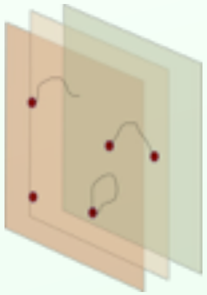
Exact results in N=2 theories

Prepotentials, Wilson Loops, chiral correlators

F.Fucito, J.F.M, R. Poghossian , 1507.05426

M. Billo, M.L. Frau, A. Lerda, F.Fucito, J.F.M , 1507.07476, 1507.08027

Motivations



- **Non-perturbative effects in gauge theories:**

✻ QCD (Instantons): Confinement, gaugino condensation , chiral symmetry breaking, etc

✻ String Theory (D-branes): String dualities, AdS/CFT, AGT, etc.

- **Circular Wilson loop in N=4:** Special theories and operators

✓ Localization:

$$W = \frac{2\, I_1\left(\sqrt{\lambda_{\rm YM}}\right)}{\sqrt{\lambda_{\rm YM}}}$$

$\lambda_{YM} = g^2_{\rm YM} N$

$$= 1 + \frac{\lambda_{\rm YM}}{8} + \frac{\lambda^2_{\rm YM}}{192} + \ldots$$

loops — Weak coupling

$$= \frac{e^{\sqrt{\lambda_{\rm YM}}}}{4\sqrt{2\pi}\,\lambda_{\rm YM}^{\frac{5}{4}}}\left(-3 + 8\sqrt{\lambda_{\rm YM}} + \ldots\right)$$

holography — Strong coupling

Exact results !

- **N=2 theories**

$$S_{\rm eff} = \int d^4x d^4\theta \mathcal{F}(\Phi)$$

Prepotential

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_{1-loop} + \sum_{k=1}^{\infty} \mathcal{F}_k q^k$$

Gauge instantons

✓ Seiberg Witten Theory, localization: ⟹ $\mathcal{F}(\Phi)$

Outline

- **Chiral deformations of N=4:** $\delta\mathcal{L} = \int d^4\theta\, \tau_J \mathrm{tr}\Phi^J$

✻ Wilson loops, gauge partition function : Interacting Matrix Models

⇒ ✓ q-Exact formula for $\langle W \rangle_{\delta\mathcal{L}}$ or $\langle W\, \mathrm{tr}\Phi^{J_1}\, \mathrm{tr}\Phi^{J_2} \ldots \rangle_{\mathcal{L}}$

cusp anomalous dimension

- **Mass deformations of N=4:** $\delta\mathcal{L} = m \int d^2\theta (\Phi_2^2 + \Phi_3^2)$

✻ Prepotentials for small mass: Recursion relations

⇒ ✓ q-Exact formulas for $\mathcal{F}$ Self-duality of ADE & SO/Sp duality !

- **SU(2)+funds SCFT at rational squashed sphe res**

✻ AGT dual of Minimal Models : Localization

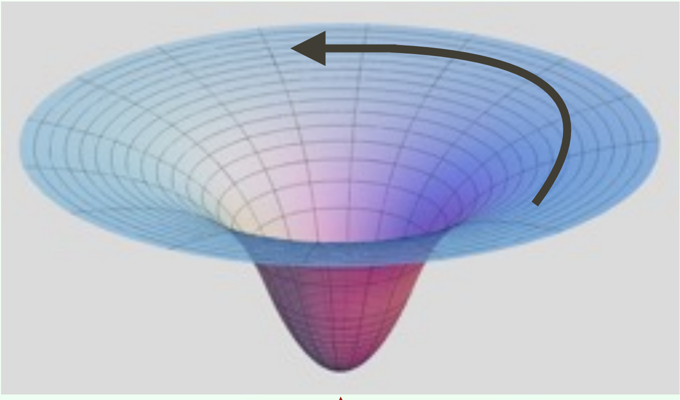
⇒ ✓ q-Exact formulas for the gauge partition function $Z_{S^4} = \sum_{a_0} |Z(a_0)|^2$

- **Conclusions**

Localization

- **Localization formula:**

$$Q_\xi \equiv d + i_\xi \qquad i_\xi dx^i \equiv \delta_\xi x^i \qquad Q_\xi^2 = \delta_\xi$$



$$Q_\xi \, \alpha = 0 \quad \Longrightarrow$$

$$\int_M \alpha = (-2\pi)^\ell \sum_s \frac{\alpha_0(x_0^s)}{\det^{\frac{1}{2}} Q_\xi^2(x_0^s)}$$

- **Example:** Gaussian integral: $I = \int_{\mathbb{R}^2} e^{-a(x^2+y^2)} \, dx dy$

$$\xi = \epsilon \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \qquad Q^2 = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \qquad \alpha = e^{-a(x^2+y^2)} \, dx dy - \frac{\epsilon}{2a} e^{-a(x^2+y^2)}$$

$$I = \int_{\mathbb{R}^2} \alpha = 2\pi \frac{\epsilon \, e^{-a(x_0^2+y_0^2)}}{2a\epsilon} = \frac{\pi}{a}$$

- **Supersymmetric theories:** Q_ξ Supersymmetry charge and $\alpha = e^{-S}$

τ_J-deformations

- The N=2 gauge action:

$$S_{\rm class} = \left[\int d^4x d^4\theta \sum_{J=2}^{n} \frac{i\tau_J}{2\pi J!} \, {\rm tr}\, \Phi^J + {\rm h.c.} \right] + \ldots = \int d^4x \, {\rm Im}\tau(\varphi) F^2 + \int {\rm Re}\tau(\varphi) F \wedge F + {\rm susy} + \ldots$$

scalar dependent coupling

- The gauge partition function:

Localization

$$Z(\tau_J) = \int \mathcal{D}\Phi e^{-S_{\rm YM}(\Phi,\tau_J)} = Z_{\rm one-loop} Z_{\rm inst+tree}(\tau_J)$$

fluctuations around instanton configuration

$$Z_{\rm inst+tree}(\tau) = \sum_{k=0}^{\infty} \int d\mathfrak{M}_k \, e^{-S_{\rm inst}(\vec{\tau})} = \sum_Y \frac{1}{\det_Y Q^2} \exp\left(-\frac{2\pi i}{\epsilon_1 \epsilon_2} \sum_{J=2}^{p} \frac{\tau_J}{J!} \mathcal{O}_{J,Y} \right)$$

Sum over Young tableaux (fixed points)

τ_J-deformed YM action

$$ {\rm tr}\, e^{z\tilde{\varphi}_0} \big|_Y = \sum_J \frac{z^J \mathcal{O}_J}{J!} = \sum_u \left(e^{z a_u} - (1 - e^{z\,\epsilon_1})(1 - e^{z\,\epsilon_2}) \sum_{(i,j)\in Y_u} e^{z\chi(i,j)} \right)$$

$$\mathcal{O}_{2,Y} = {\rm tr} a^2 - 2\, k\, \epsilon_1\, \epsilon_2 \qquad \mathcal{O}_{3,Y} = {\rm tr} a^3 - 3\, \epsilon_1\, \epsilon_2 \sum_{u=1}^{N} \sum_{(i,j)\in Y_u} (\epsilon + 2\chi_{(i,j)})$$

* Instanton partition function :

$$Z_Y = \frac{\prod_{u,v}^N Z_{Y_u,Y_v}(a_{uv}+m)}{\prod_{u,v}^N Z_{Y_u,Y_v}(a_{uv})}$$

$$Z_{Y_u,Y_v}(x) = \prod_{(i,j)\in Y_u} \left(x - \epsilon_1(k_{vj}-i) + \epsilon_2\,(1+\tilde{k}_{ui}-j)\right) \times \prod_{(i,j)\in Y_v} \left(x + \epsilon_1(1+k_{uj}-i) - \epsilon_2\,(\tilde{k}_{vi}-j)\right)$$

* One-loop partitionn function :

$$Z_{\rm one-loop} = \frac{\prod_{u<v}^N \Gamma_2(a_{uv}-m)\Gamma_2(a_{uv}+m+\epsilon)}{\prod_{u<v}^N \Gamma_2(a_{uv})\Gamma_2(a_{uv}+\epsilon)}$$

$$\Gamma_2(x) = \prod_{i,j=0}^{\infty} \left(\frac{\Lambda}{x+i\epsilon_1+j\epsilon_2}\right)$$

On the 4-sphere

- Partition function:

$$Z_{S^4}(\vec{\tau}) = c \int_\gamma d^N a \, |Z_{\rm one-loop}(a)\, Z_{\rm tree+inst}(a,\vec{\tau})|^2$$

Wilson loops

- **Supersymmetric Wilson loop:**

$$\mathcal{C} = i\int_0^L (A_m\,\dot{x}^m + |\dot{x}|\,\varphi_1)ds$$

$$z_\ell(s) = r_\ell\, e^{i\epsilon_\ell s} \qquad \frac{\epsilon_1}{\epsilon_2} = \frac{n_1}{n_2}$$

Squashed parameter

Circular Wilson loops

$$|\dot{x}| = 1$$

- **localization:**

$$e^{\mathcal{C}_0}\Big|_Y = \operatorname{tr} e^{i\,L\,\tilde{\varphi}_0}\,\big|_Y = \sum_u e^{\frac{2\pi i\, n_1}{\epsilon_1}\, a_u}$$

$$\langle \operatorname{tr} e^{\mathcal{C}} \rangle_{S^4} = \frac{1}{Z}\int_\gamma d^N a \operatorname{tr} e^{\frac{2\pi i n_1\, a}{\epsilon_1}}\,|Z_{\text{one-loop}}(a) Z_{\text{tree+inst}}(a,\vec{\tau})\,|^2$$

Wilson loop

Deformed N=4 theory

- N=2* theory ⟹ N=4 theory $\quad m = \epsilon_1$

$$Z_{\text{inst}} = 1 \qquad |Z_{\text{oneloop}}|^2 = \Delta(a) = \prod_{u<v} a_{uv}^2$$

⟹

$$\langle e^{\mathcal{C}} \rangle_{S^4} = \frac{1}{Z} \int d^N a \, \Delta(a) \, \mathrm{tr}\, e^{\frac{2\pi i a}{\epsilon_1}} \, e^{-N\, V(a,\vec{\tau})}$$

Potential

Interacting Matrix Models

- Example: $V(a) = \frac{1}{2\lambda} a^2 + g_4\, a^4$

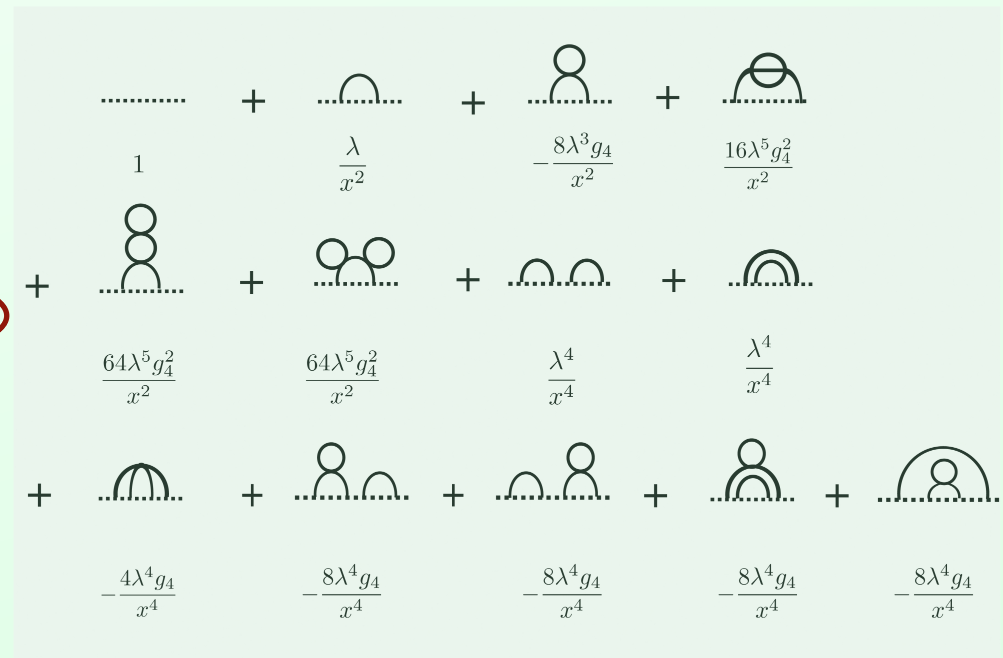
⟹

$$\langle e^{\mathcal{C}} \rangle = \frac{2}{\sqrt{\lambda_{\text{eff}}}} \left[I_1\left(\sqrt{\lambda_{\text{eff}}}\right) - \frac{1-\delta}{1+\delta} I_3\left(\sqrt{\lambda_{\text{eff}}}\right) \right] \sim e^{\sqrt{\lambda_{\text{eff}}}}$$

$$\lambda_{\text{eff}} = -\frac{2\, g_{YM}^2 N\, n_1 n_2}{1+\delta} \qquad \delta = \sqrt{1 + 48\, g_4\, \lambda^2}$$

Minimal Area String AdS

Exact formula for W !



N=2* theory

- **Periods :** (a, a_D) prepotential $\mathcal{F}$

$$a_D = \frac{1}{2\pi i}\frac{\partial F}{\partial a}$$ dual periods

- **S-duality :** ADE groups $\mathcal{F}(a) \to \tilde{\mathcal{F}}(a_D) = \mathcal{F}(a) - \pi\mathrm{i}\, a \cdot a_D$

- **Recursion relation :** $\mathcal{F} = \pi\mathrm{i}\tau\, a^2 + f(\tau, a, m)$ $\quad f = \sum_{n=1}^{\infty} f_n m^{2n}$

tree One-loop+instantons

$$\frac{\partial f_n}{\partial E_2} = -\frac{1}{24}\sum_{\ell=1}^{n-1}\frac{\partial f_\ell}{\partial a}\frac{\partial f_{n-\ell}}{\partial a}$$

$f_n(E_2, E_4, E_6)$ quasi-Modular function of weight 2n-2

$f_1^{\mathrm{one-loop}} \neq 0 \quad f_1^{\mathrm{inst}} = 0$!

the prepotential

- One loop :

$$f_{\text{one-loop}} = \frac{1}{4}\sum_{\alpha\in\Psi}\left[-(\alpha\cdot a)^2\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2 + (\alpha\cdot a+m)^2\log\left(\frac{\alpha\cdot a+m}{\Lambda}\right)^2\right]$$

$$= \frac{m^2}{4}\sum_{\alpha\in\Psi}\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2 - \sum_{n=2}^{\infty}\frac{m^{2n}}{4n(n-1)(2n-1)}C_{2n-2} \qquad C_k = \sum_{\alpha\in\Psi}\frac{1}{(\alpha\cdot a)^k}$$

- Solving the recursion

$$C_{n,r_1,r_2,\ldots} = \sum_{\alpha\in\Psi}\frac{1}{(\alpha\cdot a)^n}\sum_{\beta_1\neq\beta_2\ldots\in\Psi(\alpha)}\frac{1}{(\beta_1\cdot a)^{r_1}(\beta_2\cdot a)^{r_2}\ldots}\,.$$

⇒

$$f_1 = \frac{m^2}{4}\sum_{\alpha\in\Psi}\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2$$

$$f_2 = -\frac{m^4}{24}\left(C_2 + \frac{1}{4}C_{1,1}\right)E_2(q)$$

$$f_3 = -\frac{m^6}{720}\left(5E_2(q)^2 + E_4\right)C_4 - \frac{m^6}{576}\left(E_2(q)^2 - E_4(q)\right)C_{2,1,1}$$

q-Exact results for any ADE group !

SO(2n+1)/Sp(2n) duality

- Long and Short root sums

$$L_{n;m_1,m_2,\cdots,m_\ell} = \sum_{\alpha\in\Psi_{\rm L}} \sum_{\beta_1\neq\beta_2\neq\cdots\beta_\ell\in\Psi(\alpha)} \frac{1}{(\alpha\cdot a)^n(\beta_1\cdot a)^{m_1}(\beta_2\cdot a)^{m_2}\cdots(\beta_\ell\cdot a)^{m_\ell}}\,,$$

$$S_{n;m_1,m_2,\cdots,m_\ell} = \sum_{\alpha\in\Psi_{\rm S}} \sum_{\beta_1\neq\beta_2\neq\cdots\beta_\ell\in\Psi^\vee(\alpha)} \frac{1}{(\alpha\cdot a)^n(\beta_1^\vee\cdot a)^{m_1}(\beta_2^\vee\cdot a)^{m_2}\cdots(\beta_\ell^\vee\cdot a)^{m_\ell}}\,,$$

- Solving the recursion

$$\Longrightarrow \quad f_1 = \frac{m^2}{4}\sum_{\alpha\in\Psi_L}\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2 + \frac{m^2}{4}\sum_{\alpha\in\Psi_S}\log\left(\frac{\alpha\cdot a}{\Lambda}\right)^2$$

$$f_2 = -\frac{m^4}{24}L_2\,E_2(q) - \frac{m^4}{24}S_2\,E_2(q^2)$$

$$f_3 = -\frac{m^6}{720}\left(5E_2^2+E_4\right)(q)L_4 - \frac{m^6}{576}\left(E_2^2-E_4\right)(q)L_{2,1,1}$$

$$-\frac{m^6}{720}\left(5E_2^2+E_4\right)(q^2)S_4 - \frac{m^6}{576}\left(E_2^2-E_4\right)(q^2)S_{2,1,1}$$

- S-duality: $\quad \tau\to-\frac{1}{2\tau} \quad SO\leftrightarrow Sp \quad$!

AGT correspondence

- **Gauge/CFT dictionary:** SU(2) gauge +4 fundamentals. = Liouville CFT

$$\alpha_1 = \frac{\epsilon}{2} + \frac{1}{2}(\bar{m}_1 - \bar{m}_2) \quad \alpha_2 = -\frac{1}{2}(\bar{m}_1 + \bar{m}_2)$$

$$\alpha_3 = \epsilon + \frac{1}{2}(\bar{m}_3 + \bar{m}_4) \quad \alpha_4 = \frac{\epsilon}{2} + \frac{1}{2}(\bar{m}_3 - \bar{m}_4) \qquad \Delta_i = \alpha_i(Q - \alpha_i)$$

$$\alpha = \frac{\epsilon}{2} + a \quad \epsilon = \epsilon_1 + \epsilon_2 = Q \quad \epsilon_1 = b \quad \epsilon_2 = b^{-1} \quad z = q$$

- **Partition function vs four-point function :**

$$Z_{S^4}(m_i|q) = \int_\gamma d^N a \, |Z_{\mathbb{R}^4}^{SU(2)}(a, m_i|q)|^2 = \langle \mathcal{O}_{\alpha_1} \, \mathcal{O}_{\alpha_2} \, \mathcal{O}_{\alpha_3} \, \mathcal{O}_{\alpha_4} \rangle$$

$$Z_{\mathbb{R}^4}^{SU(2)}(a, m_i|q) = q^{\Delta - \Delta_3 - \Delta_4} \, (1-q)^{-2\alpha_2(Q-\alpha_3)} \, Z_{\text{one-loop}}^{U(2)}(a, m_i) Z_{\text{inst}}^{U(2)}(a, m_i q)$$

tree — U(1) — one loop — instantons

✻ Critical values of masses & rational Ω -backgd : $\bar{m}_3 + \bar{m}_4 = \epsilon_{1,2}$

⇒ q-Exact results ! $\quad \frac{\epsilon_1}{\epsilon_2} = -\frac{p}{q} \quad$ ⇒ Minimal models !

The gauge partition function

- **Critical masses:** $\bar{m}_3 + \bar{m}_4 = \epsilon_1 \quad \Longrightarrow \quad a_\pm = -\bar{m}_3, -\bar{m}_4$

The one-loop partition function vanishes

double poles

$$Z^{U(2)}_{\text{inst},\pm} = {}_2F_1\left({}^{(a_\pm - \bar{m}_1)\epsilon_1, (a_\pm - \bar{m}_2)\epsilon_1}_{(2a_\pm + \epsilon)\epsilon_1} \Big| q \right)$$

Only single-column Young diagrams contribute !

- **Ising model:** $b^2 = \dfrac{\epsilon_1}{\epsilon_2} = -\dfrac{3}{4}$

$$\alpha_i \in \{\alpha_o, \alpha_\sigma, \alpha_\varepsilon\} = \left\{0, -\frac{b}{2}, -\frac{1}{2b}\right\} \cup \left\{Q, Q + \frac{b}{2}, Q + \frac{1}{2b}\right\}$$

$$Z^{U(2),\text{inst}}_{\langle\sigma\sigma\sigma\sigma\rangle_\varepsilon} = {}_2F_1\left(\frac{3}{4}, \frac{5}{4}, \frac{3}{2}, q\right) = \frac{\sqrt{2}\,\sqrt{1-\sqrt{1-q}}}{\sqrt{q(1-q)}}$$

$$Z^{U(2),\text{inst}}_{\langle\sigma\sigma\sigma\sigma\rangle_o} = {}_2F_1(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}; q) = \frac{\sqrt{1+\sqrt{1-q}}}{\sqrt{2}\,\sqrt{1-q}}$$

$$\frac{C_-}{C_+} = \frac{\text{Res}_{a_-} |Z^{\text{one-loop}}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_a}|^2}{\text{Res}_{a_+} |Z^{\text{one-loop}}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_a}|^2} = 4$$

$$Z^{U(2)}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle} = C_+ |q^{\Delta_+ - \Delta_3 - \Delta_4}\, Z^{U(2)}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{\mathcal{O}_+}}|^2 + C_- |q^{\Delta_- - \Delta_3 - \Delta_4}\, Z^{U(2)}_{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{\mathcal{O}_-}}|^2$$

$$= \frac{|1+\sqrt{1-q}| + |1-\sqrt{1-q}|}{2|q|^{1/4}|1-q|}$$

Reproduce the Ising 4-pt correlator !

Conclusions

- Chiral deformations of N=4 :

✓ τ_J-deformations of N=4: Exact formula for Z and W (Matrix model)

⇒ S=ln W (prediction for the Area of the dual string world sheet on AdS) !

- Mass deformations of N=4

✓ N=2* at small mass: q-Exact formula for F for any gauge group (Recursion relation)

⇒ S-duality (self duality of ADE , B <—> C) !

- SU(2) + 4 fundamentals

✓ AGT dual of minimal models: q-Exact formula for Z on the four-sphere (Localization)

⇒ AGT-duality (match with Ising correlators) !

- Open questions: Other Wilson loops, perturbative tests W, 5d recursions, pert+insts