Integrability in $\mathcal{N} = 2$ gauge theories

J.E. Bourgine

INFN

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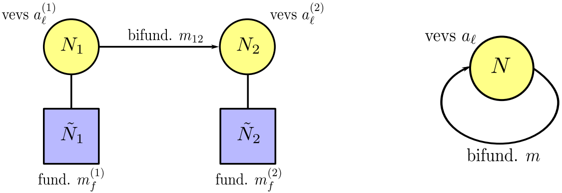
$\mathcal{N} = 2$ class $\mathcal{S}$ SUSY gauge theories

- Gauge group $\otimes_i SU(N_i)$, classified by ADE (+affine) quiver diagrams



- Field content:
  - To each node i: Gauge multiplet $(A_\mu^{(i)}, \phi^{(i)})$ in the adjoint representation of $SU(N_i)$, exponentiated gauge coupling $q_i = e^{-8\pi^2/g_i^2} e^{i\theta_i}$ (+renormalization).
  - To each node i: Chiral multiplets $(\varphi_f^{(i)})$ in the fundamental representation of $SU(N_i)$ with masses $m_f^{(i)}$, $f = 1 \cdots \tilde{N}_i$.
  - To each link $i \to j$: Chiral multiplets $(\varphi^{(ij)})$ in the bifundamental representation of $SU(N_i) \times SU(N_j)$ with mass m_{ij}.
- Coulomb branch: $\phi_{\text{cl}}^{(i)} = \text{diag}(a_1^{(i)}, \cdots, a_{N_i}^{(i)})$.

Seiber-Witten theory

[Seiberg-Witten 1994]

Effective IR theory defined by the prepotential $\mathcal{F}$,

$$\mathcal{L}_{\text{eff}} \sim \text{Im} \int d^4\theta \mathcal{F}[\Phi^i]$$

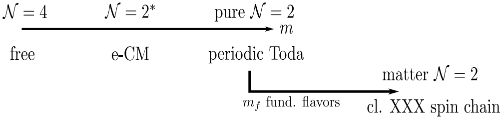
The prepotential is determined by an algebraic curve and a meromorphic differential form dS from the relations

$$a_\ell = \oint_{A_\ell} dS, \quad \frac{\partial \mathcal{F}}{\partial a_\ell} = \oint_{B_\ell} dS.$$

For instance, for pure $SU(N)$ SYM,

$$w + \frac{q^{2N}}{w} = 2\prod_{\ell=1}^{N}(z - a_\ell), \quad dS = z\frac{dw}{w}.$$

☞ Recognize a classical integrable system (Hitchin system, Whitham tau functions)



Localization in the Omega-background

[Nekrasov 2003]

♠ To regularize the infinite volume of $\mathbb{R}^4$ consider the theory in the Omega background $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2_{\epsilon_2}$ ($\Rightarrow$ break Lorentz invariance but keep enough SUSY).

♠ Non-perturbative computation of the partition function

$$\mathcal{Z} = \mathcal{Z}_{\text{cl.}} \mathcal{Z}_{\text{1-loop}} \mathcal{Z}_{\text{inst.}}$$

For instance, for $SU(N)$ SYM ($\epsilon_+ = \epsilon_1 + \epsilon_2$),

$$\mathcal{Z}_{\text{inst.}} = \sum_{n=0}^{\infty} q^n \int \prod_{i=1}^{n} Q(\phi_i) \frac{d\phi_i}{2i\pi} \prod_{i<j} K(\phi_i - \phi_j) = \sum_{\vec{Y}} q^{|\vec{Y}|} \mathcal{Z}_{\text{vect.}}(\vec{a}, \vec{Y}) \mathcal{Z}_{\text{fund.}}(\vec{m}; \vec{a}, \vec{Y}).$$

☞ In the limit $\epsilon_1, \epsilon_2 \to 0$, recover Seiberg-Witten theory [Nekrasov-Okounkov 2003] (continuous profile of Young diagrams, minimize an effective action).

$$\mathcal{Z} \sim e^{\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}}$$

AGT correspondence

[Alday, Gaiotto, Tachikawa 2009]

♠ Correspondence 4d/2d between $\mathcal{N}=2$ $SU(2)$ partition functions and correlator of Liouville theory, a 2d Conformal Field Theory with Virasoro symmetry.

4d $\mathcal{N}=2$ Gauge theory	$\longleftrightarrow$	Liouville
$SU(2)$ $\tilde{N}=4$ fund. flavors		Sphere with 4 punctures
$\mathcal{N}=2^*$ (adjoint matter)		Torus with one punctures
Quiver theories		Higher genus Riemann surfaces

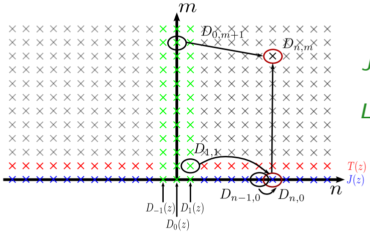
♠ Extended to $SU(N)$ gauge groups and Toda CFTs (with $W^{(N)}$ symmetry) by [Wyllard 2009]

♠ Recently extended to 5d ($\mathbb{R}^2_{\epsilon_1}\times\mathbb{R}^2_{\epsilon_2}\times\mathcal{C}^1$) and 6d ($\mathbb{R}^2_{\epsilon_1}\times\mathbb{R}^2_{\epsilon_2}\times\mathbb{T}^2$) theories with q-Virasoro and e-Virasoro conformal blocks.

♠ Proofs [Alba, Fateev, Litvinov, Tarnopolsky 2011], [Maulik-Okounkov 2012], [Morozov et. al.],...

SHc algebra

♠ Hopf algebra constructed by [Vasserot-Schiffmann 2012] from DDAHA algebra.



$$J_{\pm n} = (-\sqrt{-\epsilon_1/\epsilon_2})^{-n} D_{\mp n,0},$$

$$L_{\pm n} = \frac{1}{n}(-\sqrt{-\epsilon_1/\epsilon_2})^{-n} D_{\mp n,1} + \frac{1}{2}(1-n)N_c \frac{\epsilon_+}{\epsilon_2} J_{\pm n}.$$

♠ Covariance of the building blocks for $\mathcal{Z}_{\text{inst}}$
[Kanno, Mastuo, Zhang 2013], [JEB, Matsuo, Zhang to appear].

♠ Contains $U(1)$, Virasoro, $W^{(N)}$ subalgebras ⇒ proof of AGT:
AFLT basis $|\vec{a}, \vec{Y}\rangle \propto$ Generalized Jack polynomials [Morozov-Smirnov 2014]

♠ Integrability: $D_{0,2}$ Calogero-Moser Hamiltonian

★★★ Recently: qq-character = double deformation of SW curve.
[JEB, Matsuo, Zhang to appear]

Intermezzo

Integrability for $\mathcal{N} = 2$ on $\mathbb{R}^4$ ✓

Integrability for $\mathcal{N} = 2$ on $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2_{\epsilon_2}$ ✓

☞ What lies between the two ?

Nekrasov-Shatashvili limit

♠ Consider the $\epsilon_2 \to 0$ limit of the Omega-background: $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2$

♠ Observe the quantization of the classical integrable systems ($\mathbb{R}^4$) in the form of a TBA-like NLIE [Nekrasov-Shatashvili 2009], confirmed using Mayer cluster expansion in [Meneghelli-Yang 2013],[JEB 2014]

♠ Derivation of Bethe and TQ equations (minimizing the sum over $\vec{Y}$).
For instance, matter $SU(N)$ SYM:

$$1 = q\frac{m(u_r)}{\Xi(u_r)\Xi(u_r+\epsilon_1)}\prod_{s=1}^{N_B}\frac{u_r-u_s-\epsilon_1}{u_r-u_s+\epsilon_1},\quad \Xi(x)=\prod_{\ell=1}^{N}(x-\xi_\ell),$$

[Poghossian 2010], [Fucito, Morales, Ricci-Pacifici, Poghossian 2011],...

☞ Connection between the two approaches ? [JEB-Fioravanti to appear]

☞ Semiclassical limit of Liouville theory under AGT-correspondence.

TQ equation $\sim$ Schrödinger equation $\sim$ Quantum SW curve ($\epsilon_1 = \hbar$)

Bispectral duality [Mironov, Morozov Shakirov 2009-2010],[JEB 2012]

Understanding the ϵ_2-deformation

Second quantization of the QIM

- NS limit of the SHc transformation [JEB 2014]
- qq-characters and quantum Seiberg-Witten geometry [JEB, Matsuo, Zhang to appear]
- Derivation of the first ϵ_2-correction [JEB-Fioravanti 2015]^2

$$\mathcal{Z}_{\text{inst}} \sim \int D[\rho,\varphi] \exp\left(\epsilon_2^{-1}\mathcal{S}_0[\rho,\varphi] + \mathcal{S}_1[\rho,\varphi]\right)$$

where the NS action $\mathcal{S}_0[\rho,\varphi]$ (EOM produces the TBA-like NLIE),

$$\mathcal{S}_0[\rho,\varphi] = \frac{1}{2}\int \rho(x)\rho(y)G(x-y)dxdy + \int \rho(x)\varphi(x)dx + \int \text{Li}_2(Q_0(x)e^{-\varphi(x)})\frac{dx}{2i\pi},$$

gives the correct one-loop result, up to an quantum extra correction

$$\mathcal{S}_1[\rho,\varphi] = \frac{i\pi}{2}\int dx\rho(x)\nabla\rho(x),$$

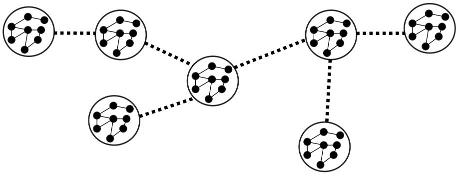
with $\nabla\rho = \partial\rho_{\text{reg}} - \partial\rho_{\text{sing}}$.

Confinement of instantons

♠ Pole pinching the contour of integration ($\epsilon_+ = \epsilon_1 + \epsilon_2$)

$$K(\phi) = \frac{\phi^2(\phi^2 - \epsilon_+^2)}{(\phi^2 - \epsilon_1^2)(\phi^2 - \epsilon_2^2)}$$

⇒ Decompose into long- and short-range interaction (G and p links)



♠ Field theory argument for the factorization

$$\mathcal{Z}_{\text{inst}} = \int DXe^{-\mathcal{S}_{\text{long}}[X]}\, \mathcal{Z}_{\text{short}}[X], \quad \mathcal{S}_{\text{long}}[X] = \frac{1}{2}\int \frac{dxdy}{(2i\pi)^2} G^{-1}(x-y)X(x)X(y)$$

$$\mathcal{Z}_{\text{short}}[X] = \sum_{N=0}^{\infty} \frac{q^N \epsilon_2^{-N}}{N!} \int \prod_{i=1}^{N} Q(\phi_i) e^{X(\phi_i)} \frac{d\phi_i}{2i\pi} \prod_{\substack{i,j=1\\ i<j}}^{N} \frac{\phi_{ij}^2}{\phi_{ij}^2 - \epsilon_2^2},$$

Perspectives

- Computation of $\mathcal{Z}_{\text{short}}[X]$ at all orders in ϵ_2 ?
- Confinement of mesons in $\mathcal{N} = 4$ amplitudes ?
  [Fioravanti, Piscaglia, Rossi 2015], [Bonini, Fioravanti, Piscaglia, Rossi 2015]
- Lifting to 5d/6d cases (Ding-Iohara, DAHA, Macdonald polynomials,...)
- How about $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2_{\epsilon_2} \times \mathbb{R}^2_{\epsilon_3}$?

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Thank you !!!