Supersymmetry on curved spaces and exact partition functions

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Abstract

- I'll review the technique of equivariant localization of supersymmetric path integrals and discuss some properties of space-time manifolds where supersymmetry allows the exact evaluation of gauge theories partition functions.

  As notable examples, we discuss the general structure of partition functions of $\mathcal{N} = 2$ $D = 4$ gauge theories on compact (and non compact) toric surfaces and give some new results about the path integral evaluation for $\mathcal{N} = 1^*$ and $\mathcal{N} = 2$ supersymmetric gauge theories on the five sphere.

Introduction

- Quantum Field Theory is a tool of paramount importance in the description of reality at subatomic level (High Energy Particle Physics) as well as at larger scales (Condensed Matter and Statistical Mechanics).
- While its Perturbative formulation is very well understood and fruitful, from both the above viewpoints it is very desirable to be able to understand and keep under control strong coupling effects.
- Perturbative series, when concretely calculable at higher order, are only asymptotic ones and can not be resummed as they are (neither in principle) to give a reliable picture at finite coupling.
- In the last years the exploration of non perturbative aspects of QFTs got boosted at least in two directions:

  $\rightarrow$ new exact results in supersymmetric gauge theories

  and
  $\rightarrow$ analysis of complex saddles in path integrals [G. Dunne and many others]

Introduction

New exact results in supersymmetric gauge theories

These new results consist of

- Exact computation of novel observables in QFTs
- New insights into non-perturbative dynamics (RGE flows / dualities)
- Connections with other areas of physics and mathematics

and are technically obtained as exact path integral evaluations.
$\rightarrow$ The computation technique is a very elegant refinement of the non-rinormalisation theorems in supersymmetric quantum field theories.
$\rightarrow$ Mathematically, it is an adaptation of the equivariant localization to the path-integral which allows, in certain cases, its exact calculation. [much more on this later]

The main theme is *supersymmetric quantum field theories on non-trivial background geometries*.

Introduction

- A first example is the Witten index:

$$Z(T^D) = Tr_{\mathcal{H}}(-1)^F e^{-\beta H}$$

$\rightarrow$ Order Parameter for Susy breaking in strongly coupled gauge theories
$\rightarrow$ up to subtleties, captures $N^B_{E=0} - N^F_{E=0}$ and is exactly computed

- A second example is the Nekrasov partition function for $\mathcal{N} = 2$ theories on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$

$$Z\left(\mathbb{R}^4_{\epsilon_1,\epsilon_2}\right) = \sum_n q^n \int_{\mathcal{M}_n} d\mu = e^{-\frac{1}{\epsilon_1\epsilon_2}(\mathcal{F}_{SW}+O(\epsilon_1,\epsilon_2))}$$

$\rightarrow$ computes the equivariant volumes of instanton moduli spaces
$\rightarrow$ captures the exact IR physics by the Seiberg-Witten prepotential
$\rightarrow$ is the generating function of an infinite class of observables
$\star$ Equivariant vortex counting analog on $\mathbb{R}^2_\varepsilon$

Introduction

- More in general, gauge theories in curved spacetime yield novel observables
$\rightarrow$ These probe the theory with new scale parameters
$\rightarrow$ The generating functions are non trivial functions of the parameters of the theory
- Partition function on $S^1 \times S^{D-1}$ [Romelsberger]
$\rightarrow$ measures short representations of susy, not just the vacuum sector
$\rightarrow$ is a non trivial function of the chemical potentials for global charges
$\rightarrow$ if the theory flows in the IR to a SCFT, computes the superconformal index
- Partition function on S^D [Komalgorsky]
$\rightarrow$ unified description of [c-, F-, a-]theorem as

$$Z(S^D) = e^{-F} \qquad F_{UV} \geq F_{IR}$$

$\rightarrow$ entanglement entropy of SCFT with spherical entangling surface in $\mathbb{R}^{1,D+1}$
• for $\mathcal{N} = 2$ and $D = 4$ it is a non trivial computable function of the gauge coupling [more on this later][Pestun]. Proof of Drukker-Gross conjecture on circular WL in $\mathcal{N} = 4$ and gaussian Matrix Model.

Introduction

New insights into dynamics

- check of S-duality
  → exact matching of S-dual partition functions
  → [Wilson loops] ↔ ['t Hooft loops]
- dynamics of strongly coupled IR fixed points of renormalization group
  → Infrared dualities (Seiberg dualities, Mirror symmetry)
  → matching of conformal dimensions of operators in dual phases

Introduction

New connections in math and phys

- AGT correspondence [Alday-Gaiotto-Tachikawa]

$$Z^{\mathcal{N}=2}_{gauge}(S^4) = Z_{Toda\ field\ th.}(\Sigma_2)$$

- 3d/3d correspondences [Dimofte-Gaiotto–Gukov]

$$Z_{quiver}(S^3) = Z_{Chern-Simons}(M_3)$$

- proof of mirror symmetry, exact Kählex potentials [Benini-Cremonesi,Gomis-Lee]

$$e^{-\mathcal{K}} = Z^{\mathcal{N}=(2,2)}_{GL\sigma M}(S^2) = Z_{mirror\ LG}$$

- quantum integrable systems and gauge theories [Nekrasov-Okounkov-Shatashvili, B.-Sciarappa-Vasko-Tanzini]

Introduction

- These new results follow from the application of *equivariant localization* to *supersymmetric path integrals*.
- This is an extension of the Duistermaat-Heckman theorem to infinite dimensions which we are now going to discuss.

Equivariant Localization in a nut-shell

- Consider a supersymmetric path integral

$$\mathcal{Z} := \int D[X] e^{-S[X]} \tag{1}$$

that is it exists a *scalar supercharge* Q such that the measure in (1) is invariant under the infinitesimal redefinition $\delta X = QX$.

- $Q^2 = R$ is a bosonic symmetry of the theory. We assume it to be *compact* [a torus action with compact orbits].
- Then the path integral is supported at the fixed loci of the odd symmetry Q. [Proof: away from fixed loci the field space can be parameterized by an odd parameter θ describing the Q-flow plus transverse modes, but the integrand – being Q-invariant – is θ-independent and $\int d\theta[const] = 0$.]
- To evaluate (1) one deforms the action S by a Q-exact term $S \to S + tQV$, where V is an odd R-invariant fuctional $Q^2 V = RV = 0$ (localizing fermion) as

$$\mathcal{Z}_t := \int D[X] e^{-S[X] - tQV[X]}.$$

This is t-independent [Proof: $-\frac{d}{dt}\mathcal{Z}_t := \int D[X] e^{-S[X]-tQV[X]} QV[X] = \int D[X] Q \left[e^{-S[X]-tQV[X]} V[X] \right] = 0$.] and can be evaluated at large t

$$\mathcal{Z} = lim_{t\to\infty} \mathcal{Z}_t$$

via semiclassical approximation (as if $\hbar = \frac{1}{t}$).

Equivariant Localization in a nut-shell

Note that

- By different choices of V one gets different integral forms of the same object.
- The semiclassical expansion is well defined if along with a properly chosen V a proper choice of reality conditions on the fields is given.

The path integral then localizes on the space M_V of minima of the localizing action $\{QV = 0\}$ as

$$\mathcal{Z} = \int_{\mathcal{M}_V} dX_0 e^{-S[X_0]} \left[\mathrm{sdet}(QV)^{(2)}_{X_0}\right]^{-1/2}$$

There is a common choice of localizing fermion V in supersymmetric theories: distinguish the field coordinate in $X = (\Phi, \Psi)$ in bosons and fermions (Grassmann even and odd fields) and consider $V =< \Psi, Q\Psi >$ where $< \cdot, \cdot >$ is a R-invariant scalar product.

The localizing action can be expanded in the fermions as

$$QV = \left[|Q\Psi|^2\right]_{\Psi=0} + < \Psi, D\Psi > + O(\Psi^4)$$

so that the localization explicitly takes place on the (R-invariant) BPS configurations

$$\mathcal{M}_{BPS} = \left\{\Phi | \, [Q\Psi]_{\Psi=0} \text{ and } \Psi \in Ker[D] \oplus coKer[D]\right\}.$$

The one loop contribution can be determined by making use of the equivariant index theorem.

Equivariant Localization in a nut-shell

To compute the one-loop contribution

- Find coordinates such that $QX_{e,o} = \hat{X}_{o,e}$, $Q\hat{X}_{o,e} = RX_{e,o}$ [the field space is organized in multiplets] and the gauge fixing fermion expands as $V = X_o D X_e + \dots$
- After huge boson-fermion cancelation one stays with (unpaired eigenvalues)

$$[one - loop] = \frac{det_{coker(D)} R_o}{det_{ker(D)} R_e} = \mathcal{E}\mathrm{char} R = \prod_\alpha \rho_\alpha(\epsilon)^{\mu_\alpha}$$

- To compute it: read the weights $(\rho_\alpha, \mu_\alpha)$ from the equivariant index $[indexD] = tr_{ker(D)} e^R - tr_{coker(D)} e^R = \mathcal{C}\mathrm{char} R = \sum_\alpha \mu_\alpha e^{\rho_\alpha(\epsilon)}$
- The index is computed via Atiyah-Singer index formula

$$[indexD] = tr_{KerD} R - tr_{coKerD} R = \sum_{P \in \{fix.pts.\}} \frac{tr_{KerD(P)} R - tr_{coKerD(P)} R}{det_{T\mathcal{M}_{\mathcal{BPS}}(P)}(1 - R)}$$

The final result typically is in the form

$$\mathcal{Z}(\zeta, \epsilon) = \int_{\mathcal{M}_{BPS}} dm\; e^{-S(\zeta, m)} [one - loop](\epsilon, m) \tag{2}$$

where ζ are background couplings by which the original action depends and ϵ are the equivariant parameters in the maximal torus of the global symmetry as $R = e^\epsilon$.

Supersymmetric observables & equivariant cohomology

Analogously to the partition function one can evaluate the correlation functions of supersymmetric observables, that is the Q-cohomology ring $Ker\, Q / Im\, Q$ on the R–invariant fields space.

- On the field space therefore susy acts as an R-equivariant differential
- The ring of susy observables corresponds to the R-equivariant cohomology of the supercharge Q.
- Equivariant localization is basically an equivariant contracting homotopy which reduces the computation of the equivariant Q-cohomolgy

  $$[Ker\, Q / Im\, Q]_R \sim H^\bullet \left(\mathcal{M}_{BPS}\right)_R$$

  to the moduli space of BPS configurations
- The susy path-integral accordingly reduces to the integration over the moduli space and defines the intersection theory of the ring of susy observables.

Supersymmetric gauge theories and integrable systems

- In the general situation the BPS moduli space is stratified by discrete topological charges (e.g. instanton number, vortex number, etc.) as $\mathcal{M}_{BPS} \sim \cup_n X_n$.
- The topological charge is typically additive in the sense of clustering of finite action solutions

$$X_n \times X_{n'} \hookrightarrow X_{n+n'}$$

- This makes $\mathcal{M}_{BPS}$ an H-space.
- The construction is equivariant, so promotes to equivariant cohomology.
- By restriction and Kunneth theorem one obtains a graded coproduct on the resummed equivariant cohomology $H_T^\bullet(\mathcal{M}_{BPS}) \equiv \oplus_n q^n H_T^\bullet(X_n)$
- promoted to a full Hopf algebra structure: Quantum Integrable system at work!

Rigid supersymmetry and curved spaces

- Let's now look for concrete realizations of the above situation.
- Curved spaces where rigid supersymmetry is defined
- The program can be realized in different space-time dimensions and with different amounts of supersymmetries
- Let's focus on four and five dimensions

Introduction: 4d $\mathcal{N} = 1$

Let's start with an *appetizer* by approaching in a constructive perspective the question: which properties should a (euclidean) four manifold have to host a quantum field theory with rigid $\mathcal{N} = 1$ supersymmetry?
In order to answer, let's study the superalgebra on multiplets.
[N.B. This is just $\frac{1}{2}$ of the question: the other half is then when these properties are strong enough to use equivariant localization to compute the supersymmetric path integral in a closed form.]

Introduction: 4d $\mathcal{N} = 1$

Let's obtain the supersymmetry algebra with one supercharge ($\delta^2 = 0$) parametrized by a chiral spinor ξ_α of R-charge $+1$, as represented on a vector multiplet (in Wess and Bagger notation a gauge field $A_{\alpha\dot{\alpha}}$, gauginos λ_α and $\tilde{\lambda}_{\dot{\alpha}}$, and an auxiliary field D). Supersymmetric variations of the gauge field and the gauginos are fixed by Lorentz covariance and R-charge conservation to be:

$$\delta A_{\alpha\dot{\alpha}} = \xi_\alpha \tilde{\lambda}_{\dot{\alpha}}, \ \delta\tilde{\lambda}_{\dot{\alpha}} = 0, \ \delta\lambda_\alpha = i\xi_\alpha D + (F^+)_{(\alpha\beta)}\xi_\beta.$$

- $\delta^2 A_{\alpha\dot{\alpha}} = 0$ and $\delta^2 \tilde{\lambda}_{\dot{\alpha}} = 0$ by construction.
- $\delta^2\lambda_\alpha = i\xi_\alpha\delta D + [D(\xi\tilde{\lambda})]^+_{(\alpha\beta)}\xi_\beta$
  $= i\xi_\alpha\delta D + \nabla_{(\alpha\dot{\gamma}}\xi_{\beta)}\tilde{\lambda}_{\dot{\gamma}}\xi_\beta + (\xi_{(\alpha} D_{\beta)\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}})\xi_\beta$
  $= i\xi_\alpha\delta D + \nabla_{(\alpha\dot{\gamma}}\xi_{\beta)}\tilde{\lambda}_{\dot{\gamma}}\xi_\beta + (\xi_\alpha D_{\beta\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}})\xi_\beta$
  [$\xi^2 = \xi^\alpha\xi^\beta\epsilon_{\alpha\beta} = 0$ is used, ∇ = covariant derivative containing the spin connection and $D = \nabla + A$]
  To get $\delta^2\lambda_\alpha = 0$, the middle term should align along ξ_α, so that all the terms can be compensated by δD. This happens if $\exists\hat{V}$ such that

  $$\nabla_{(\alpha\dot{\gamma}}\xi_{\beta)} = i\hat{V}_{\alpha\dot{\gamma}}\xi_\beta + i\hat{V}_{\beta\dot{\gamma}}\xi_\alpha \qquad (3)$$

  and if $\delta D = iD_{\beta\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}}\xi_\beta - \hat{V}_{\beta\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}}\xi_\beta$.
- $\delta^2 D = 0$ identically.

Introduction: 4d $\mathcal{N} = 1$

Let's check the algebra on an anti-chiral multiplet $(\tilde{\phi}, \tilde{\psi}_{\dot\alpha}, \tilde{F})$ to see if there are further conditions. Possible Susy variations are
$\delta\tilde{\phi} = 0,\ \delta\tilde{\psi}_{\dot\alpha} = i\xi_\alpha D_{\alpha\dot\alpha}\tilde{\phi},\ \delta\tilde{F} = i\xi_\alpha D_{\alpha\dot\alpha}\tilde{\psi}_{\dot\alpha} + \xi_\alpha[\lambda_\alpha, \tilde{\phi}] + \xi_\alpha V_{\alpha\dot\alpha}\tilde{\psi}_{\dot\alpha}$

- $\delta^2\tilde{\phi} = 0$ identically.
- $\delta^2\tilde{\psi}_{\dot\alpha} = i\xi_\alpha \left(D_{\alpha\dot\alpha}\delta\tilde{\phi} + [\delta A_{\alpha\dot\alpha}, \tilde{\phi}]\right) = i\xi_\alpha \left(D_{\alpha\dot\alpha}\delta\tilde{\phi} + [\xi_\alpha\tilde{\lambda}_{\dot\alpha}, \tilde{\phi}]\right) = 0$ [$\delta\tilde{\phi} = 0$ and $\xi^2 = 0$]
- $\delta^2\tilde{F} =$

$$
\begin{aligned}
=& i\xi_\alpha[\delta A_{\alpha\dot\alpha}, \tilde{\psi}_{\dot\alpha}] + i\xi_\alpha D_{\alpha\dot\alpha}(i\xi_\beta D_{\beta\dot\alpha}\tilde{\phi}) + \xi_\alpha[i\xi_\alpha D + (F^+)_{\alpha\beta}\xi_\beta, \tilde{\phi}] + \xi_\alpha V_{\alpha\dot\alpha}(i\xi_\beta D_{\beta\dot\alpha}\tilde{\phi})\\
=& -\xi_\alpha\nabla_{\alpha\dot\alpha}\xi_\beta D_{\beta\dot\alpha}\tilde{\phi} - \xi_\alpha\xi_\beta D_{\alpha\dot\alpha}D_{\beta\dot\alpha}\tilde{\phi} + \xi_\alpha\xi_\beta[(F^+)_{\alpha\beta}, \tilde{\phi}] + i\xi_\alpha V_{\alpha\dot\alpha}\xi_\beta D_{\beta\dot\alpha}\tilde{\phi}\\
=& -\xi_\alpha\nabla_{\alpha\dot\alpha}\xi_\beta D_{\beta\dot\alpha}\tilde{\phi} + i\xi_\alpha V_{\alpha\dot\alpha}\xi_\beta D_{\beta\dot\alpha}\tilde{\phi}.
\end{aligned}
$$

which is vanishing if

$$\boxed{\nabla_{\alpha\dot\alpha}\xi_\beta = iV_{\alpha\dot\alpha}\xi_\beta + iW_{\beta\dot\alpha}\xi_\alpha,} \tag{4}$$

$*$ Eq. (4) implies (3) (by symmetrization and setting $\hat{V} = V + W$). Therefore the latter is necessary and sufficient. [One can check that also the linear multiplet closes.]

Introduction: 4d $\mathcal{N} = 1$

$$\boxed{\nabla_{\alpha\dot{\alpha}}\xi_\beta = iV_{\alpha\dot{\alpha}}\xi_\beta + iW_{\beta\dot{\alpha}}\xi_\alpha,} \tag{5}$$

$**$ Eq.(5) is the generalized Killing spinor equation of [Klare, Tomasiello & Zaffaroni] and [Dumitrescu, Festuccia & Seiberg]. There it is obtained by requiring $\delta(\mathit{gravitinos}) = 0$ in supergravity.
In equation (5) $\hat{V}$ is a background gauge connection for the $\mathcal{R}$-symmetry $U(1)$ bundle. The existence of a solution of equation (4) on a four manifold has some consequences. It can be shown that [KTZ& DFS] this implies that the almost complex structure $J = \xi \otimes \xi$ is integrable and that J is self-dual w.r.t. a compatible hermitean metric. [W turns to be a co-closed [$d \star W = 0$] background 1-form.]
$*$ *Nota Bene*: One does not require a Kähler structure.
Therefore one concludes that on any hermitean four manifold one can formulate one chiral supersymmetry.

Introduction: generalizations

One can generalize this kind of analysis to

- $d \neq 4$
- more supercharges (with different $\mathcal{R}$-charges)
- more supersymmetry (more spinorial components that is larger $\mathcal{R}$-symmetry bundle.

The localizing supercharge is effective if $\delta^2 = \mathcal{R}$ *is a compact (bosonic) symmetry to be used in the process of equivariant localization.*

Introduction: generalizations

- The minimal amount for $\mathcal{N} = 1$ and $D = 4$ is a linear combination of *two* supercharges.
- $Q = \delta_\zeta + \delta_{\bar{\zeta}}$
- $Q^2 = \{\delta_\zeta, \delta_{\tilde{\zeta}}\} = \mathcal{L}_v$, where $V_{\alpha\dot{\alpha}} = \zeta_\alpha \tilde{\zeta}_{\dot{\alpha}}$ is a vector on the four manifold.
- The four manifold has then to be a T^2 fibration on a Riemann surface.
- As first examples, one can have $M_4 = T^2 \times S^2$ or $M_4 = S^1 \times S^3$.

4d $\mathcal{N} = 2$ theories on $\mathbb{C}^2$

- $D = 4$ $\mathcal{N} = 2$ gauge theories have an exactly solvable moduli space of order parameters while the RG-flow in the coupling constants is still non trivial.
- The IR effective dynamics of asymptotically free theories can be computed from some spectral data (Seiberg-Witten curve & differential) defining an IR integrable system. (Toda lattice)
- The microscopic derivation of the SW solution has been obtained by Nekrasov using Equivariant Localization.
- The Ω-background – a specific gravitational coupling – provides a regularization of the theory on $\mathbb{C}^2$, by making the moduli space compact by equivariance $\mathbb{C}^2 \to \mathbb{C}^2_{\epsilon_1 \epsilon_2}$ (physically, one turns on an attractive gravitational background potential towards the origin).
- The instanton part of the Nekrasov partition function $\mathcal{Z}^{Nek}_{inst}$ computes the equivariant volume of the ADHM moduli space of instantons and encodes the data of its equivariant cohomology.
- Adding the perturbative part is crucial to reproduce the full SW geometry in the $\epsilon_1, \epsilon_2 \to 0$ limit as

$$\mathcal{Z}^{Nek}_{full} = e^{-\frac{1}{\epsilon_1 \epsilon_2}[\mathcal{F}_{SW} + \mathcal{O}(\epsilon)]}$$

- The Nekrasov partition function is the building block of the $\mathcal{N} = 2$ $D = 4$ partition functions on more general geometries.

$\mathcal{N} = 2$ susy gauge theories on four manifolds

- The exact partition function depends in general on all the possible background couplings that one can turn on by preserving susy up to those removable by a susy transformation.
- in particular, if the stress-energy tensor of the gauge theory is susy exact, then the partition function is independent on the volume moduli/Kähler moduli of the four manifold, so it is a topological invariant (or better, depends on a reduced set of geometric moduli)
- long ago ('89) E. Witten proposed that $\mathcal{N} = 2$ $D = 4$ gauge theories indeed compute some topological invariants of four manifolds, known as Donaldson Invariants, and later Moore and Witten proposed a way to make this effective (u-plane integral) in few test cases
- Some mathematicians (mainly Goettsche, Nakajima and Yoshioka) checked the MW solution and gave some more cases via a very involved technique in algebric geometry
- this proposal got roughly extended by N. Nekrasov to general toric complex surfaces to explain the computational techniques of GNY
- our aim (in the 4D case) is to make Nekrasov's proposal complete and precise, to extend it to equivariant Donaldson invariants and check it
- the idea is that indeed the gauge theory, via equivariant localization, is the natural way to organize those math computations by making them simpler

Let's study the conditions for the existence of $\mathcal{N}=2$ D=4 susy on curved manifolds. The computations are much more involved than in the $\mathcal{N}=1$ case. One can show that

$$
\begin{aligned}
\mathcal{Q}A_\mu &= i\xi^A\sigma_\mu\bar{\lambda}_A - i\bar{\xi}^A\bar{\sigma}_\mu\lambda_A,\ \mathcal{Q}\phi = -i\xi^A\lambda_A,\ \mathcal{Q}\bar{\phi} = +i\bar{\xi}^A\bar{\lambda}_A,\\
\mathcal{Q}\lambda_A &= \frac{1}{2}\sigma^{\mu\nu}\xi_A(F_{\mu\nu} + 8\bar{\phi}T_{\mu\nu}) + 2\sigma^\mu\bar{\xi}_A D_\mu\phi + \sigma^\mu D_\mu\bar{\xi}_A\phi + 2i\xi_A[\phi,\bar{\phi}] + D_{AB}\xi^B,\\
\mathcal{Q}\bar{\lambda}_A &= \frac{1}{2}\bar{\sigma}^{\mu\nu}\bar{\xi}_A(F_{\mu\nu} + 8\phi\bar{T}_{\mu\nu}) + 2\bar{\sigma}^\mu\xi_A D_\mu\bar{\phi} + \bar{\sigma}^\mu D_\mu\xi_A\bar{\phi} - 2i\bar{\xi}_A[\phi,\bar{\phi}] + D_{AB}\bar{\xi}^B,\\
\mathcal{Q}D_{AB} &= -i\bar{\xi}_A\bar{\sigma}^m D_m\lambda_B - i\bar{\xi}_B\bar{\sigma}^m D_m\lambda_A + i\xi_A\sigma^m D_m\bar{\lambda}_B + i\xi_B\sigma^m D_m\bar{\lambda}_A\\
&\quad - 2[\phi,\bar{\xi}_A\bar{\lambda}_B + \bar{\xi}_B\bar{\lambda}_A] + 2[\bar{\phi},\xi_A\lambda_B + \xi_B\lambda_A].
\end{aligned} \tag{6}
$$

(T and $\bar{T}$ are background tensors) squares to

$$
\begin{aligned}
\mathcal{Q}^2 A_\mu &= i\iota_V F + iD\Phi,\\
\mathcal{Q}^2\phi &= i\iota_V D\phi + i[\Phi,\phi] + (w + 2\Theta)\phi,\\
\mathcal{Q}^2\bar{\phi} &= i\iota_V D\bar{\phi} + i[\Phi,\bar{\phi}] + (w - 2\Theta)\bar{\phi},\\
\mathcal{Q}^2\lambda_A &= i\iota_V D\lambda_A + i[\Phi,\lambda_A] + (\frac{3}{2}w + \Theta)\lambda_A + \frac{i}{4}(D_\rho V_\tau)\sigma^{\rho\tau}\lambda_A + \Theta_{AB}\lambda^B,\\
\mathcal{Q}^2\bar{\lambda}_A &= i\iota_V D\bar{\lambda}_A + i[\Phi,\bar{\lambda}_A] + (\frac{3}{2}w - \Theta)\bar{\lambda}_A + \frac{i}{4}(D_\rho V_\tau)\bar{\sigma}^{\rho\tau}\bar{\lambda}_A + \Theta_{AB}\bar{\lambda}^B,\\
\mathcal{Q}^2 D_{AB} &= i\iota_V D(D_{AB}) + i[\Phi,D_{AB}] + 2wD_{AB} + \Theta_{AC}D^C{}_B + \Theta_{BC}D^C{}_A,
\end{aligned} \tag{7}
$$

Introduction: $\mathcal{N}=2$ D=4 on curved manifolds

where the parameters

$$\begin{aligned} V^\mu &= 2\bar{\xi}^A\bar{\sigma}^\mu\xi_A, && U(1)-isometry \\ \Phi &= 2i\bar{\xi}_A\bar{\xi}^A\phi + 2i\xi^A\xi_A\bar{\phi}, && gauge\ transf. \\ \Theta_{AB} &= -i\xi_{(A}\sigma^\mu D_\mu\bar{\xi}_{B)} + iD_\mu\xi_{(A}\sigma^\mu\bar{\xi}_{B)}, && SU(2)\ R-symm. \\ w &= -\frac{i}{2}(\xi^A\sigma^\mu D_\mu\bar{\xi}_A + D_\mu\xi^A\sigma^\mu\bar{\xi}_A), && dilatation \\ \Theta &= -\frac{i}{4}(\xi^A\sigma^\mu D_\mu\bar{\xi}_A - D_\mu\xi^A\sigma^\mu\bar{\xi}_A) && U(1)\ R-symm.. \end{aligned} \tag{8}$$

The spinorial parameters satisfy the generalized Killing equations

$$\begin{aligned} D_\mu\xi_B + T^{\rho\sigma}\sigma_{\rho\sigma}\sigma_\mu\bar{\xi}_B - \frac{1}{4}\sigma_\mu\bar{\sigma}_\nu D^\nu\xi_B = 0 \\ D_\mu\bar{\xi}_B + \bar{T}^{\rho\sigma}\bar{\sigma}_{\rho\sigma}\bar{\sigma}_\mu\xi_B - \frac{1}{4}\bar{\sigma}_\mu\sigma_\nu D^\nu\bar{\xi}_B = 0 \end{aligned} \tag{9}$$

and the auxiliary equations

$$\begin{aligned} \sigma^\mu\bar{\sigma}^\nu D_\mu D_\nu\xi_A + 4D_\lambda T_{\mu\nu}\sigma^{\mu\nu}\sigma^\lambda\bar{\xi}_A &= M_1\xi_A, \\ \bar{\sigma}^\mu\sigma^\nu D_\mu D_\nu\bar{\xi}_A + 4D_\lambda\bar{T}_{\mu\nu}\bar{\sigma}^{\mu\nu}\bar{\sigma}^\lambda\xi_A &= M_2\bar{\xi}_A. \end{aligned}$$

$\mathcal{N} = 2$ D=4 on curved manifolds

- $\xi \in \Gamma\left(S^{+} \otimes \mathcal{R} \otimes \mathcal{L}_R\right)$ and $\bar{\xi} \in \Gamma\left(S^{-} \otimes \mathcal{R}^{\dagger} \otimes \mathcal{L}_R^{-1}\right)$, where $S^{\pm}$ are the spinor bundles of chirality $\pm$, $\mathcal{R}$ is the $SU(2)$ R-symmetry vector bundle and $\mathcal{L}_R$ is the $U(1)$ R-symmetry line bundle.
- The four manifold is subject to the conditions that the above product bundles are well defined and that a solution to the generalized Killing spinor equations exists and is everywhere well defined.
- These conditions differently constrain the space-time four manifold depending on the choice of $\mathcal{R}$ and $\mathcal{L}_R$ (twisted spinors).
- The choice leading to the topologically twisted theory is to set $\mathcal{L}_R = \mathcal{O}$ to be the trivial line bundle and $\mathcal{R} = S^{-}$.
- Therefore, for this choice of the R-symmetry bundles, $S^{+} \otimes S^{-} \sim T$ and $S^{-} \otimes S^{-} \sim \mathcal{O} + T^{(2,+)}$ with T the tangent bundle and $T^{(2,+)}$ the bundle of selfdual forms.
- The simplest solution is then $\bar{\xi}^{A}_{\dot{\alpha}} = \delta^{A}_{\dot{\alpha}}$ and $\xi_{\alpha A} = 0$ (Witten's topological twist)

New solution: $\mathcal{N}=2$ D=4 on Manifolds with $U(1)$ compact action

- If the spacetime manifold admits a compact $U(1)$ action, then one can always choose a metric such that the generating vector field is Killing.
- Then one can solve the $\mathcal{N}=2$ susy conditions with

$$\bar{\xi}^{A}_{\dot{\alpha}} = \delta^{A}_{\dot{\alpha}} \text{ and } \quad \xi_{\alpha A} = v_{\mu}\sigma^{\mu}_{\ \alpha A}. \tag{10}$$

- These satisfy the equations with

$$\bar{T}=0\ , \quad M_1 = M_2 = 0 \text{ and } \quad T=(d\lambda)^{-} \text{ where } \quad \lambda = \star i_V \star 1 \tag{11}$$

- The equations reduce to the Killing vector equation for V.

$\mathcal{N} = 2$ D=4 with more natural variables

- Having a smooth solution of the generalized Killing spinor equations, it is then natural to work out the gauge theory with adapted variables

$$
\begin{aligned}
\bar{\Phi} :=& \phi - \bar{\phi} \ , \Phi := 2i\bar{\xi}^2\phi + 2i\xi^2\bar{\phi} \\
B^+_{\mu\nu} :=& 2(\bar{\xi}^2)^2(F^+_{\mu\nu} + 8\phi\bar{T}_{\mu\nu} - 8\bar{\phi}\bar{S}_{\mu\nu}) \\
& - (\bar{\xi}^A\bar{\sigma}_{\mu\nu}\bar{\xi}^B)(\xi_A\sigma^{\kappa\lambda}\xi_B)(F_{\kappa\lambda} + 8\bar{\phi}T_{\kappa\lambda} - 8\phi S_{\kappa\lambda}) \\
& - 4\bar{\xi}^2 V_{[\mu}D^+_{\nu]}\bar{\Phi} + \frac{1}{2}(\xi^2 + \bar{\xi}^2)(\bar{\xi}^A\bar{\sigma}_{\mu\nu}\bar{\xi}^B)D_{AB} \qquad (12) \\
\eta :=& -i(\xi^A\lambda_A + \bar{\xi}^A\bar{\lambda}_A), \\
\Psi_\mu :=& i(\xi^A\sigma_\mu\bar{\lambda}_A - \bar{\xi}^A\bar{\sigma}_\mu\lambda_A), \\
\chi^+_{\mu\nu} :=& 2\bar{\xi}^A\bar{\sigma}_{\mu\nu}\bar{\xi}^B(\bar{\xi}_A\bar{\lambda}_B - \xi_A\lambda_B).
\end{aligned}
$$

- This change of variables has to be everywhere invertible and well defined. Its Jacobian is given by

$$
\mathcal{J} = \frac{\mathcal{J}_{bos}}{\mathcal{J}_{ferm}} = 1 \ , \quad \mathcal{J}_{bos} = \mathcal{J}_{ferm} \sim \left(\bar{\xi}^2 + \xi^2\right)^4 (\bar{\xi}^2)^3. \qquad (13)
$$

  Therefore the change of variables is everywhere well defined if $\mathcal{J}_{bos} = \mathcal{J}_{ferm}$ is never vanishing.

- In our case we have $\bar{\xi}^2 = 1$ and $\bar{\xi}^2 + \xi^2 = 1 + v^2$ which are nowhere vanishing.

$\mathcal{N} = 2$ D=4 with more natural variables (cohomological form)

The supersymmetry algebra in terms of the new variables is

$$\begin{aligned} &\mathcal{Q}A = \Psi, && \mathcal{Q}\Psi = i\iota_V F + D\Phi, && \mathcal{Q}\Phi = i\iota_V\Psi, \\ &\mathcal{Q}\bar{\Phi} = \eta, && \mathcal{Q}\eta = i\,\iota_V D\bar{\Phi} + i[\Phi, \bar{\Phi}], && \\ &\mathcal{Q}\chi^+ = B^+, && \mathcal{Q}B^+ = i\mathcal{L}_V\chi^+ + i[\Phi, \chi^+]. && \end{aligned} \tag{14}$$

- These are the equivariant extension of Witten's twisted supersymmetry.
- In (14) ι_V is the contraction with the vector V and $\mathcal{L}_V = D\iota_V + \iota_V D$ is the covariant Lie derivative.
- The supercharge (14) manifestly satisfies $\mathcal{Q}^2 = i\mathcal{L}_V + \delta_\Phi^{gauge}$. [There's still a consistency condition on the last line, that is the action has to preserve the self-duality of B^+ and χ^+. This is satisfied iff $L_V\star = \star L_V$, where $\star$ is the Hodge-$\star$ and $L_V = d\iota_V + \iota_V d$ is the Lie derivative. This condition coincides with the requirement that V is an isometry of the four manifold.]

Therefore, we proved that for any four-manifold with a compact $U(1)$ action, once the R-symmetry bundle is properly chosen to fit the equivariant twist, there is a well defined realization of the corresponding $\mathcal{N} = 2$ equivariant supersymmetry algebra [the twist holds at quantum level! – the Jacobian].

Nekrasov Instanton Partition Function

Consider $\mathcal{N} = 2$ Yang-Mills theory on $\mathbb{C}^2_{\epsilon_1,\epsilon_2}$ with gauge group $SU(N)$.

- The maximal torus of global symmetries is $\mathbb{C}_\star^{2+N}/\mathbb{C}_\star$
- The BPS moduli space is the moduli space of selfdual configurations $F^+ = 0$, that is the solutions of the ADHM constraints (at fixed instanton number k) $[B_1, B_2] + IJ = 0$, where $B_i \in \mathrm{Mat}(k \times k, \mathbb{C})$ and $I, J^t \in \mathrm{Mat}(k \times N, \mathbb{C})$, modulo $GL(k, \mathbb{C})$.
- The fixed points are the singular configurations where the instantons are packed at the origin (fixed point of the $\mathbb{C}^2_\star$)
- The partition function reads

$$\mathcal{Z}^{full}_{Nek}(\vec{a}, \epsilon_1, \epsilon_2; \Lambda) = \mathcal{Z}_{cl}\mathcal{Z}_{1-loop}\mathcal{Z}_{inst}$$

where

$$\mathcal{Z}_{cl} = e^{-\frac{\vec{a}^2}{\epsilon_1\epsilon_2}} \; ; \;\; \mathcal{Z}_{1-loop} = e^{\sum_{i<j}\gamma_\Lambda(a_i - a_j;\epsilon_1,\epsilon_2)} \; , \;\; \gamma = log\Gamma_2$$

$$\mathcal{Z}_{inst} = \sum_k \Lambda^{2Nk} \frac{1}{k!} \frac{\epsilon^k}{(2\pi i \epsilon_1\epsilon_2)^k} \oint \prod_{s=1}^{k} \frac{d\sigma_s}{P(\sigma_s)P(\sigma_s + \epsilon)} \prod_{s<t}^{k} \frac{\sigma_{st}^2(\sigma_{st}^2 - \epsilon^2)}{(\sigma_{st}^2 - \epsilon_1^2)(\sigma_{st}^2 - \epsilon_2^2)} \quad (15)$$

with $P(\sigma_s) = \prod_{j=1}^{N}(\sigma_s - a_j)$, $\epsilon = \epsilon_1 + \epsilon_2$ and Λ a reference scale.

Nekrasov Instanton Partition Function

The link with the equivariant cohmology of the ADHM moduli space becomes cleaner by (carefully) performing the above integrals and obtain

$$\mathcal{Z}_{inst} = \sum_{\vec{Y}} \Lambda^{2N|\vec{Y}|} \prod_{i,j=1}^{N} \prod_{s \in Y_i} \frac{1}{E(s)\,(E(s)-\epsilon)}$$

where $E(s) = a_i - a_j - \epsilon_1 h(s) + \epsilon_2 (v(s)+1)$ and $h(s)$ and $v(s)$ are the horizontal and vertical relative position of the box $s \in Y_i$ in the Young diagram Y_j.

- (I) D-brane realization D(-1)D3 system at the tip of the $\mathbb{C}^2/\mathbb{Z}_2$ quotient: the k $D(-1)$branes are divided in N groups $k = \sum_j k_j$, one for each $D3$brane. Each group is in a state labeled by a conjugacy class Y_j of the permutations of the $k_j = |Y_j|$ $D(-1)$branes.
- (II) Via AGT correspondence $\mathcal{Z}_{inst}$ is related to Liouville conformal blocks. This will be crucial later on! [The link then is in the fact that Virasoro algebra acts on the equivariant cohomology of ADHM]

Susy Gauge theory on toric compact surfaces

- Toric surfaces are complex four manifolds acted on by $\mathbb{C}_\star^2$ with isolated fixed points.
- These can be covered by toric $\mathbb{C}^2$ patches glued by $\mathbb{C}_\star^2$-equivariant transition functions, each patch corresponding to a fixed point.
- These are usually presented in terms of a "toric fan" which encodes the full geometric data.
- Compact toric surfaces are obtained by successive blow-ups (inflating a fixed point to a $\mathbb{P}^1$) of $\mathbb{P}^2$ or of Hirzebruch-Serre $\mathbb{F}_a$ surfaces ($\mathbb{P}^1 \hookrightarrow \mathbb{F}_a \to \mathbb{P}^1$ with charge a; $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$).
- The toric structure implies on Betti numbers that $b_1 = 0$ and $b_2^+ = 1$.
- Toric surfaces come equipped with $\mathbb{C}_\star^2$ invariant Kähler metrics (for example Fubini-Study for $\mathbb{P}^2$ or the sum of round metrics for $\mathbb{P}^1 \times \mathbb{P}^1$).

Solving fixed point equtions on toric surfaces

- To compute the susy partition function we first solve the fixed points under Q
- This corresponds to set $[fermions] = 0$ and ensure the susy stability as $Q[fermions] = 0$
- From the above algebra, one gets

$$i\iota_V F + D\Phi = 0 \text{ and } \quad \iota_V D\bar{\Phi} + [\Phi, \bar{\Phi}] = 0 \tag{16}$$

- Choosing the reality condition $\bar{\Phi} + \Phi^{\dagger} = 0$ and studying the integrablity conditions of (16) one gets that all fields are Cartan valued and

$$(F + \Phi)_{\alpha} = F_{\alpha}^{point-like} + (F + \Phi)_{\alpha}^{equivariant} \tag{17}$$

  where $F^{point-like}$ is the contribution of pointlike instantons at the fixed points of the $\mathbb{C}^2_{\star}$ action (one set for toric patch) and $(F + \Phi)^{equivariant,(\ell)} = a + k^{(\ell)}\omega^{(\ell)} + k^{(\ell+1)}\omega^{(\ell+1)}$ where a is a reference Cartan label (to be integrated over), $\ell = 1, \ldots, \chi_M$ labels the toric patches, $\{k\}$'s are integer flux parameters and $\omega^{(\ell)} = \omega + H^{(\ell)}$ are equivariant extensions (that is $i\iota_V\omega + dH^{(\ell)} = 0$ and $H_i^{(\ell)}(P^{\ell}) = 0$) of the Poincare' duals of the toric divisors. [For example, for $\mathbb{P}^2$ there is a single divisor, but three equivariant extensions]

Evaluation of the path integral

- the partition function is supported on vector bundles E such that

$$\mu(E) \geq \mu(G) \quad , \quad \text{where} \quad \mu(E) \equiv \frac{\int \omega \wedge TrF_E}{2\pi rank(E)} \tag{18}$$

for any sub-bundle G, that is on semi-stable vector bundles.

The zero modes for the $U(N)$ theory split in two types: $u(1)$ and $su(N)$ zero modes.

- The $u(1)$ zero modes are due to a global $u(1)$ symmetry to gauge fix. This is done by improving the susy charge and they get eliminated as a perfect quartet.
- By correctly treating the gaugino zero modes in the $su(N)$ sector we get precise instructions about the integration on the leftover $N-1$ Cartan parameters $a_\rho = a_\alpha - a_\beta$.
- Solving the fixed point equations we bounded the field theory phase to the deep Coulomb branch by declaring Φ and $\bar{\Phi}$ to lay at a generic point in the Cartan subalgebra where the gauge symmetry is completely broken as $U(N) \to U(1)^N$.
- $\Rightarrow$ The integral over $(a, \bar{a})$ is in $\mathbb{C}^{N-1} \setminus \mathcal{T}$ where $\mathcal{T}$ is a tubular neighborhood of the hyperplanes set $\Delta = \{a_\alpha - a_\beta = 0\}$ where some extra gauge symmetry could restore.

  [For example if $N=2$ one gets a single contour integral around the origin in $\mathbb{C}$.]

$\mathcal{N} = 2$ Partition function : explicit form

The complete gauge theory partition function is then given by

$$Z^{M,U(N)}(q,z\,;\epsilon_1,\epsilon_2) = \sum_{\{k\}|\text{stable}} \oint_\Delta d\vec{a}\;\; z^{c_1[k]} \prod_\ell Z^{\mathbb{C}^2,U(N)}_{\text{full}}(q\,;\vec{a}^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}) \tag{19}$$

where

- $\sum_{\{k\}|\text{stable}}$ is over fluxes defining equivariant stable vector bundles on M.
- Δ is the set of diagonals in the root space.
- $\ell = 1,\ldots,\chi(M)$ labels the fixed points with weights given by $\epsilon_1^{(\ell)}$ and $\epsilon_2^{(\ell)}$.
- $Z^{\mathbb{C}^2,U(N)}_{\text{full}}$ is the local patch full Nekrasov partition function.
- The $U(1)^N$ equivariant weights $\vec{a}^{(\ell)} = \Phi^{(\ell)}(P^{(\ell)})$

$\star$ Formula (19) was conjectured by Nekrasov and it was proposed as a contour integral formula for the (equivariant) Donaldson invariants of toric manifolds.
$\star\star$ Formula (19) can be interpreted as a definition of a chiral CFT in d=2, generalizing the AGT correspondence to *compact* toric four manifolds. [*non-compact* ones, namely the resolution of Hirzebruch-Jung singularities $\mathbb{C}^2/\Gamma$ is already known, for example $\mathbb{C}^2/\mathbb{Z}_2$ is dual to $\mathcal{N} = 1$ Super Liouville]

We matched in detail $\star$ for $M = \mathbb{P}^2$ and $\star\star$ for $M = \mathbb{P}^1 \times \mathbb{P}^1$ which is dual to a chiral version of Liouville gravity.

Comparison with Donaldson Invariants

- If $N = 2$, then the integral over a_{12} is a contour integral around the origin.
- This prescription agrees and reproduces (in the case c_1 odd) with the wall-crossing formulae of Gottsche-Nakajima-Yoshioka.
- For manifolds with $b_2^+ = 1$ and $b_2^- = 1$, Donaldson invariants are only piece-wise metric independent: their behavior is described by a chamber structure in $H^2(M, \mathbb{R})$ with walls located at $H^2(M, \mathbb{Z}) \cap H^{2,-}(M, \mathbb{R})$. [This is encoded in the stability condition] The prototype for this case is $\mathbb{P}^1 \times \mathbb{P}^1$.
- A common strategy to calculate Donaldson invariants is then given by identifying a vanishing chamber and then compute the invariants in the other chambers via wall crossing. Our general formulas for $N = 2$ indeed reproduce the known results. [In the non equivariant limit $\epsilon_i \to 0$ limit, one compares with Moore-Witten u-plane integral.]
- Notice that for $M = \mathbb{P}^2$ there is a single chamber and the above procedure is not available. Moreover, it is neither possible to deform to $\mathcal{N} = 1$ supersymmetry with mass terms ($h^{(1,1)}(\mathbb{P}^2) = 1$). This makes this case particularly interesting since it has to be computed directly (and this will be the focus of next talk).

AGT dual interpretation

Lets go back to our general formula

$$Z^{M,U(N)}(q,z\,;\epsilon_1,\epsilon_2) = \sum_{\{k\}|\text{stable}} \oint_\Delta d\vec{a}\; z^{c_1[k]} \prod_\ell Z^{\mathbb{C}^2,U(N)}_{\text{full}}(q\,;\vec{a}^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}) \qquad (20)$$

- Have AGT correspondence in mind $\rightarrow Z^{\mathbb{C}^2}_{inst} = Z_{\mathrm{U}(1)}\mathcal{F}_W$, and $\prod Z_{1-loop} \sim C_{3-pt}$
- (20) directly imply that the fixed points of the moduli space of instantons on M provide a basis for the representation of L copies of Heisenberg plus W_N algebrae

$$\mathcal{A}_N(X_{p,q}) \equiv \oplus_{\ell=0}^{L-1}(\mathcal{H} \oplus {}^{\ell}W_N) \qquad (21)$$

with central charge of ${}^{\ell}W_N$ given by

$$c_\ell = (N-1)\left(1 + Q_\ell^2 N(N+1)\right)\ , \quad Q_\ell = \sqrt{\frac{\epsilon_1^{(\ell)}}{\epsilon_2^{(\ell)}}} + \sqrt{\frac{\epsilon_2^{(\ell)}}{\epsilon_1^{(\ell)}}}$$

- The overall central charge $c = \sum_\ell c_\ell$ coincides with the central charge of the CFT_2 that can be computed from M-theory compactification on M.
- This extends to the case of *compact* toric manifolds an analogous construction valid for the blow=up of toric local singularities $\mathbb{C}^2/\Gamma_{p,q}$ (but the compact case technically is more difficult because of the integral over the Cartan variables has to be explicitly performed)

Supersymmetry on squashed S^5

- One can study the partition function of $\mathcal{N} = 1^*$ susy gauge theories on S^5 (and more in general on 5 dimensional Riemannian manifolds) [Kallen-Zabzine,...]
- Notice that S^5 is the Hopf S^1-fibration over $\mathbb{C}P^2$. The round metric $ds^2(S^5) = ds^2(\mathbb{C}P^2) + \mathrm{k}^2$, where $\mathrm{k} = dt + A_{\mathbb{C}P^2}$ and $\mathbb{C}P^2$ is equipped with the standard Fubini-Study structure. The squashed metric is obtained by reducing $\mathbb{C}^3$ on the the level set of $\mu = \sum_{i=1}^3 \omega_i^2 |z_i|^2$ and still decomposes according to the Hopf fibration. This is invariant under a $U(1)^3$ action $z_i \to e^{i\phi_i} z_i$.
- The fixed loci of the V-action are the three S^1 at the origins of the three patches of $\mathbb{C}P^2$.
- The supersymmetry (already in cohomological form) reads

$$\delta A = \Psi \quad \delta\Psi = i_V F - D\sigma \quad \delta\sigma = -i_V \Psi$$

$$\delta B = \chi \quad \delta\chi = [B, \sigma] + \mathcal{L}_V B$$

where σ is hermitean and (χ, B) are 2-forms in $Ker(\mathrm{k} \wedge +*)$ and all in the adjoint of the gauge group and the hyper – still in the adjoint –

$$\delta\Phi^A = \Psi^A \quad \delta\Psi^A = r^A_B \Phi^B + i_V D\Phi^A + [\sigma, \Phi^A]$$

where r^A_B rotates the two doublets constrained by the reality condition $\left(\Phi^A\right)^* = \Omega_{AB\epsilon}\Phi^B$.

Supersymmetry on S^5

- The partition function then localizes on the solitonic configurations given by the "instanton-particles" with world-line the three invariant S^1s.
- It is given therefore by glueing three copies of the full Nekrasov partition function on $S^1 \times \mathbb{C}^2$.
- The single copy ($U(r)$ theory) is given by

$$\mathcal{Z}^{S^1\times\mathbb{C}^2}_{U(2),\mathcal{N}=1^*}(\epsilon_1,\epsilon_2,\beta,q,M;a) = \mathcal{Z}_{pert}\mathcal{Z}_{inst}$$

where

$$\mathcal{Z}_{pert} \propto \prod_{\alpha,\beta=1}^{r} \frac{\Gamma_e(a_{\alpha\beta}-M)}{\Gamma_e(a_{\alpha\beta})} \quad \text{and} \quad \Gamma_e(x) = \prod_{i,j\geq 0}(1-e^{-\pi i\beta(x+i\epsilon_1+j\epsilon_2)})$$

and $s[x] \equiv \sin\left(\frac{\beta}{2}x\right)$

$$\mathcal{Z}_{inst} = \sum_{k\geq 0} q^k \sum_{|\vec{Y}|=k} \prod_{\alpha,\beta} \prod_{s\in Y_\alpha} \frac{s[e_{\alpha\beta}(s)-M]s[e_{\alpha\beta}(s)-\epsilon_1-\epsilon_2+M]}{s[e_{\alpha\beta}(s)]s[e_{\alpha\beta}(s)-\epsilon_1-\epsilon_2]}$$

where $e_{\alpha\beta}(s) = a_{\alpha\beta} - \epsilon_1 L_{Y_\alpha}(s) - \epsilon_2(A_{Y_\beta}(s)+1)$

Supersymmetry on S^5

- The three copies are combined then as

$$\mathcal{Z}^{S^5}_{U(r),\mathcal{N}=1^*}(\{\omega\},\Lambda,M) = \int_{\mathbb{R}^{r-1}} d\vec{a} e^{-\Lambda^2 a^2/2} \mathcal{Z}^{S^5}_{1-loop}(a,\vec{\omega},M)\mathcal{Z}^{S^5}_{inst}(\Lambda,a,\vec{\omega},M)$$

where $\Lambda^2 \equiv \frac{2\pi}{\omega_1\omega_2\omega_3 g^2_{YM}}$

$$\mathcal{Z}^{S^5}_{1-loop}(a,\vec{\omega},M) = \prod_{\alpha<\beta} \frac{S_3(ia_{\alpha\beta})S_3(ia_{\alpha\beta}+iE)}{S_3(ia_{\alpha\beta}+M)S_3(ia_{\alpha\beta}-M+iE)}$$

and

$$\mathcal{Z}^{S^5}_{inst}(\Lambda,a,\vec{\omega},M) = \mathcal{Z}^{(1)}_{inst}\mathcal{Z}^{(2)}_{inst}\mathcal{Z}^{(3)}_{inst}$$

where

$$\mathcal{Z}^{(i)}_{inst} = \mathcal{Z}^{S^1\times\mathbb{C}^2}_{U(r),\mathcal{N}=1^*}(\epsilon^{(i)}_1,\epsilon^{(i)}_2,\beta^{(i)},q^{(i)},M;a)$$

and the $\beta^{(i)} = \frac{1}{\omega_i}$, $\epsilon^{(i)}_\ell$ are the left over ω_j, $j\neq i$ and $q^{(i)} = e^{-\frac{8\pi^3}{g^2_{YM}\omega_i}}$.

Supersymmetry on S^5

If one is able to analyze in detail the analitic properties of the integrand and finds favourable conditions, then she/he can perform the above integral.
That's what we are aiming to do for the $U(2)$ theory.
Let $\mathcal{I} = \mathcal{Z}_{1-loop}\mathcal{Z}_{inst}$, so that

$$Z^{S^5} = \int_{\mathbb{R}} e^{-\Lambda^2 a^2/2}\mathcal{I}(a)$$

Actually, via a precise recursion relation and direct analysis, we are able to prove the following points:

- The integrand $\mathcal{I}$ is an even meromorphic function in a with simple poles only
- The poles are in known positions along the imaginary axis in the a space ($a = \pm i x_L$)
- The integrand $\mathcal{I}$ behaves as a constant at large a

then, we can express it as

$$\mathcal{I}(a) = \mathcal{I}(\infty)\left(1 + \sum_L f_L\left(\frac{1}{a - i x_L} - \frac{1}{a + i x_L}\right)\right)$$

Supersymmetry on S^5

So we can compute the above integral by the formula

$$\int_{\mathbb{R}} \frac{e^{-\Lambda^2 a^2/2}}{a - ix} = i\pi \mathrm{sgn}(x) e^{\Lambda^2 x^2/2} Erfc(\Lambda|x|/\sqrt{2})$$

where $Erfc(y) \equiv -\frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$ is the complementary error function.
Notice that the location of the poles is in a subset of the lattice $\mathbb{Z}^3$ at the values $a = i \sum_{i=1}^3 n_i \omega_i$ for the vector multiplet and $a = i \sum_{i=1}^3 n_i \omega_i + M$ for the massive hyper.
Moreover the factor

$$\mathcal{I}(\infty) = \left[\mathcal{Z}^{S^5}_{U(1),\mathcal{N}=1^*}(\{\omega_i\}, \Lambda, M) \right]^2$$

where

$$\mathcal{Z}^{S^5}_{U(1),\mathcal{N}=1^*}(\{\omega_i\}, \Lambda, M) = \prod_{i=1}^{3} \mathcal{Z}^{\mathbb{R}^4 \times S^1}_{U(1),\mathcal{N}=1^*}(\epsilon_1^{(i)}, \epsilon_2^{(i)}, \beta^{(i)}, q^{(i)})$$

is the $U(1)$ gauge theory partition function.

Supersymmetry on S^5: $\mathcal{N} = 2$

The formula for the $U(2)$ partition function on S^5 is then in the form

$$\mathcal{Z}^{S^5}_{U(2),\mathcal{N}=1^*}(\{\omega_i\},\Lambda,M) =$$

$$= \left(\mathcal{Z}^{S^5}_{U(1),\mathcal{N}=1^*}(\{\omega_i\},\Lambda)\right)^2 \times \sum_{n_i\in\mathbb{Z}^3\,|\,constraints} \tilde{f}_{n_i}(\{\omega_i\},\Lambda,M) Erfc\left(\Lambda(n_i\omega_i)/\sqrt{2}\right)$$

where the coefficients $\tilde{f}_{n_i}(\{\omega_i\},\Lambda,M)$ are complicated, but known, functions.
These coefficients *simplify enormously* in the $M \to 0$ limit, that is for the $\mathcal{N} = 2$ theory.
In this limit we find the more intelligible formula

$$\mathcal{Z}^{S^5}_{U(2),\mathcal{N}=2}(\{\omega_i\},\Lambda) =$$

$$= \left(\mathcal{Z}^{S^5}_{U(1),\mathcal{N}=2}(\{\omega_i\},\Lambda)\right)^2 \left(\frac{\Lambda}{M} + \sum_{n_i\in\mathbb{Z}^3\,|\,\text{constraints}} e^{-\Lambda^2 n\cdot Q(\{\omega_i\})\cdot n} Erfc\left(\Lambda(n_i\omega_i)/\sqrt{2}\right)\right)$$

where $Q(\{\omega_i\})$ is a quadratic form which depends on the $\{\omega_i\}$.

Supersymmetry on S^5: $\mathcal{N} = 2$

One can add Wilson loop observables to our computation.
The equivariant Wilson loop observables are along the three fixed S^1s in S^5.
So we insert in the path integral

$$W(\{y_i\}) = \mathrm{Tr}_{Ad}\left(\prod_{k=1}^{3} Pe^{y_k \oint_{S^1_k}(A+i\Bbbk\sigma)}\right) \quad \rightarrow \quad e^{i\sum_k n_k y_k}$$

and obtain for the generating function the form

$$\mathcal{W}^{S^5}_{\mathcal{N}=2}(\{\omega_i\}, \Lambda|\{y_i\}) = \sum_{n_i \in \mathbb{Z}^3|\text{constraints}} e^{-\Lambda^2 n\cdot Q(\{\omega_i\})\cdot n + in\cdot y} \, Erfc\left(\Lambda(n_i\omega_i)/\sqrt{2}\right)$$

This is in the form of a (generalized) Zwegers mock theta function. [This is generalized w.r.t. Zwegers case because of the $(\{\omega_i\})$ dependence in Q, just like Siegel-Narain theta functions generalize the usual Riemann's theta function. In the round S^5 case (all squashing parameters $\{\omega_i\}$ equal) one removes the generalization because a single modulus stays, the radius.]
Modular properties in the ω's follows by Poisson resummation and the fact that $f(x) = e^{x^2/2} Erfc(x/\sqrt{2})$ is a fixed point under Fourier transform [precisely $\int_{\mathbb{R}_+} dx f(x)\sin(px) = \sqrt{\frac{\pi}{2}} f(p)$]

Interpretation as regularized Witten index

- Since many years there's a conjecture about the fact that $D = 5$ susy gauge theories should capture in the strong coupling the theory of M5-branes on a circle. (I'd say [Douglas])
- The radius of the circle is identified with the (dimensional) SYM coupling in 5d.
- Therefore, one should identify the $N = 2$ partition function on S^5 with the partition function of a set of M5-branes on $S^1 \times S^5$. [or $\mathcal{N} = (2,0)$ A_n theory on $S^1 \times S^5$ for $n+1$ M5-branes]
- Indeed the single M5-brane has been already matched with the 5d partition function on S^5 by comparing it with the Witten index of the selfdual tensor multiplet [Lockhart-Vafa]
- Nobody (by now) knows much about the multi-M5 brane, but we can try to interpret our result for 2 M5-branes.
- It should be interpreted as the regularized Witten index $Tr_{\mathcal{H}_2}\left[(-1)^F e^{-\beta H}\right]$, where β is the S^1 radius and H the Hamiltonian (plus fugacities of conserved global charges) on the Hilbert space $\mathcal{H}_2$ of 2 M5-branes.
- The two $U(1)$ terms are clearly interpreted as the long distance states (well separated M5-branes). The rest should correspond to bound states.

On the regularized Witten index

There are important subtleties in the computation of the Witten index [Cecotti-Girardello, Imbimbo-Mukhi, Akoury-Comtet, Niemi-Wijewardhana] ('83-'84).

- Consider just susy quantum mechanics of a single particle mode on the real line with superpotential $W(X)$.

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + i\psi^\dagger\dot{\psi} + \frac{1}{2}[W'(x)]^2 + \psi\psi^\dagger W''(x)$$

  and compute the regularized Witten index

$$\Delta(\beta) = \mathrm{Tr}(-1)^F e^{-\beta H} = \int_{PBC} \mathcal{D}x\, \mathcal{D}\psi e^{-\int_0^\beta \mathcal{L}(x,\psi)}$$

- The path integral is evaluated by integrating over constant modes, while fluctations cancel. One stays with

$$\Delta(\beta) = \frac{1}{\sqrt{2\pi\beta}} \int_{\mathbb{R}} dx_0\, (\beta W''(x_0))\, e^{-\frac{\beta}{2}(W'(x_0))^2} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\beta}W'(-\infty)}^{\sqrt{\beta}W'(+\infty)} dy e^{-y^2/2}$$

[Normalization check: harmonic oscillator, $W(x) = m^2X^2/2$ and $\Delta(\beta) = 1$]

Interpretation as regularized Witten index

- For a potential with bounded asymptotic values there's a dependence on β in the form of incomplete Gaussian integrals. (continuum spectrum anomalous behavior)
- Thus one finds

$$\Delta(\beta) = \frac{1}{2}\left[Erfc\left(\sqrt{\frac{\beta}{2}}W'(+\infty)\right) - Erfc\left(\sqrt{\frac{\beta}{2}}W'(-\infty)\right)\right]$$

- Therefore the *Erfc* terms that we find in the dual 2 M5-brane partition funcion have a possible interpretation as anomalous thresholds terms of an effective quantum mechanical model.
- Notice that this anomalous behavior is usually assumed not to be relevant in the index computations, but this is done without any justification.
- The study of the modular properties of the S^5 partition function and its interpretation as regularized index are still work in progress.

Open Issues D=4

- Compute the partition functions for more general $\mathcal{R}$-symmetry bundle choice ($\mathcal{R} \neq \mathcal{S}^{\pm}$). [Untwisted spinors on $\mathbb{P}^1 \times \mathbb{P}^1$ in the paper.]
- Extend the equivariant Donaldson recognition to Hirzebruch-Serre surfaces (indeed $\mathbb{P}^1 \times \mathbb{P}^1$ would be enough if a blow-up formula is then used)
- Extend the residue computation to higher rank $\rightarrow$ generalize the Zamolodchikov recursion relation to W-algebra conformal blocks
- Then one can safely use the gauge theory computation to predict equivariant Donaldson invariants at higher rank
- Chiral CFT_2 defined by a toric diagram: define it off-shell!
- Extend the chiral CFT_2 definition to non zero fluxes (degenerate field insertions?)
- Lift to higher dimensions (6 dim): non-abelian Donaldson Thomas invariants vs. D=6 susy gauge theories

Open Issues D=5

- Study the modular properties of the S^5 partition function
- Interpretation of the S^5 partition function as regularized Witten index: what can we infer from it about the multi M5-brane spectrum?
- Study the $\mathcal{N} = 1^*$ deformation and its M-theory interpretation
- Obtain higher rank and study sub-leading corrections to the large N limit
- Give an interpretation in terms of the theory of $M2$-branes suspended between $M5$-branes
- Compute the 5D partition functions on S^1-fibrations over generic compact toric complex surfaces

Thank you!