THE GROWING TOOLBOX OF PERTURBATIVE QCD

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Outline

- Introduction
- QCD at future colliders
- Selected examples of new tools
- Soft gluons beyond leading power
- Outlook

INTRODUCTION

Where we stand

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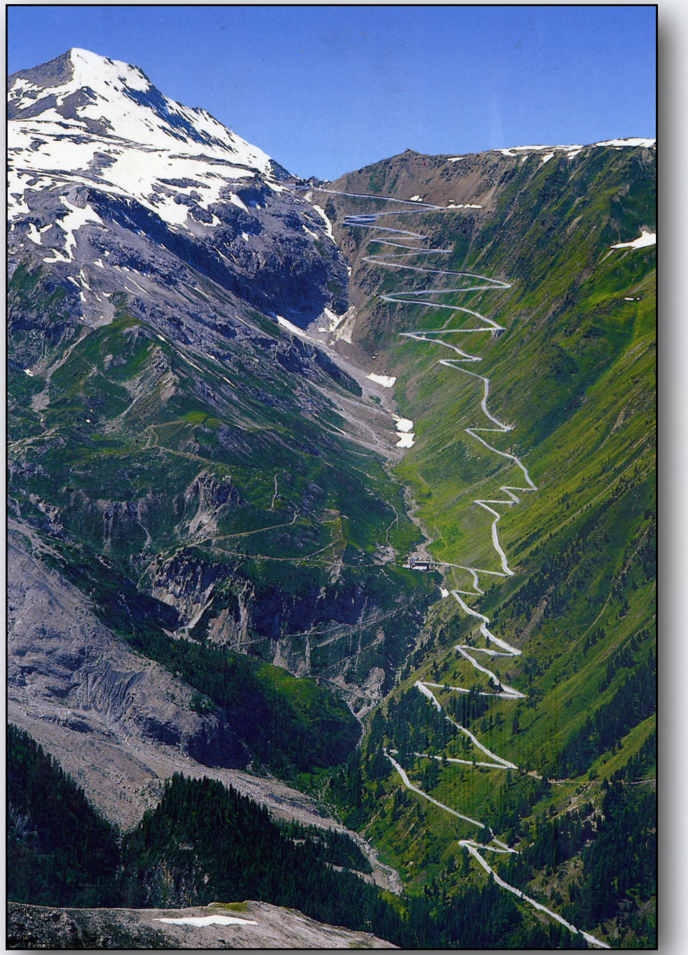
- In the coming years the need will be for precise predictions of increasingly complex SM processes.
  - A great time for the QCD community.
- The run-up to the LHC has seen a vast effort and great progress in precision phenomenology: PDF's, jets, hard cross sections, resummations, and more.
  - *A continuing effort that will hopeful pay off during Run Two!*

QCD BEYOND LHC



QCD at future colliders

Hadron colliders:

Hadron Collider
you build,
much QCD
you need.

Hadron colliders:

QCD at future colliders

Lepton colliders:

QCD at future colliders

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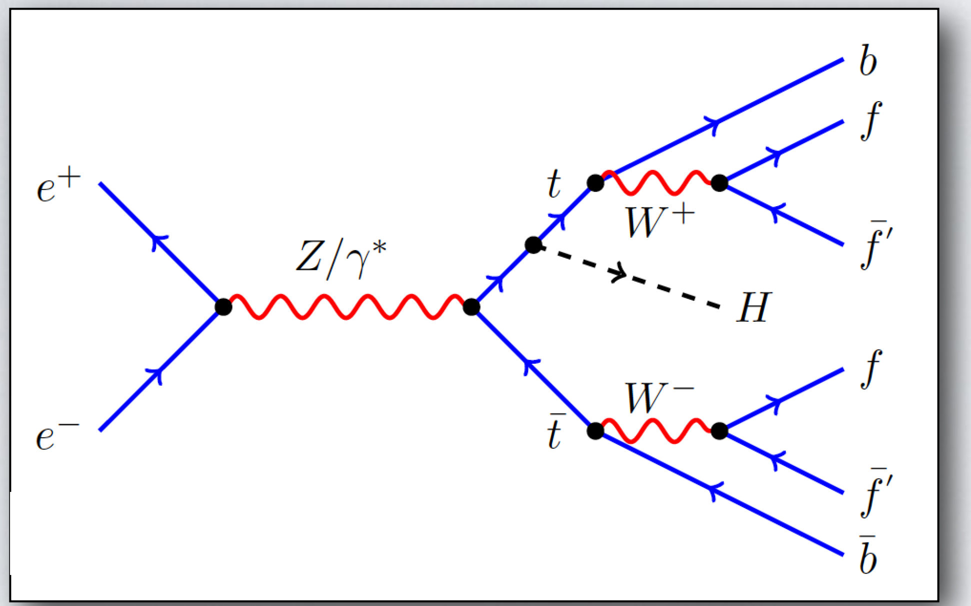
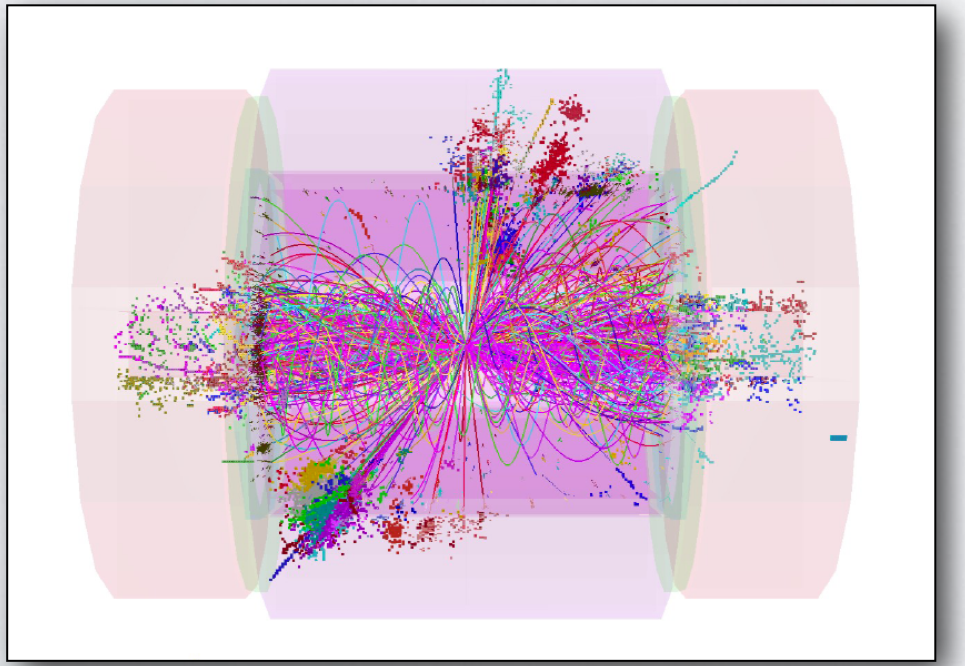
Lepton colliders:

Lepton collider jets



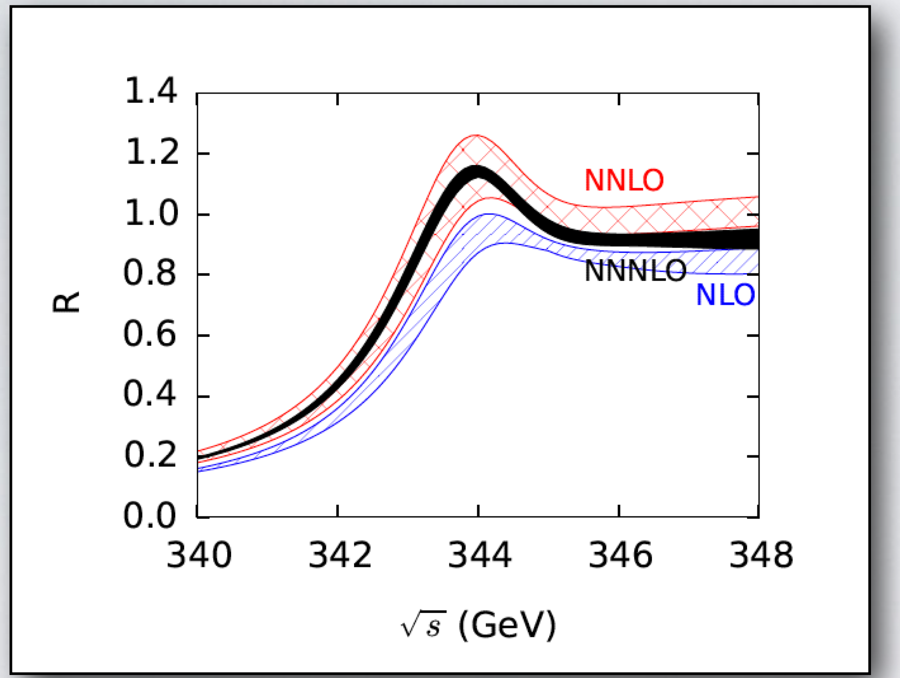
ttH productions can yield 8-jet final states



"Underlying event" at CLIC3TeV (Vos, at LCWS14)

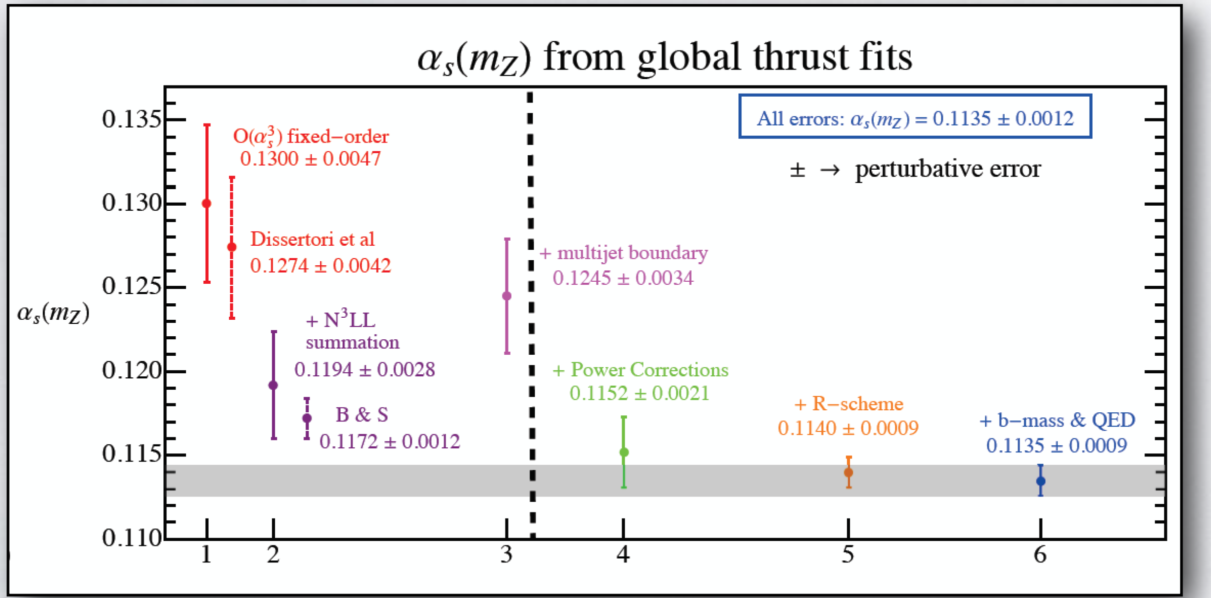
- Relatively small samples at lepton colliders
  - *Important to use hadronic final states.*
- Interesting processes yield many jets.
  - *The best available jet tool are needed.*
- There actually is an Underlying Event.
  - *γγ → hadrons, pair production.*
- Jet algorithms are important.
  - *Boost invariance is less relevant.*
- Algorithms affect jet shapes and areas.
  - *Forward region has high backgrounds.*
- Jet radius analysis must be re-tuned.
  - *Strong energy dependence of UE.*
- Jet substructure analysis still important.
  - *But fatter jets and unboosted objects.*

Standard model parameters



Threshold tt production at N^{3}LO (Beneke et al.)

- M_{top} must be precisely defined.
  - ✦ *Minimize non-perturbative effects.*
- Lepton colliders provide the best precision.
  - ✦ *Scanning the threshold for tt production.*
- The most refined tools of QCD are needed.
  - ✦ *High orders, effective theories, resummations.*
- M_{top} can be determined to within 50 MeV.



Do we really know αs as well as we think? (Mateu 2013)

- Unsolved discrepancies in event shape αs fits.
  - ✦ *Unexpected influence of power corrections.*

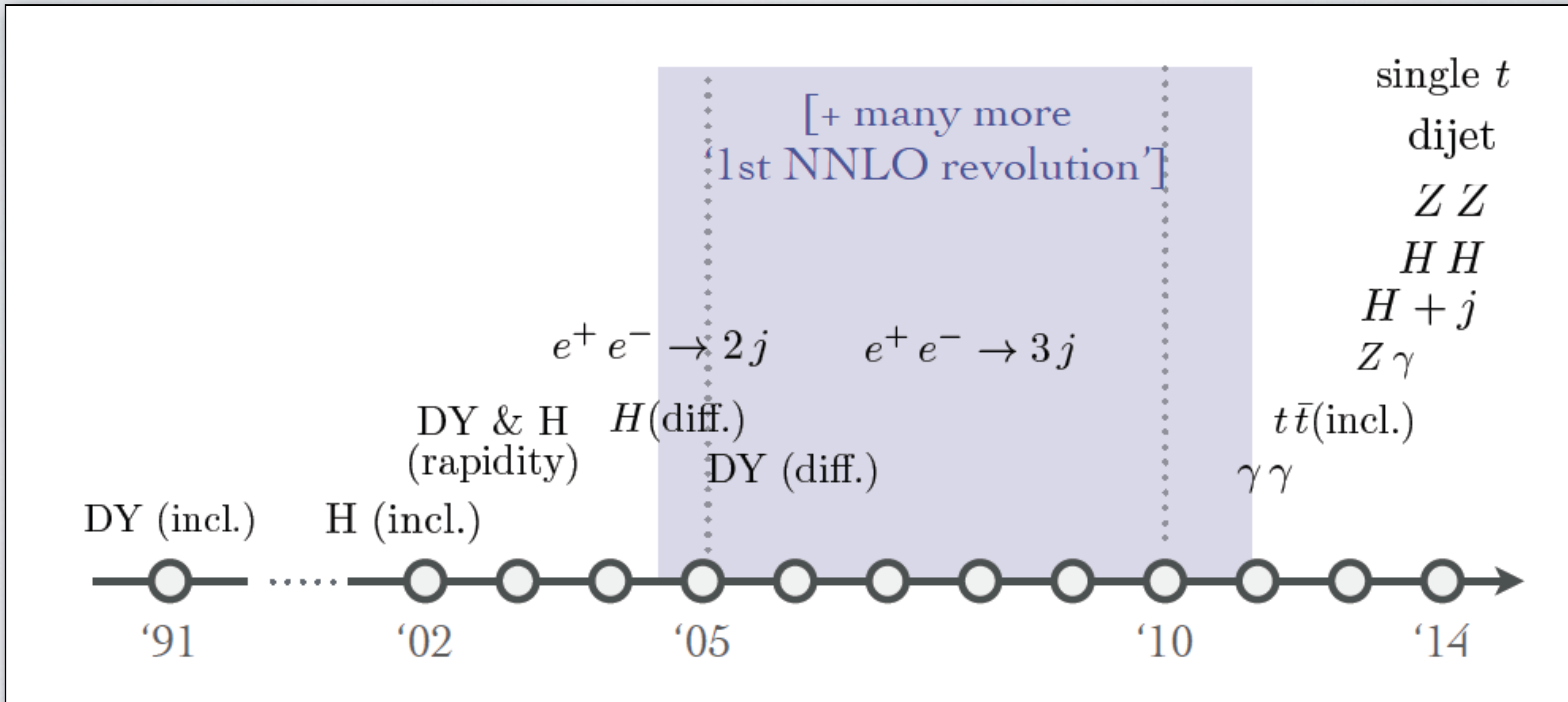
$$\alpha_s(M_Z^2) = 0.1172 \pm 0.0022 \quad \text{thrust (BS)}$$
$$\alpha_s(M_Z^2) = 0.1220 \pm 0.0031 \quad \text{jet mass (SC)}$$
$$\alpha_s(M_Z^2) = 0.1135 \pm 0.0010 \quad \text{thrust (AFHMS)}$$

- Very accurate studies: NNLO, N^{3}LL, PC's.
  - ✦ *Do we need a more precise definition of α_s?*

NEW TOOLS

Two loops is the new one loop

- Two-loop calculations are not yet a commodity: they are largely custom-made and expensive.
- A major stumbling block has been the subtraction of infrared and collinear singularities.
- Progress has been slow but is rapidly speeding up: automation is on the way.



From Claude Duhr's talk at ICHEP 2014

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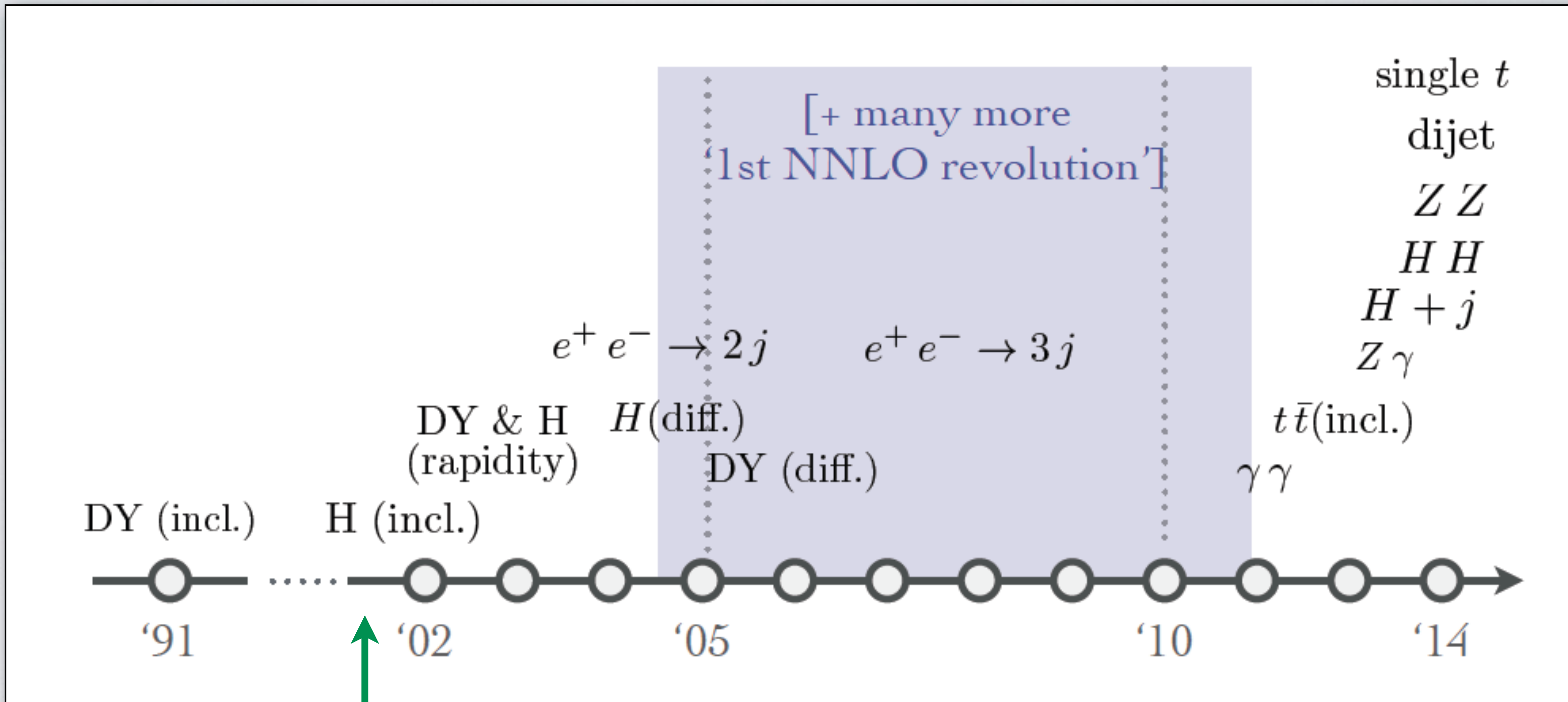

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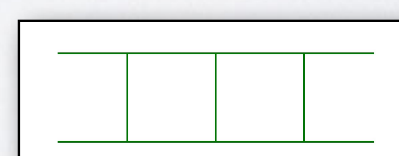
V. Smirnov hep-ph/9905323



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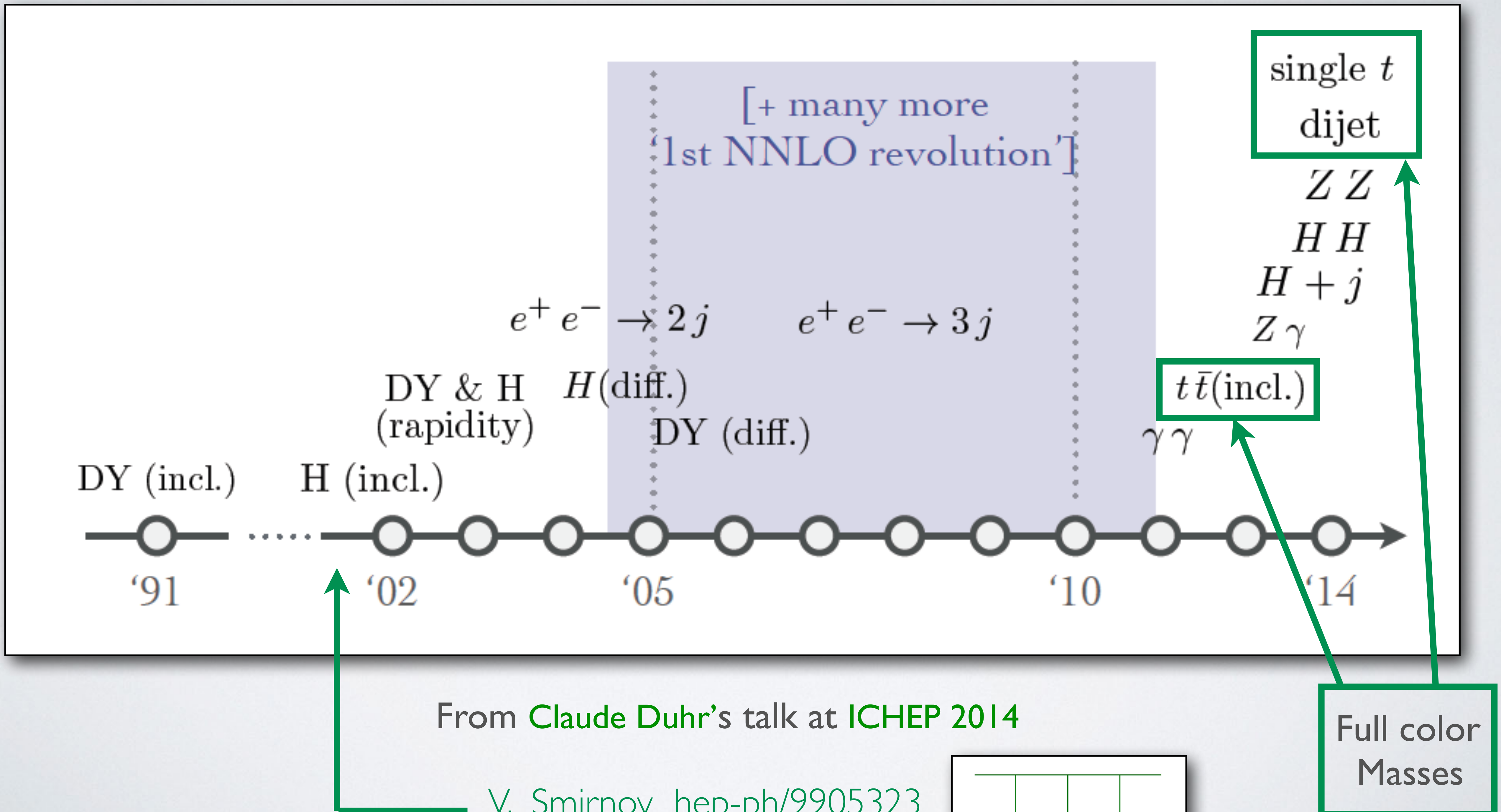

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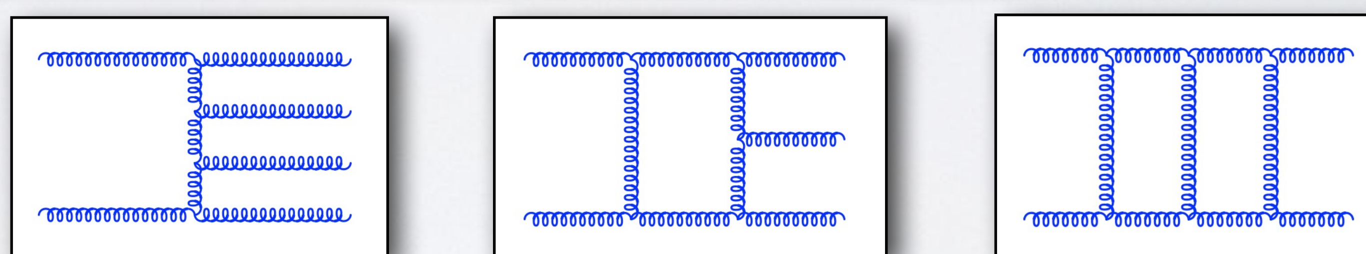
The NNLO subtraction problem

- A well-known problem: infrared and collinear divergences cancel between final states with different particle content and different phase spaces.
- The cancellation must be performed locally in phase space to allow for generic observables.
- "Simple" subtraction counterterms must be constructed in each phase space.
- A surprisingly hard problem, on the table for more than a decade.

$$\mathrm{d}\hat{\sigma}_{NNLO} \sim \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{RR}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{RV}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{VV}_{NNLO}$$

Different final-state multiplicities conspire to cancel infrared an collinear poles



	analytic	FS colour	IS colour	local
antenna subtraction	✓	✓	✓	✗
STRIPPER	✗	✓	✓	✓
q_T subtraction	✓	✗	✓	✓
reverse unitarity	✓	✗	✓	-
Trócsányi et al	✗	✓	✗	✓

Comparing subtraction algorithms, from James Currie's talk at LoopFest

- Several solutions are now available.
- Analytical vs. numerical approaches.
- Dedicated vs. general algorithms.
- Several groups at work.
- New: N-jettiness subtraction.
- No silver bullet yet.

Iterated integrals

- A large class of integrals arising from Feynman diagrams (but not all!) can be expressed as "iterated integrals", yielding functions in the class of polylogarithms. At one loop

$$\log z = -\int_0^{1-z} \frac{dt}{1-t}\,, \qquad \mathrm{Li}_2(z) = \int_0^z \frac{dt}{t} \int_0^t \frac{du}{1-u}$$

- At higher orders one ancounters more general examples, such as Harmonic Polylogarithms or Goncharov Polylogarithms

$$G_{a_1,\ldots,a_n}(z) \equiv \int_0^z \frac{dt}{t-a_1}\, G_{a_2,\ldots,a_n}(t)\,,$$

- Notice that all these integrals are of a "d log" form: at each step one integrates over the logarithm of a simple (here linear) function of the integration variables.
- The parameters a_n are the locations of singular points and have physical meaning.
- Iterated integrals are organized by a powerful underlying algebraic structure, described by the "Symbol" map or by a Hopf algebra with a notion of "Co-product" (Duhr).
- In particular each such function can be assigned a "weight" w, equal to the number of iterations. For example $\mathrm{Li}_2(z)$ has weight $w = 2$, and $\zeta(n)$ has weight $w = n$.
- These structures were uncovered in the context of studies of N=4 Super Yang-Mills theory amplitudes, where they have played a pivotal role.
- We now see powerful new applications to ordinary QCD (Henn, Smirnov, Von Manteuffel)

A good N^{p}LO harvest

- A wealth of results for on-shell production of electroweak boson pairs: ZZ, W^+W^-, Wγ, Zγ (Zurich group, see e.g. 1405.2219, 1507.062570).
- All helicity amplitudes for pp → VV' at two-loop now known. (Caola, Henn, Melnikov, Smirnov^2, 1404.5590, 1408.6409, 1503.08759).
- Preliminary results for γ* γ* production (Anastasiou *et al.* 1408.4546).
- Differential distributions for associated ZH production (Ferrera, Grazzini, Tramontano, 1407.4747).
- Differential distributions in Higgs + one jet production (Chen *et al.* 1408.5325; Boughezal et al. 1504.07922).
- Differential distributions in W,Z + one jet production (Boughezal et al. 1504.02131; Gehrmann *et al.* 1507.02850;).
- Differential distributions for t-channel single top production in the `structure function' approximation (Brucherseifer, Caola, Melnikov, 1404.7116).
- Progress towards the construction of a general basis for two-loop master integrals (Mastrolia et al.; Badger, Frellesvig, Zhang, 1407.3133).
- The three-loop angle-dependent cusp anomalous dimension in QCD (Grozin, Henn, Korchemsky, Marquard, 1409.0023).
- The three-loop massless multi-parton soft anomalous dimension in QCD (Almelid, Duhr, Gardi 1507.00047).

Three loops is the new two loops

- After the landmark calculation of three-loop DIS structure functions by Moch, Vermaseren and Vogt a decade ago, the next PQCD challenge has been the computation of a cross section without an OPE at three loops. The "Drell-Yan" process is the best candidate.
- At LHC, "Drell-Yan" means vector boson production and Higgs production via gluon fusion. The phenomenological impact is evident, especially given the large corrections to Higgs production at one and two loops.
- Approximate three-loop results using threshold and Regge limits exist (Moch, Vogt, 2005; LM, Laenen, 2005; Ball, Bonvini, Forte, Marzani, Ridolfi, 2013).
- The full calculation is now complete (Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistiberger 1503.06056), in a threshold expansion to essentially arbitrary order.
- The threshold expansion is also a "soft gluon" expansion

$$\widehat{\sigma} = \widehat{\sigma}(z)\,, \qquad z = \frac{Q^2}{\hat{s}}\,, \qquad \widehat{\sigma}(z) = \widehat{\sigma}_{SV} + \widehat{\sigma}_0 + (1-z)\,\widehat{\sigma}_1 + \mathcal{O}\big[(1-z)^2\big]$$

- The three-loop soft-virtual contribution is fully predicted by threshold resummation except for $\delta(1 - z)$ contributions. Resummation at next-to-leading power is under study.
- The "Drell-Yan" timeline: 1979 - 1991 - 2002 - 2015.

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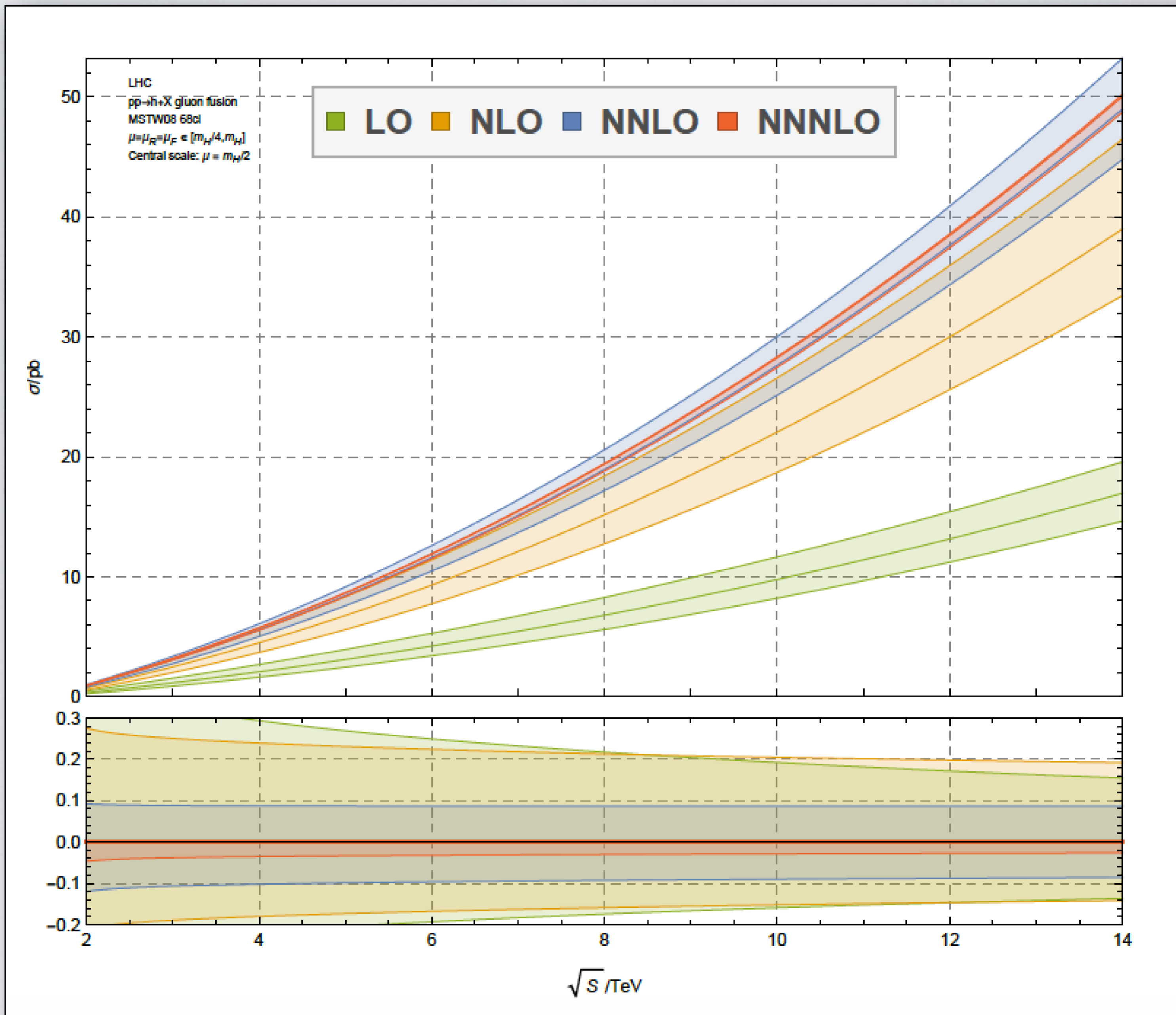

The partonic cross section for SM Higgs production through gluon fusion up to N^{3}LO, with scale uncertainties.

Bold projections

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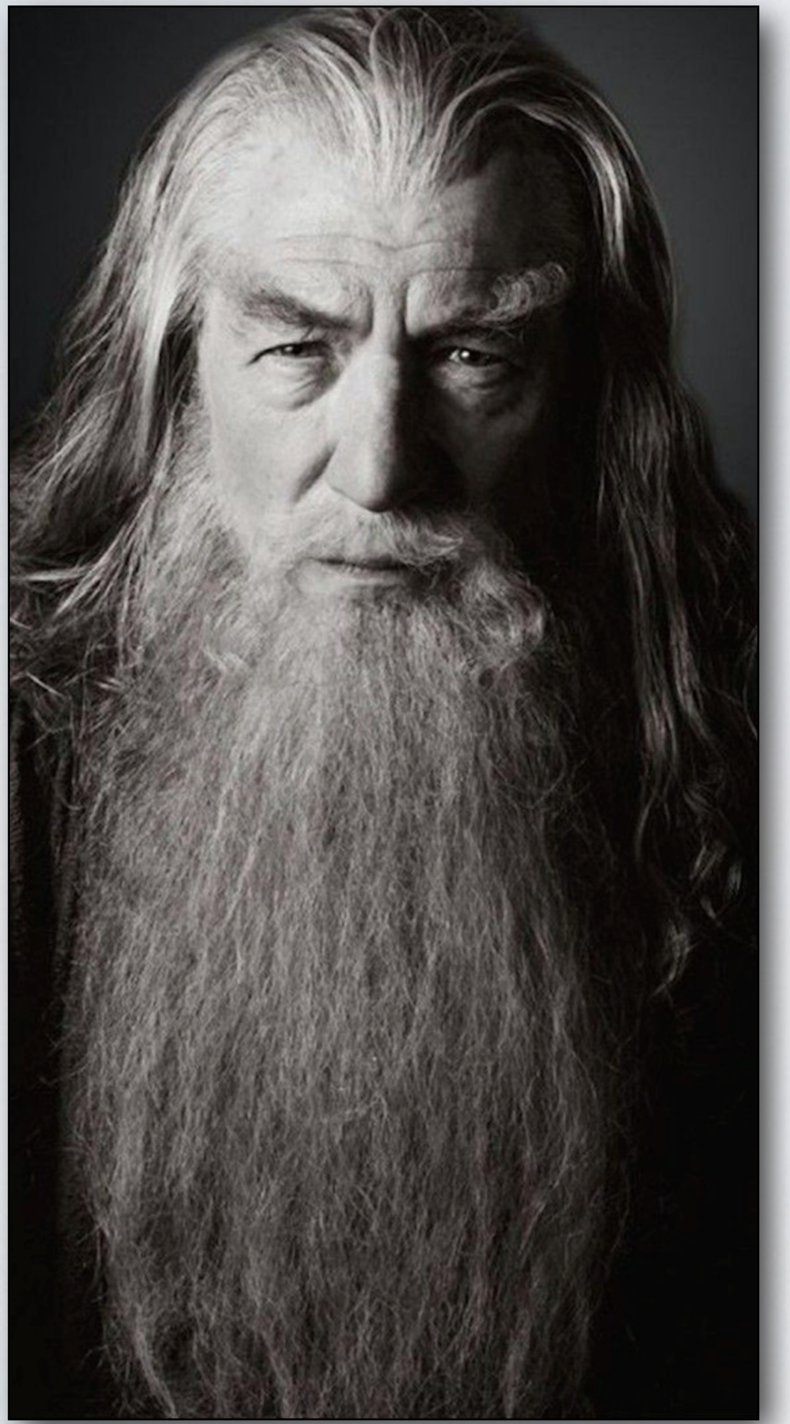

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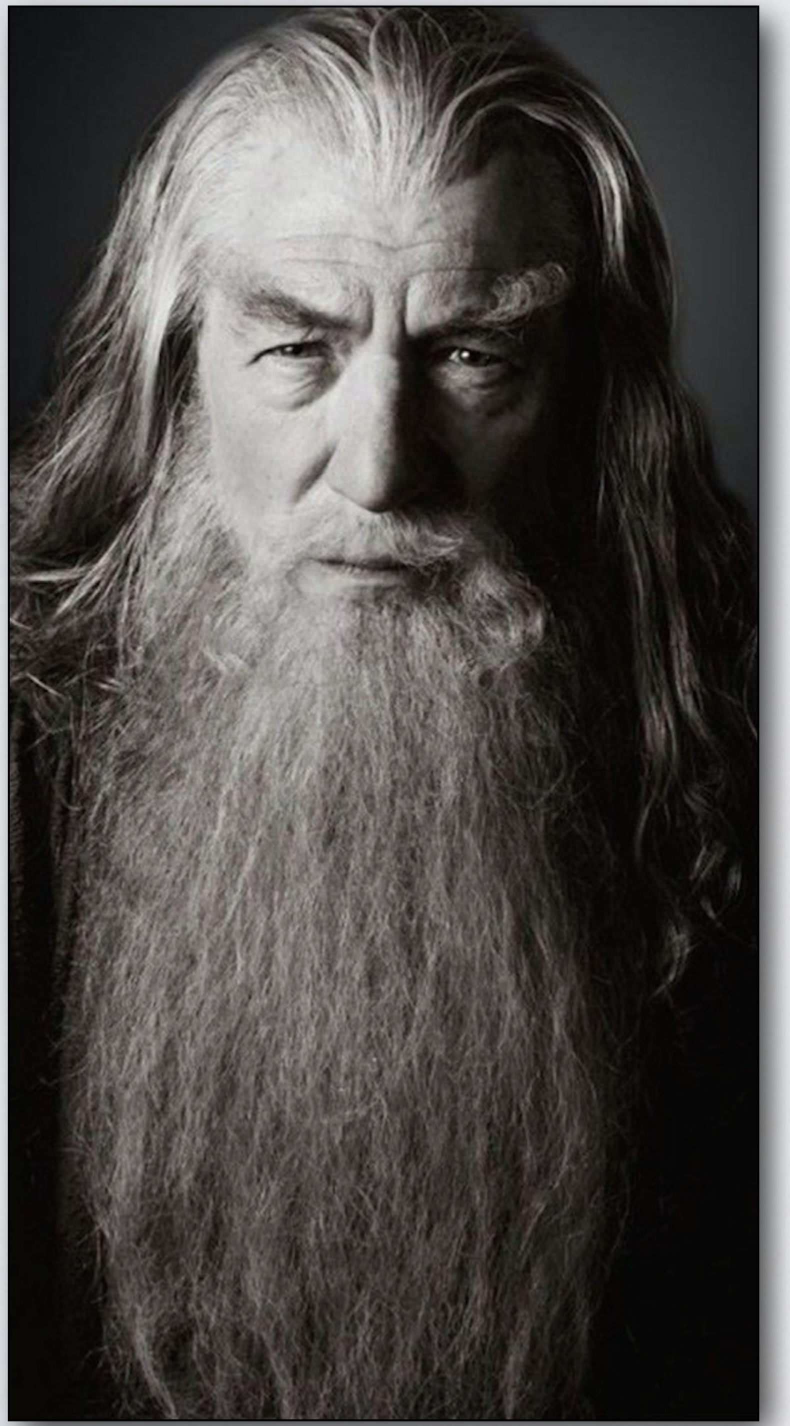

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  - *Structure functions at four loops, N^4LL threshold resummation, detailed analytic power corrections.*
  - *Significant impact on parton distributions.*
- Electron-positron annihilation (inclusive).
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  - *Control of threshold and non-threshold logarithms.*

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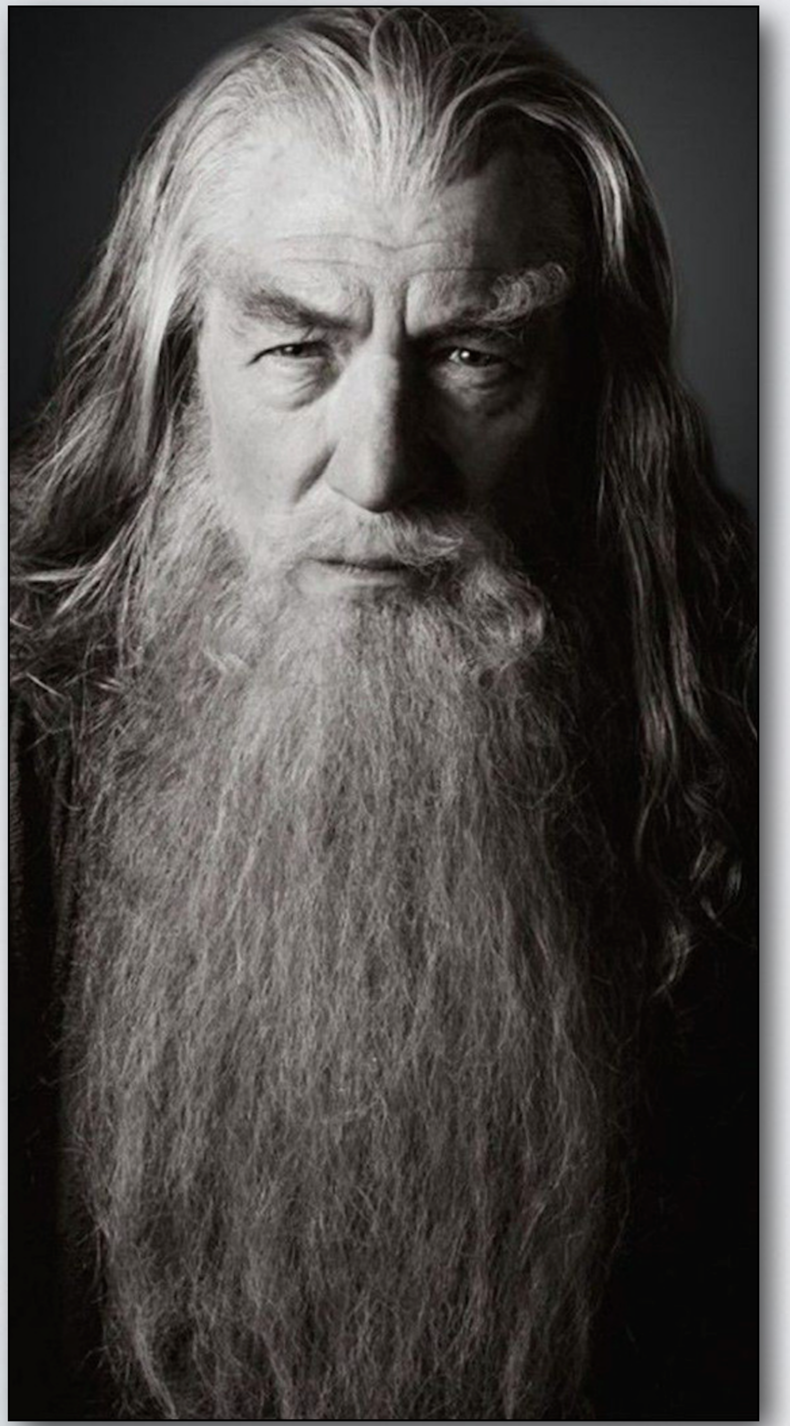

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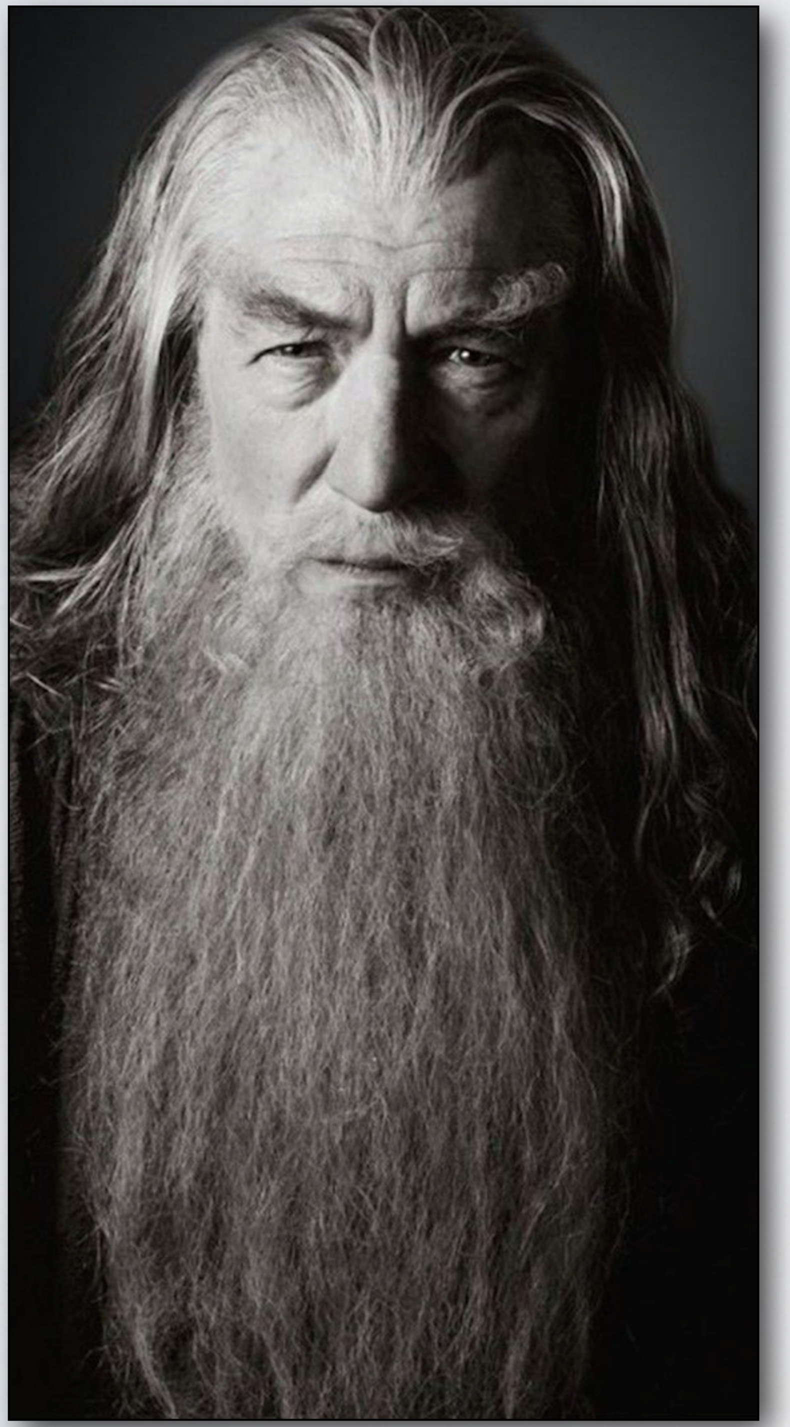

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- Full control of SM phenomenology below percent level.
  - *We must hope we will NOT need this.*

THRESHOLDS

Logarithms

- Multi-scale problems in renormalizable quantum field theories have perturbative corrections of the form $\alpha_s^n \log^k \left(Q_i^2/Q_j^2\right)$, which may spoil the reliability of the perturbative expansion. However, they carry important physical information.
  - Renormalization and factorization logs: $\alpha_s^n \log^n \left(Q^2/\mu^2\right)$
  - High-energy logs: $\alpha_s^n \log^{n-1} \left(s/t\right)$
  - Sudakov logs: $\alpha_s^n \log^{2n-1} \left(1-z\right) , \quad 1-z = W^2/Q^2, \, 1 - M^2/\hat{s} \, , \, Q_\perp^2/Q^2, \ldots$

- Logarithms encode process-independent features of perturbation theory. For Sudakov logs: the structure of infrared and collinear divergences.

$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \qquad \Longrightarrow \qquad \ln(m^2/Q^2)$$

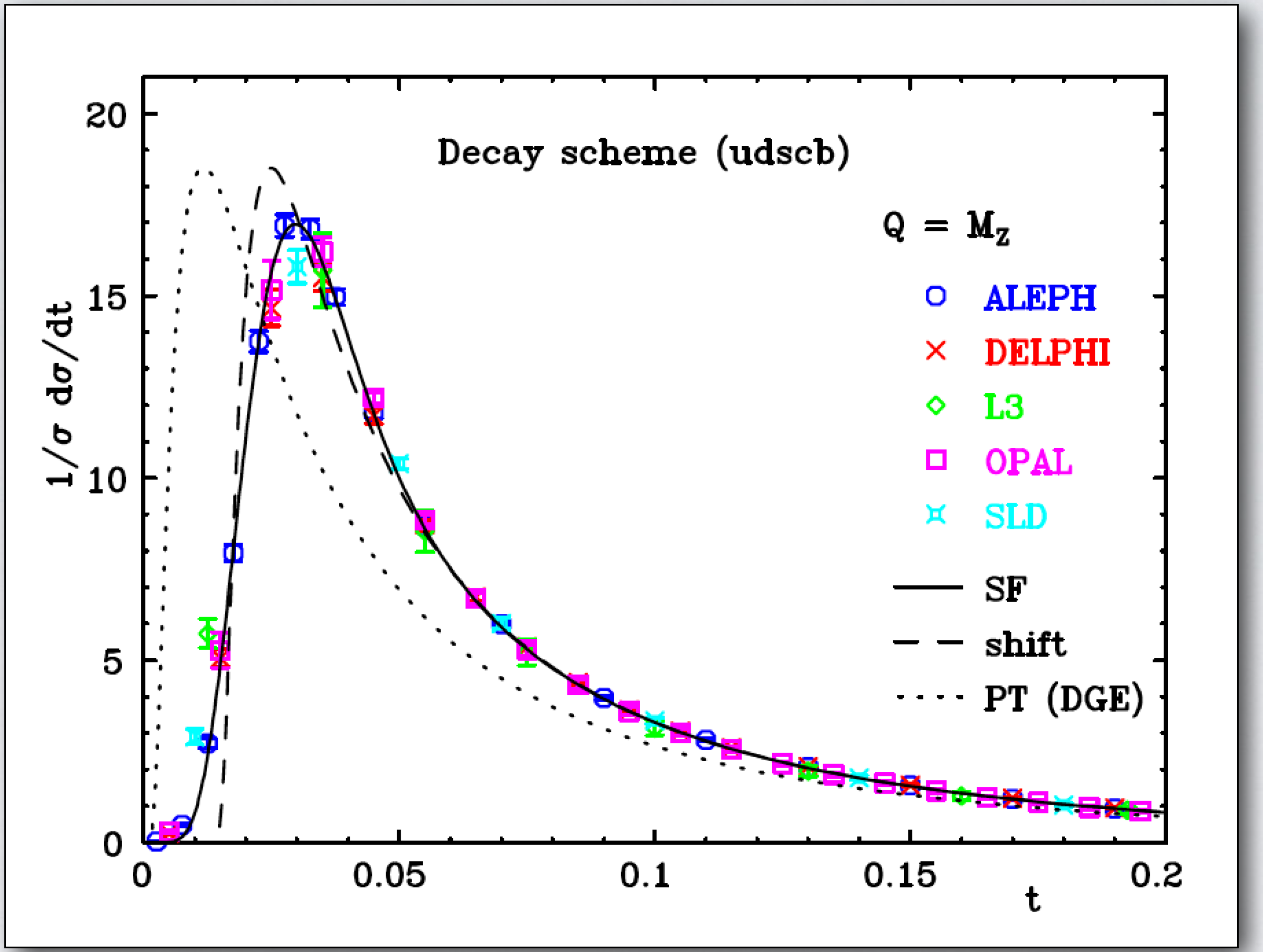
  - For inclusive observables: analytic resummation to high logarithmic accuracy.
  - For exclusive final states: parton shower event generators, (N(N))LL accuracy.

Gradually merging

- Resummation probes the all-order structure of perturbation theory.
  - Non-perturbative contributions to QCD cross sections can be estimated.
  - Links to the strong coupling regime can be established for special gauge theories.

Logarithms at work

Predictions for the thrust distribution at LEP (E. Gardi, J. Rathsmann, 02)

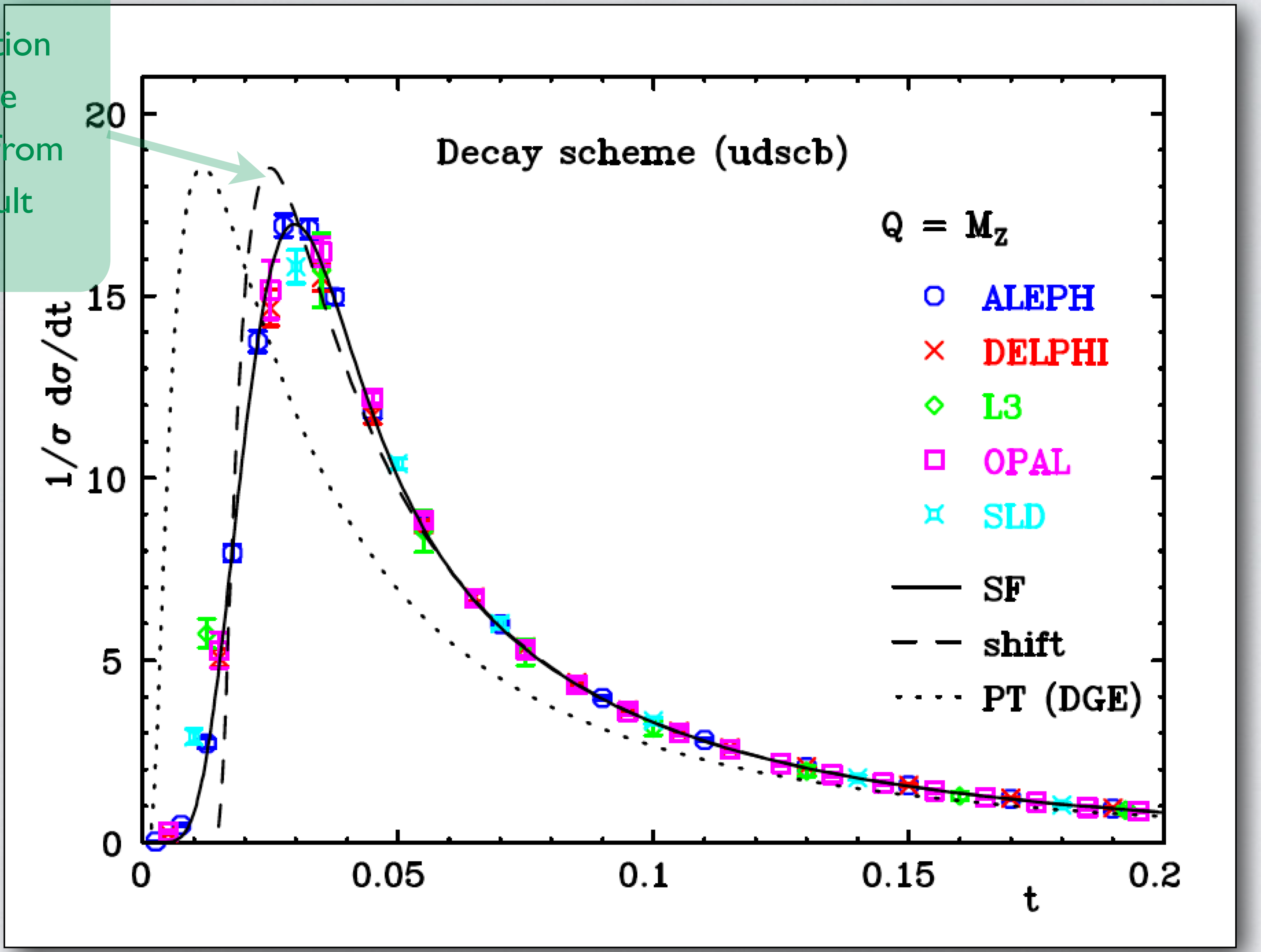


The thrust distribution is computed with NLL soft gluon resummations and with various models of power correction (shape functions). For illustrative purposes: many improvements in later work.

Logarithms at work

Predictions for the thrust distribution at LEP (E. Gardi, J. Rathsmann, 02)

Note shift in the distribution due to non-perturbative corrections extrapolated from all-order resummed result



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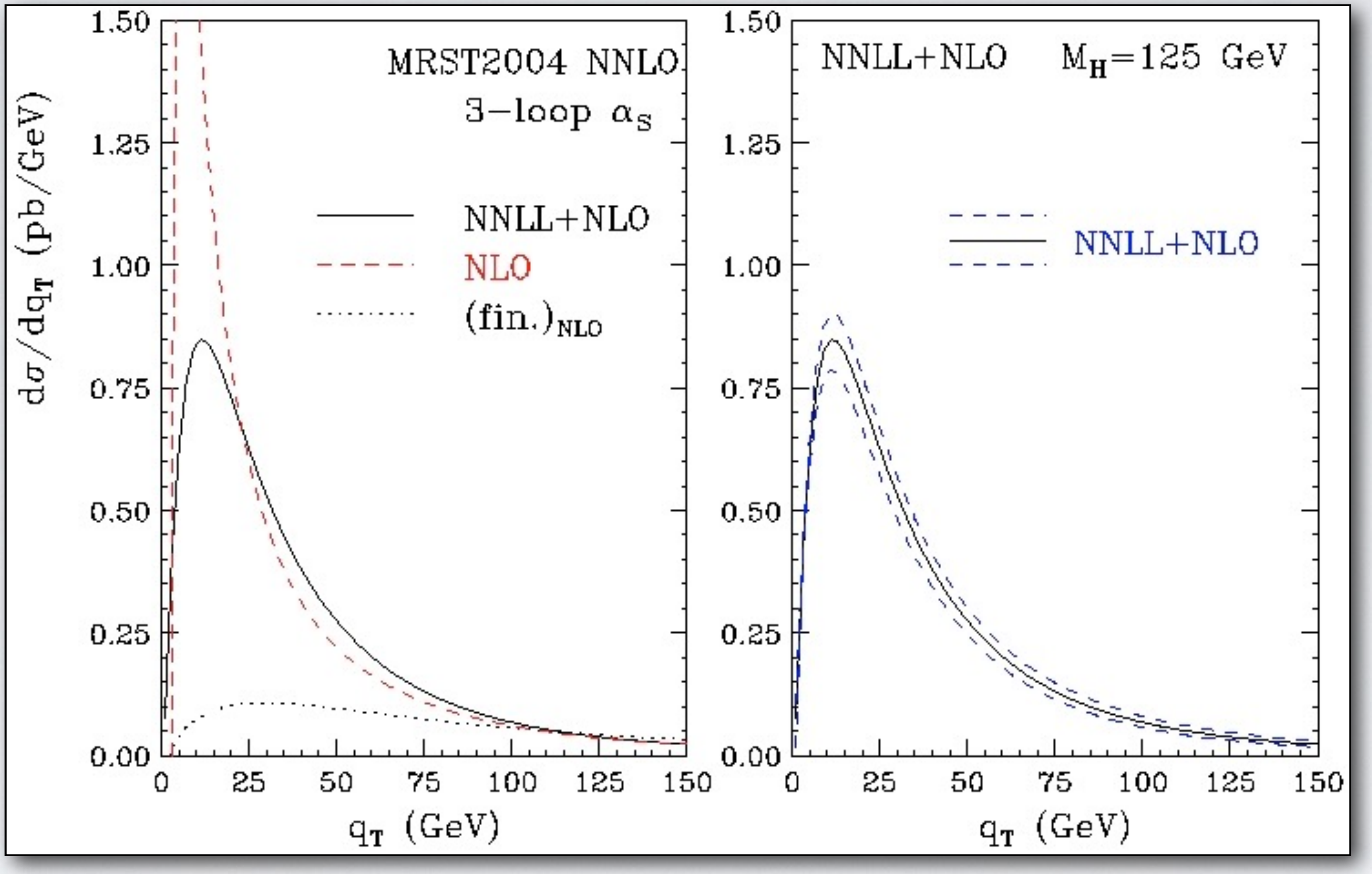
Predictions for the Higgs boson q_T spectrum at LHC (M. Grazzini, 05)



Predictions for the q_T spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation, and theoretical uncertainty band of the resummed prediction.

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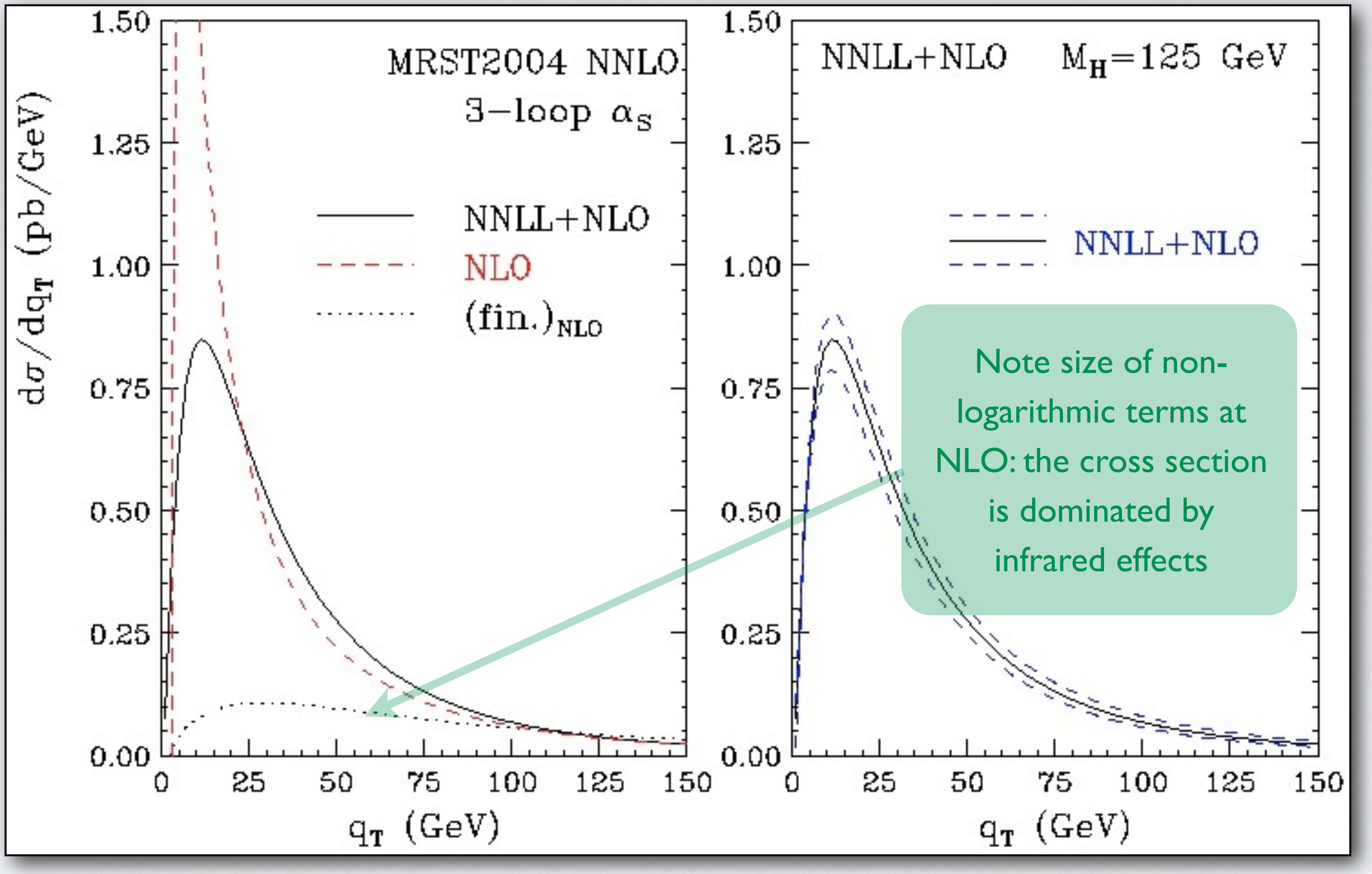

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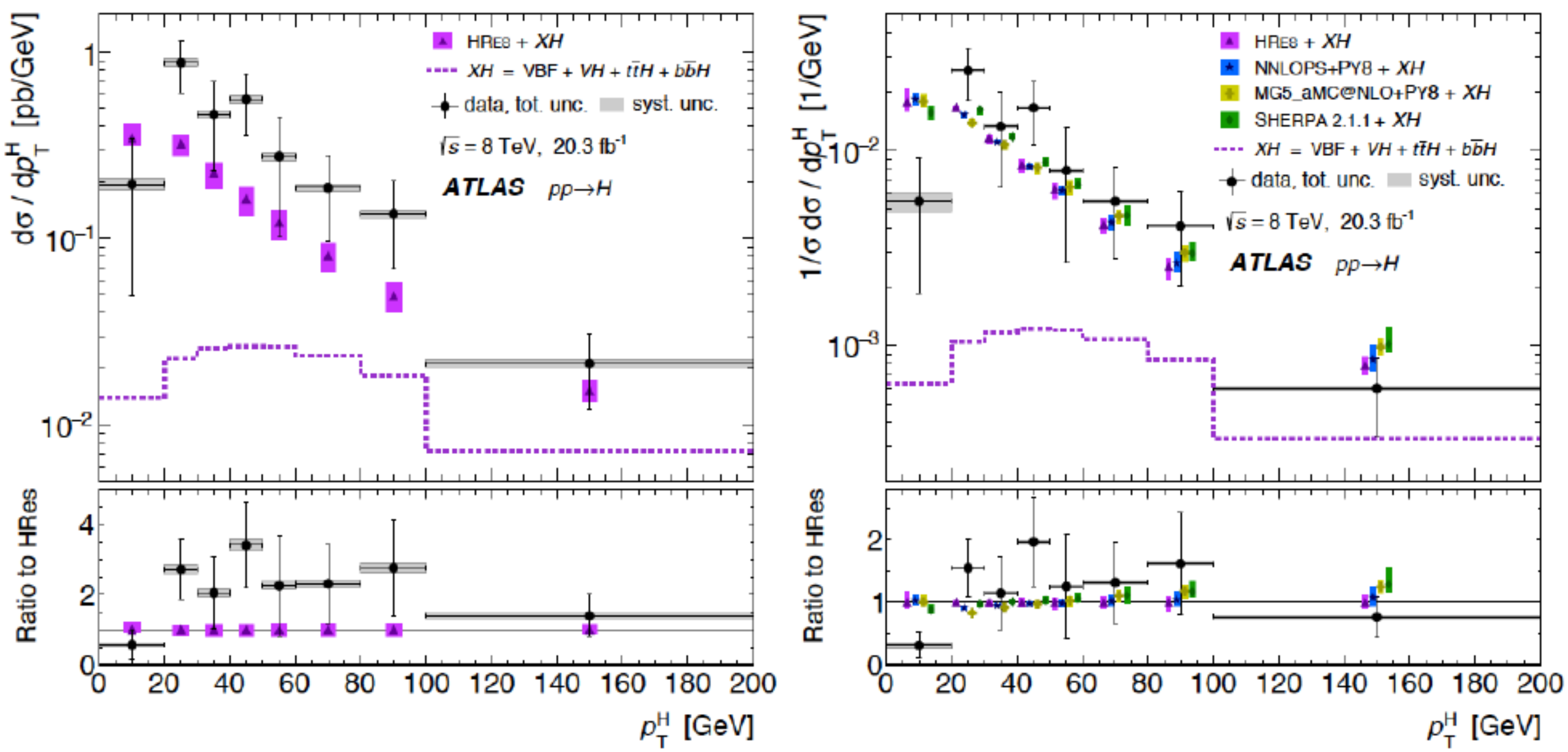
Logarithms at work

Higgs p_T distribution somewhat shifted to higher transverse momentum but of course statistically limited.



Preliminary ATLAS data on the Higgs p_T distribution from Run I
(shown by Joey Huston at Radcor-Loopfest 2015)

The perturbative exponent

A classic way to organize Sudakov logarithms in terms of the Mellin (Laplace) transform of the momentum space cross section (Catani et al. 93) is to write

$$
\begin{aligned}
d\sigma(\alpha_s, N) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{k=0}^{2n} c_{nk} \log^k N \;+\; \mathcal{O}(1/N) \\
&= H(\alpha_s)\, \exp\left[\log N\, g_1(\alpha_s \log N) + g_2(\alpha_s \log N) + \alpha_s\, g_3(\alpha_s \log N) + \ldots\right] + \mathcal{O}(1/N)
\end{aligned}
$$

This displays the main features of Sudakov resummation

- Predictive: a k-loop calculation determines g_k and thus a whole tower of logarithms to all orders in perturbation theory.
- Effective:
  - the range of applicability of perturbation theory is extended (finite order: $\alpha_s \log^2 N$ small. NLL resummed: α_s small);
  - the renormalization scale dependence is naturally reduced.
- Theoretically interesting: resummation ambiguities related to the Landau pole give access to non-perturbative power-suppressed corrections.
- Well understood:
  - NLL Sudakov resummations exist for most inclusive observables at hadron colliders, NNLL and approximate N^3LL in simple cases.

BEYOND THRESHOLD

With D. Bonocore, E. Laenen, L. Vernazza and C. White

More logarithms

- Threshold logarithms are associated with kinematic variables ξ that vanish at Born level and get corrections that are enhanced because phase space for real radiation is restricted near partonic threshold: examples are $1-T$, $1-M^2/\hat{s}$, $1-x_{BJ}$.

- At leading power in the threshold variable ξ logarithms are directly related to soft and collinear divergences: real radiation is proportional to factors of

$$\frac{1}{\xi^{1+p\epsilon}} = -\frac{1}{p\epsilon}\,\delta(\xi) + \left(\frac{1}{\xi}\right)_+ - p\epsilon\left(\frac{\log\xi}{\xi}\right)_+ + \ldots$$

Cancels virtual IR poles

Leading power threshold logs

- Beyond the leading power, $1/\xi$, the perturbative cross section takes the form

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1}\left[c_{nm}^{(-1)}\left(\frac{\log^m\xi}{\xi}\right)_+ + c_n^{(\delta)}\,\delta(\xi) + c_{nm}^{(0)}\log^m\xi + \ldots\right]$$

Resummed to high accuracy

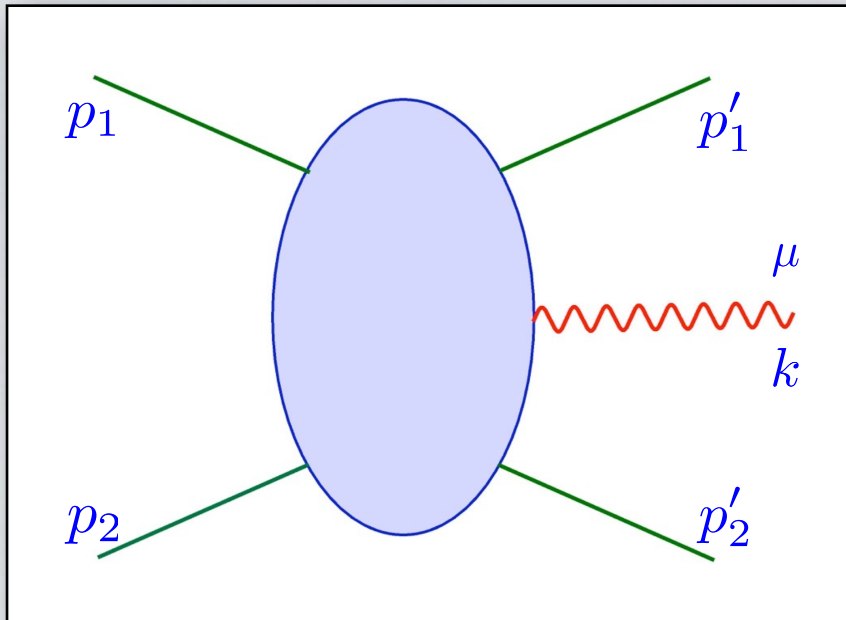
All-order structure in some cases

NLP threshold logs

- The structure of NLP threshold logarithms may be understood to all orders.

The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low's theorem (Low 58)



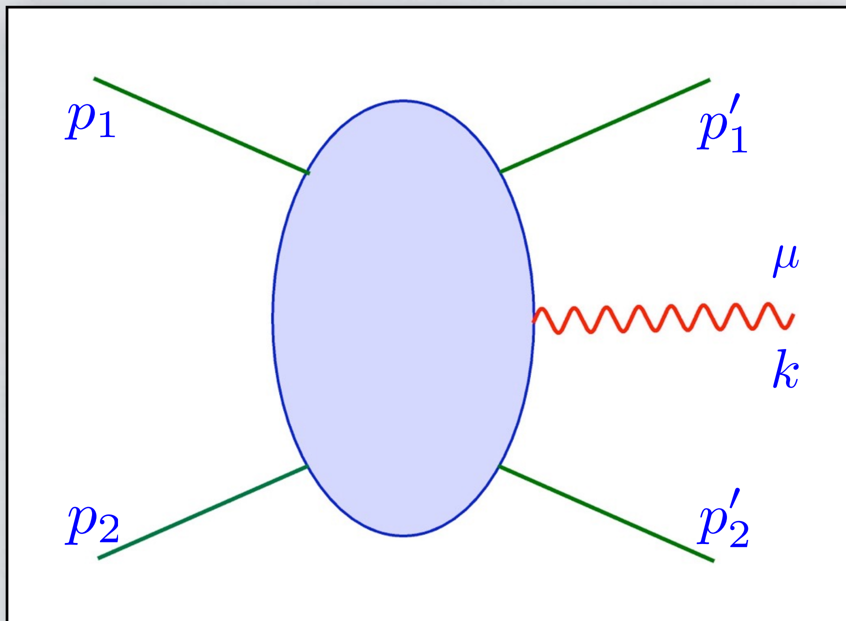
A radiative matrix element

$$M_\mu = e\left(\frac{p_{1\mu}'}{p_1'\cdot k} - \frac{p_{1\mu}}{p_1\cdot k}\right) T(\nu,\Delta)$$
$$+ e\left(\frac{p_{1\mu}' p_2'\cdot k}{p_1'\cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_2\cdot k}{p_1\cdot k} - p_{2\mu}\right)\frac{\partial T(\nu,\Delta)}{\partial \nu} + O(k),$$

Low's original expression for the radiative matrix element

The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low's theorem (Low 58)



A radiative matrix element

Eikonal approximation

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Next-to-eikonal contribution

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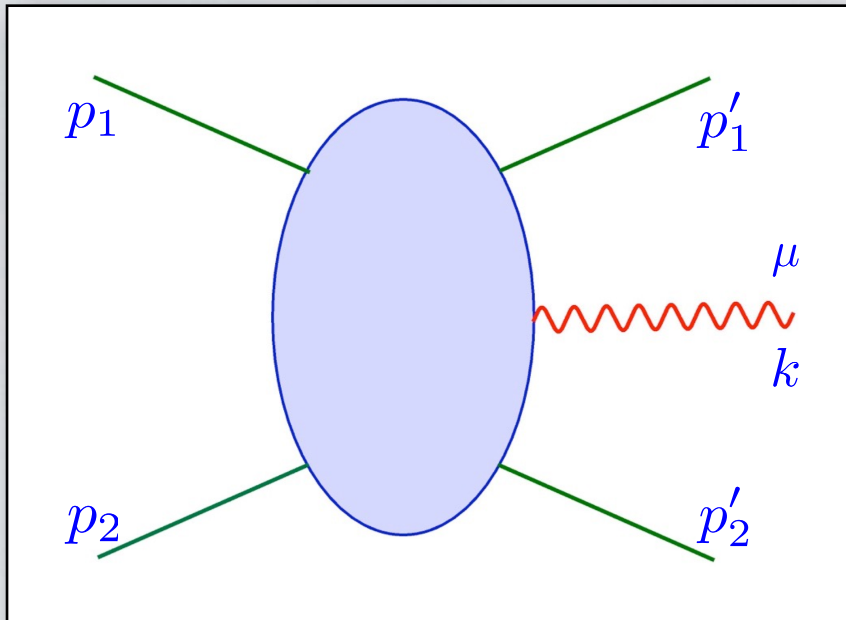

A radiative matrix element

Eikonal approximation

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Low's original expression for the radiative matrix element

Next-to-eikonal contribution

The radiative matrix element for the emission of a (next-to-) soft photon is determined by the Born amplitude T and its first derivative with respect to external momenta.

- Low's result established for a single charged scalar particle, follows from gauge invariance.
- It generalizes the well known properties of soft emissions in the eikonal approximation.
- The theorem was extended by (Burnett, Kroll 68) to particles with spin.
- The LBK theorem applies to massive particles and uses the mass as a collinear cutoff.
- It was extended to massless particles by (Del Duca 90), as discussed below.

Towards systematics

The problem of NLP threshold logarithms has been of interest for a long time, and several different approaches have been proposed. Recent years have seen a resurgence of interest, both from a theoretical point of view and for phenomenology.

- Early attempts include a study of the impact of NLP logs on the Higgs cross section by Kraemer, Laenen, Spira (98); work on F_L by Akhoury and Sterman (99) (logs without plus distributions are however leading) and work by Grunberg *et al.* (07-09) on DIS.
- Important results can be obtained by using physical kernels (Vogt *et al.* 09-14) which are conjectured to be single-logarithmic at large z, which poses constraints on their factorized expression. Note in particular a recent application to Higgs production by De Florian, Mazzitelli, Moch, Vogt (14).
- Useful approximations can be obtained by combining constraints from large N with high-energy constraints for $N \sim 1$ and analiticity (Ball, Bonvini, Forte, Marzani, Ridolfi, 13), together with phase space refinements.
- SCET techniques can be applied and indeed may be well-suited to the problem: a thorough one-loop analysis was given in (Larkoski, Neill, Stewart, 15).
- A lot of recent formal work on the behavior of gauge and gravity scattering amplitudes beyond the eikonal limit was triggered by a link to asymptotic symmetries of the S matrix (many authors from A(ndy Strominger) to Z(vi Bern), 14-15).

Beyond the eikonal

The soft expansion can be organized beyond leading power using either path integral techniques (Laenen, Stavenga, White 08) or diagrammatic techniques (Laenen, LM, Stavenga, White 10). The basic idea is simple, but the combinatorics cumbersome. For spinors

$$\frac{\not p + \not k}{2p \cdot k + k^2} \gamma^\mu u(p) = \left[\frac{p^\mu}{p \cdot k} + \frac{\not k \, \gamma^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2} \right] u(p) + \mathcal{O}(k)$$

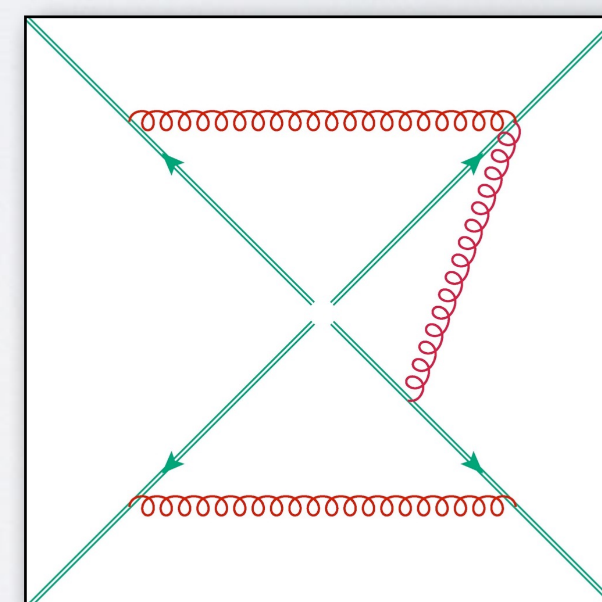
Eikonal — NE, spin-dependent — NE, spin-independent

- A class of factorizable contributions exponentiate via NE webs

$$\mathcal{M} = \mathcal{M}_0 \exp \left[\sum_{D_{\rm eik}} \tilde{C}(D_{\rm eik}) \, \mathcal{F}(D_{\rm eik}) + \sum_{D_{\rm NE}} \tilde{C}(D_{\rm NE}) \, \mathcal{F}(D_{\rm NE}) \right] .$$

- Feynman rules exist for the NE exponent, including "seagull" vertices.

$$\mathcal{M} = \mathcal{M}_0 \exp \left[\mathcal{M}_{\rm eik} + \mathcal{M}_{\rm NE} \right] \left(1 + \mathcal{M}_r \right) + \mathcal{O}\left({\rm NNE} \right) .$$

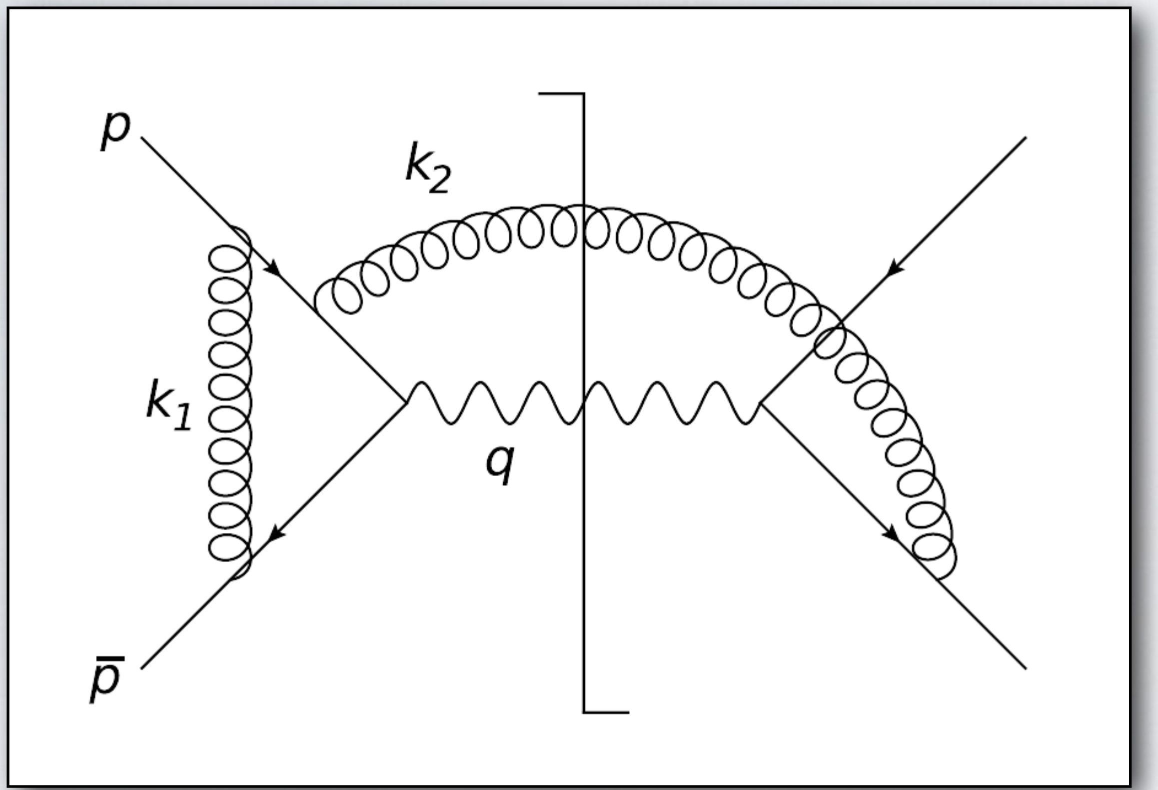


A next-to-eikonal web

- Non-factorizable contributions involve single gluon emission from inside the hard function, and must be studied using LBDK's theorem.

A collinear problem

Non-factorizable contributions start at NNLO. For massive particles they can be traced to the original LBK theorem. For massless particles a new contribution to NLP logs emerges.



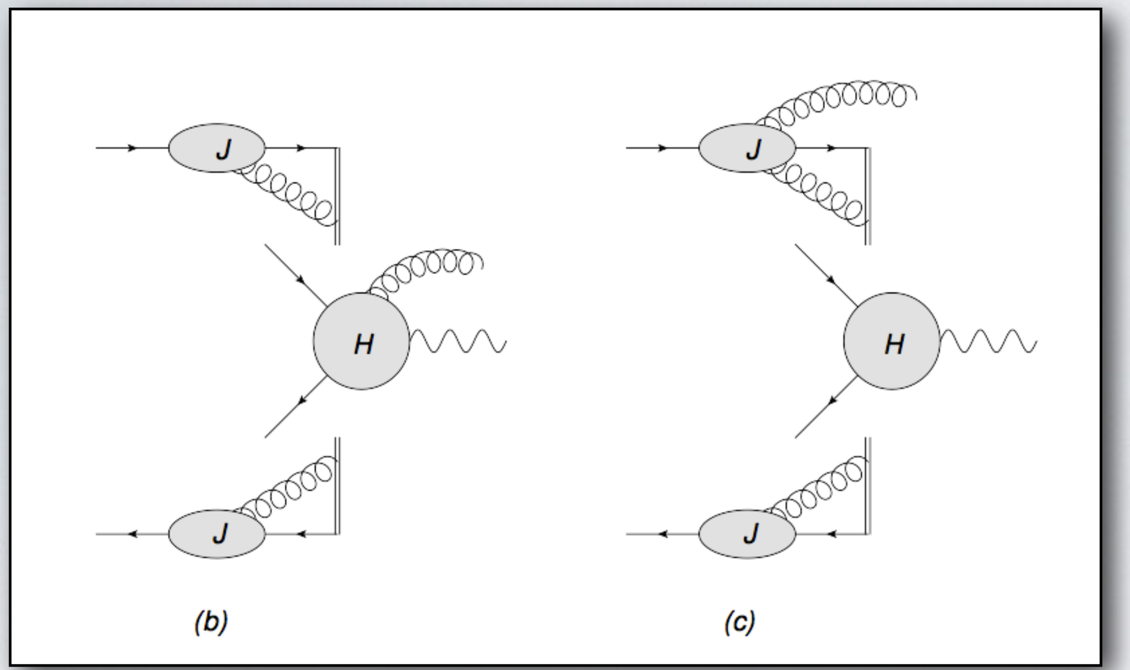
A Feynman diagram containing a collinear enhancement

- Gluon k_2 is always (next-to) soft for EW annihilation near threshold.
- When k_1 is (next-to) soft all logs are captured by NE rules.
- Contributions with k_1 hard and collinear are missed by the soft expansion.
- The collinear pole interferes with soft emission and generates NLP logs.
- The problem first arises at NNLO

- These contributions are missed by the LBK theorem: it applies to an expansion in E_k/m.
- They can be analized using the method of regions: the relevant factor is $(p \cdot k_2)^{-\varepsilon}/\varepsilon$.
- They cause the breakdown of next-to-soft theorems for amplitudes beyond tree level.

  ➡ the soft expansion and the limit $\varepsilon \to 0$ do not commute.
- They require an extension of LBK to $m^2/Q < E_k < m$. It was provided by Del Duca (90).

NLP factorization: a new jet



Factorized contributions to the radiative amplitude

Soft radiation can arise either from the jets or from the hard function

$$\mathcal{A}_\mu \, \epsilon^\mu(k) \; = \; \mathcal{A}^J_\mu \, \epsilon^\mu(k) + \mathcal{A}^H_\mu \, \epsilon^\mu(k) \, ,$$

The amplitude for emission from the jets can be precisely defined in terms of a new jet function

$$\mathcal{A}^J_\mu \; = \; \sum_{i=1}^{2} H(p_i - k; p_j, n_j) \, J_\mu(p_i, k, n_i) \prod_{j \neq i} J(p_j, n_j) \; \equiv \; \sum_{i=1}^{2} \mathcal{A}^{J_i}_\mu \, .$$

$$J_\mu \left(p, n, k, \alpha_s(\mu^2), \epsilon \right) u(p) \; = \; \int d^d y \; \mathrm{e}^{-\mathrm{i}(p-k)\cdot y} \; \langle 0 \, | \, \Phi_n(y, \infty) \, \psi(y) \, j_\mu(0) \, | \, p \rangle \; ,$$

defines the radiative jet.

- At tree level the radiative jet displays the expected dependence on spin.
- Dependence on the gauge vector n^μ starts at loop level: simplifications arise for $n^2 = 0$.

$$J^{\nu(0)} \left(p, n, k \right) \; = \; \frac{\not{k} \gamma^\nu}{2 p \cdot k} - \frac{p^\nu}{p \cdot k}$$
$$= \; - \, \frac{p^\nu}{p \cdot k} + \frac{k^\nu}{2 p \cdot k} - \frac{\mathrm{i} \, k_\alpha \Sigma^{\alpha \mu}}{2 p \cdot k} \, .$$

Beyond Low's theorem

A slightly modified version of Del Duca's result gives the radiative amplitude in terms of the non-radiative one, its derivatives, and the two "jet" functions.

$$\mathcal{A}^\mu(p_j,k) = \sum_{i=1}^{2} \Bigg\{ q_i \left(\frac{(2p_i-k)^\mu}{2p_i\cdot k - k^2} + G_i^{\nu\mu}\frac{\partial}{\partial p_i^\nu} \right) + G_i^{\nu\mu}\left[\frac{J_\nu(p_i,k,n_i)}{J(p_i,n_i)} - q_i \frac{\partial}{\partial p_i^\nu}\Big(\ln J(p_i,n_i)\Big) \right] \Bigg\} \mathcal{A}(p_i;p_j)\,.$$

The tensors $G^{\mu\nu}$ project out the eikonal contribution present in the first term.

$$\eta^{\mu\nu} = G^{\mu\nu} + K^{\mu\nu}\,, \qquad K^{\mu\nu}(p;k) = \frac{(2p-k)^\nu}{2p\cdot k - k^2}\,k^\mu\,,$$

The factorized expression for the radiative amplitude can be simplified.

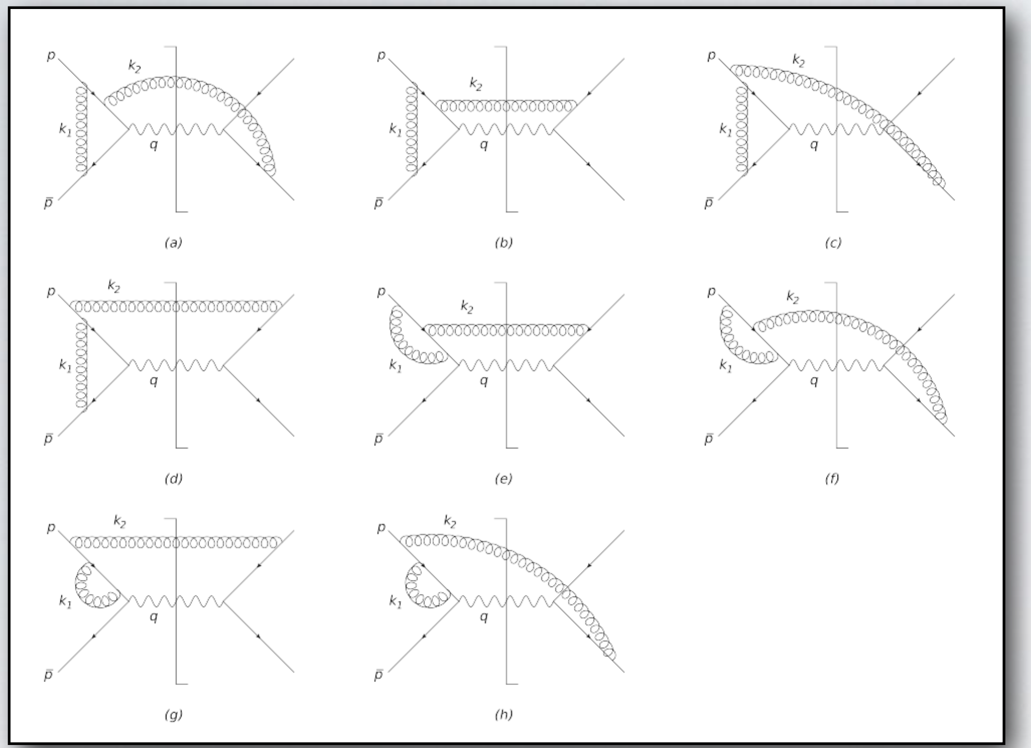
- The jet factor is RG invariant: it can be computed in bare perturbation theory.
- With this choice one can use that $J(p,n) = 1$ for $n^2 = 0$: it is a pure counterterm.
- The choice of reference vectors is then physically motivated (and confirmed by a complete analysis using the method of regions): we take $n_1 = p_2$ and $n_2 = p_1$.

$$\mathcal{A}^\mu(p_j,k) = \sum_{i=1}^{2} \left(q_i \frac{(2p_i-k)^\mu}{2p_i\cdot k - k^2} + q_i\, G_i^{\nu\mu}\frac{\partial}{\partial p_i^\nu} + G_i^{\nu\mu} J_\nu(p_i,k) \right) \mathcal{A}(p_i;p_j)\,.$$

For general amplitudes, a full subtraction of the residual n dependence should be aimed at.

Real-virtual two-loop Drell-Yan

Real-virtual corrections to EW annihilation processes involve non-factorizable contributions. NE rules cannot reproduce the perturbative result at NLP, due to collinear interference.



Real-virtual Feynman diagrams for the abelian part of the NNLO K-factor.

As a test of the LBDK factorization, we computed the C_F^2 part of the real-virtual K-factor at NNLO from ordinary Feynman diagrams, and then using the radiative amplitude integrated over phase space. As expected, plus distributions arise from the eikonal approximation, fully determined by the dressed non-radiative amplitude. Derivative terms and the projected radiative jet contribute at NLP.

- All NLP terms are correctly reproduced, including those with no logarithms.
- The radiative jet reproduces exactly the NLP collinear contribution derived by the method of regions.

$$K^{(2)}_{\rm rv}(z) = \left(\frac{\alpha_s}{4\pi}C_F\right)^2 \Bigg\{ \frac{32}{\epsilon^3}\Big[\mathcal{D}_0(z) - 1\Big] + \frac{16}{\epsilon^2}\Big[-4\mathcal{D}_1(z) + 3\mathcal{D}_0(z) + 4L(z) - 6\Big]$$
$$+ \frac{4}{\epsilon}\Big[16\mathcal{D}_2(z) - 24\mathcal{D}_1(z) + 32\mathcal{D}_0(z) - 16L^2(z) + 52L(z) - 49\Big]$$
$$- \frac{128}{3}\mathcal{D}_3(z) + 96\mathcal{D}_2(z) - 256\mathcal{D}_1(z) + 256\mathcal{D}_0(z)$$
$$+ \frac{128}{3}L^3(z) - 232L^2(z) + 412L(z) - 408 \Bigg\},$$

The abelian part of the NNLO K-factor from real-virtual diagrams, omitting constants

OUTLOOK

A Perspective

- Perturbative QCD will play a central role in any future high-energy collider.
- Through the vast and remarkable effort of an entire community, ubiquitous and consistent NLO phenomenology at LHC is a reality.
- An ongoing NNLO (r)evolution makes it possible to imagine that in a few years we will be able to state the same about NNLO phenomenology.
- N^3LO is the new NNLO.
- At some point we will have to stop … and just resum ...
- Leading power threshold resummation is highly developed and provides some of the most precise predictions in perturbative QCD.
- Low's theorem is the first of many hints that NLP logs can be understood and organized.
- Hard collinear emissions spoil Low's theorem: a new radiative jet function emerges.
- A complete treatment of NLP threshold logs is at hand.
- Much work to do to organize a true resummation formula, even for EW annihilation: we have a more intricate "factorization", we must make sure to control double countings.
- In order to achieve complete generality, we will need to include final state jets.

THANK YOU!