On Vector ARMA Models Consistent with a Finite Matrix Covariance Sequence

Bin Zhu
Department of Information Engineering (DEI), University of Padova

VARMA covariance matching problem

A forward unilateral ARMA model for an m-dimensional stationary process $\{y(t)\}$

$$\sum_{k=0}^{n} A_k y(t-k) = \sum_{k=0}^{n} B_k w(t-k), \quad t \in \mathbb{Z}, \tag{1}$$

where $\{A_k, B_k \in \mathbb{R}^{m\times m}\}$ are matrix parameters and $\{w(t)\}$ is an m-dimensional white noise with variance $\mathbb{E}[w(t)w(t)^\top] = I_m$.

Define two matrix polynomials

$$A(z) := \sum_{k=0}^{n} A_k z^{-k}, \quad B(z) := \sum_{k=0}^{n} B_k z^{-k}.$$

The *spectral density* of $\{y(t)\}$

$$\Phi(z) = A(z)^{-1} B(z) B(z^{-1})^\top A(z^{-1})^{-\top}.$$

Define the set $\mathfrak{S}_{m,n}$ of matrix Schur polynomials

$$M(z) = \sum_{k=0}^{n} M_k z^{-k}, \quad M_k \in \mathbb{R}^{m\times m}$$

such that

- M_0 is lower triangular with positive diagonal elements;
- $\det M(z) = 0$ implies $z \in \mathbb{D}$;
- $$\int_{-\pi}^{\pi} \mathrm{tr}\,[M(e^{i\theta})M(e^{i\theta})^*]\frac{d\theta}{2\pi} = \mathrm{tr}(\mathbf{M}\mathbf{M}^\top) < \mu.$$

where $\mathbf{M} = [\,M_0, \dots, M_n\,] \in \mathbb{R}^{\frac{1}{2}m(m+1)+m^2 n}$ and μ is an arbitrarily large positive constant. The set $\mathfrak{S}_{m,n}$ is *open and bounded* if identified as a subset of the Euclidean space.

Define the set $\mathfrak{P}_{m,n}$ of Hermitian matrix pseudo-polynomials of order n

$$P(z) = \sum_{k=-n}^{n} P_k z^{-k}, \quad P_{-k} = P_k^\top \in \mathbb{R}^{m\times m},$$

and the subset that contains positive ones on the unit circle

$$\mathfrak{P}^+_{m,n} := \{P(z) \in \mathfrak{P}_{m,n} \,:\, P(z) > 0,\ \forall z \in \mathbb{T}\}.$$

Problem 1 (VARMA Covariance Matching)**.** *Suppose we are given an MA polynomial $B(z) \in \mathfrak{S}_{m,n}$ and $n+1$ real $m\times m$ matrices $C_0, C_1, \dots, C_n$, such that the block-Toeplitz matrix*

$$\mathbf{T}_n = \begin{bmatrix} C_0 & C_1 & C_2 & \cdots & C_n \\ C_1^\top & C_0 & C_1 & \cdots & C_{n-1} \\ C_2^\top & C_1^\top & C_0 & \cdots & C_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n^\top & C_{n-1}^\top & C_{n-2}^\top & \cdots & C_0 \end{bmatrix}, \qquad n \in \mathbb{Z}_+$$

is positive definite. Determine the AR polynomial $A(z) \in \mathfrak{S}_{m,n}$ such that the first $n+1$ covariance matrices of the process $\{y(t)\}$ described by the VARMA model (1) *match the sequence $\{C_k\}$.*

An equivalent algebraic problem

Given covariance data organized as a block row vector

$$\mathbf{C} = [C_0\ C_1\ \dots\ C_n,]$$

define a map

$$f_{\mathbf{C}}:\ \mathfrak{S}_{m,n} \to \mathfrak{P}_{m,n}$$
$$A(z) \mapsto P(z)$$

as follows. First compute the polynomial $H(z) := \sum_{k=0}^{n} H_k z^{-k}$ of order n by polynomial multiplication and truncation

$$H(z) := \lceil\, A(z)(C_0 + 2C_1 z^{-1} + \cdots + 2C_n z^{-n})\,\rceil_{-n}^{0},$$

where the operation $\lceil\cdot\rceil_{-n}^{0}$ means retaining only the terms with powers from $-n$ up to 0. Then proceed to define

$$P(z) := \frac{1}{2}[H(z)A(z^{-1})^\top + A(z)H(z^{-1})^\top].$$

In particular, we have

$$P_0 = \mathbf{A}\mathbf{T}_n\mathbf{A}^\top,\ P_n = \frac{1}{2}(H_n + A_n C_0)A_0^\top.$$

Problem 2. *Given $B(z) \in \mathfrak{S}_{m,n}$, let $P(z) = B(z)B(z^{-1})^\top$. Find a solution $A(z) \in \mathfrak{S}_{m,n}$ to the system of quadratic equations*

$$f_{\mathbf{C}}(A(z)) = P(z). \tag{2}$$

Proposition 1. *A solution (if it exists) to Problem 2 solves Problem 1, and vice versa.*

Review of the degree theory

Assume $D \subset \mathbb{R}^n$ is bounded open and $f: \overline{D} \to \mathbb{R}^n$ is smooth (in C^∞). Consider the solvability of the equation

$$f(x) = y.$$

We call $y \in \mathbb{R}^n$ a *regular value* of f if either

(i) for any $x \in f^{-1}(y)$, $\det f'(x) \neq 0$ or

(ii) $f^{-1}(y)$ is empty.

Here $f'(x)$ denotes the Jacobian matrix of f evaluated at x. Let y be a regular value of type (i) and $y \notin f(\partial D)$, the degree of f at y is defined as

$$\deg(f, y, D) := \sum_{f(x)=y} \mathrm{sign}\ \det f'(x),$$

where the sign function

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and not define at 0. For regular values of type (ii), we set $\deg(f, y, D) = 0$.

Moreover, the set of regular values is dense in $\mathbb{R}^n$ by Sard's theorem. When y is not a regular value (also called a *critical value*), the degree can also be defined by means of limit. Further properties of the degree related to our problem:

- If y_1 and y_2 belong to the same component of $\mathbb{R}^n - f(\partial D)$, then $\deg(f, y_1, D) = \deg(f, y_2, D)$.
- If $\deg(f, y, D) \neq 0$, there exists $x \in D$ such that $f(x) = y$.
- Homotopy invariance. If $F: \overline{D} \times [a, b] \to \mathbb{R}^n$ is continuous and if $F(x, t) \neq y$ for $x \in \partial D$ and $t \in [a, b]$, then $\deg(F_t, y, D)$ is defined and independent of $t \in [a, b]$. Here F_t is the map of D into $\mathbb{R}^n$ defined by $F_t(x) = F(x, t)$.

Existence of a solution

Given the block Toeplitz matrix $\mathbf{T}_n > 0$, let $\lambda_{\min} > 0$ be its smallest eigenvalue. Define the subset

$$\mathfrak{D}^+_{m,n} := \{P(z) \in \mathfrak{P}^+_{m,n} \,:\, \det P_n \neq 0 \text{ and } \mathrm{tr}\, P_0 < \min\{1, \lambda_{\min}\}\mu\}.$$

Theorem 1. *For any fixed $B(z) \in \mathfrak{S}_{m,n}$ such that $P(z) := B(z)B(z^{-1})^\top$ belongs to $\mathfrak{D}^+_{m,n}$, there exists a matrix polynomial $A(z) \in \mathfrak{S}_{m,n}$ such that the first $n+1$ covariance matrices of the VARMA model* (1) *defined by the coefficients of $A(z), B(z)$ match the given data $C_0, \dots, C_n$.*

Sketch of the proof. It is sufficient to show that $\deg(f_{\mathbf{C}}, P(z), \mathfrak{S}_{m,n}) \neq 0$ for $P(z) \in \mathfrak{D}^+_{m,n}$. Compute the degree using the homotopy invariance property.

Consider the data $\mathbf{O} = [I_m, 0, \dots, 0]$, and the corresponding map becomes

$$P(z) = f_{\mathbf{O}}(A(z)) := A(z)A(z^{-1})^\top,$$

and the problem is reduced to the matrix spectral factorization.

Lemma 1. *Define $\mathbf{\Sigma}(t) := t\mathbf{C} + (1-t)\mathbf{O}$, $t \in [0, 1]$, and the corresponding function $f_{\mathbf{\Sigma}(t)}$ in the same way as $f_{\mathbf{C}}$. Then the function*

$$\mathcal{F}:\ \mathfrak{S}_{m,n} \times [0, 1] \to \mathfrak{P}_{m,n}$$
$$(\,A(z), t\,) \mapsto f_{\mathbf{\Sigma}(t)}(A(z))$$

is a smooth homotopy between $f_{\mathbf{C}}$ and $f_{\mathbf{O}}$.

Lemma 2. *For any $P(z) \in \mathfrak{D}^+_{m,n}$,*

$$|\deg(f_{\mathbf{O}}, P(z), \mathfrak{S}_{m,n})| = 1.$$

In order to conclude that for $P(z) \in \mathfrak{D}^+_{m,n}$,

$$\deg(f_{\mathbf{C}}, P(z), \mathfrak{S}_{m,n}) = \deg(f_{\mathbf{O}}, P(z), \mathfrak{S}_{m,n}),$$

we need to ensure that

$$f_{\mathbf{\Sigma}(t)}(A(z)) \neq P(z) \text{ for any } A(z) \in \partial\mathfrak{S}_{m,n} \text{ and } t \in [0, 1].$$

□

References

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