\oint da\prod_{\ell=1}^{\chi(M)}\mathcal{Z}^{\mathbb{C}^2}(a,\epsilon_1,\epsilon_2,q)\quad\rightsquigarrow\quad\int du\,\mathcal{F}(u,q)$$

$$\uparrow\qquad\qquad\qquad\qquad\uparrow$$

residue at $a=0$ | residue at $u=\infty$

u-plane integral calculating Donaldson invariants and wallcrossing formulae in terms of modular forms

Case of $\mathbb{CP}^2$: maximally supersymmetric case

We can add a hypermultiplet in the adjoint representation with mass m, so called $\mathcal{N} = 2^\star$. Performing the limit $m \to 0$ (after the integration) we recover the maximally supersymmetric case $\mathcal{N} = 4$. The result is

$$\mathcal{Z}^{\mathbb{CP}^2}_{\mathcal{N}=4}(q) = \left(q^{-1/24}\eta(q)\right)^{-6} \sum_{\{k^{(\ell)}\}} q^{-\frac{1}{4}\left[c_1^2 + \sum_{\ell=1}^{3}(k^{(\ell)})^2 - \sum_{\ell' \neq \ell} k^{(\ell)} k^{(\ell')}\right]}$$

That reproduce the Vafa-Witten mock modular form ($\sim$ holomorphic part of non-holomorphic modular form) generating Euler characteristic of instanton moduli spaces

$$\mathcal{Z}_0(q) = \left(q^{-1/24}\eta(q)\right)^{-6} \sum_{n=0}^{\infty} 3H(4n)q^n \qquad c_1 = 0$$

$$\mathcal{Z}_1(q) = \left(q^{-1/24}\eta(q)\right)^{-6} \sum_{n=0}^{\infty} 3H(4n-1)q^n \qquad c_1 = 1$$

($H(n)$ is the Hurwitz class number)

Holomorphic anomaly (working progress)

In principle the partition function depends on both $\tau, \bar{\tau}$. Anyway in the twisted theory the dependence on $\bar{\tau}$ is $\bar{\tau}\mathcal{Q}(\cdots)$, so the partition function actually does not depend on it. It is holomorphic in τ but it is not modular.

Vafa and Witten argued that under certain circumstances the $\bar{\tau}$ independence could be anomalous restoring the modular property of the partition function. And that the anomaly could be connected with the presence of abelian connections.

Holomorphic anomaly (working progress)

If we localize on (anti)-instanton $F^{\mp} = 0 \Rightarrow F = F^{\pm}$ we have

$$S = \frac{i\tau}{4\pi} \int_M \mathrm{Tr} F \wedge F \rightsquigarrow i \frac{\theta}{16\pi^2} (\pm |k|) = \begin{cases} -\bar{\tau}|k| & \text{instanton} \\ -\tau|k| & \text{anti-instanton} \end{cases}$$

But we have Hermitian-Yang-Mills equation

$$F^{+} = \lambda \omega_K \mathbb{1} \quad \Rightarrow \quad F = F^{-} + \lambda \omega_K \mathbb{1}$$

so there is a self-dual component parallel to the Kähler form that has to be counted with the anti-holomorphic coupling $\bar{\tau}$, maybe restoring modular properties.

Conclusion

- Significant balance between perturbative and non perturbative part ($U(2)$) 4d and 5d.

  Moreover: [pert] + [non-pert] recovers modular S-duality properties (Francisco's talk and 5d $U(1)$)

- (Test for the correctness of the instanton partition function factorization) thanks!