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International Student Workshop on
"Neutrinoless Double Beta Decay"

III. Quasiparticle Random Phase Approximation: Formalism

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$0\nu\beta\beta$-decay matrix element (two ways of calculation)

0νββ-decay matrix elements

$$M^{0\nu} = \frac{4\pi R}{g_A^2} \int \left(\frac{1}{(2\pi)^3} \int \frac{e^{-i\vec{q}.(\vec{x}_1-\vec{x}_2)}}{|q|} \right) \times$$
$$\sum_m \frac{< 0_f^+ | J_\alpha^\dagger(\vec{x}_1) | m >< m | J^{\alpha\dagger}(\vec{x}_2) | 0_i^+ >}{E_m - (E_i + E_f)/2 + |q|} d\vec{q} d\vec{x}_1 d\vec{x}_2$$

Weak hadron current

$$j^{\rho\dagger} = \overline{\Psi}\tau^+ \left[g_V(q^2)\gamma^\rho + i g_M(q^2)\frac{\sigma^{\rho\nu}}{2m_p}q_\nu - g_A(q^2)\gamma^\rho\gamma_5 - g_P(q^2)q^\rho\gamma_5 \right] \Psi,$$

Formfactor

$$g_V(\vec{q}^{\,2}) = g_V/(1+\vec{q}^{\,2}/M_V^2)^2$$
$$g_A(\vec{q}^{\,2}) = g_A/(1+\vec{q}^{\,2}/M_A^2)^2$$

Weak hadron current in a Breit frame

$$J^{\rho\dagger}(\vec{x}) = \sum_{n=1}^{A} \tau_n^+ [g^{\rho 0} J^0(\vec{q}^{\,2}) + \sum_k g^{\rho k} J_n^k(\vec{q}^{\,2})] \delta(\vec{x} - \vec{r}_n)$$
$$J^0(\vec{q}^{\,2}) = g_V(q^2)$$
$$\vec{J}_n(\vec{q}^{\,2}) = g_M(\vec{q}^{\,2}) i \frac{\vec{\sigma}_n \times \vec{q}}{2m_p} + g_A(\vec{q}^{\,2})\vec{\sigma} - g_P(\vec{q}^{\,2}) \frac{\vec{q}\,\vec{\sigma}_n \cdot \vec{q}}{2m_p}$$

Two one-body operators

One-body operator

$$\hat{\mathcal{O}}_{JM} = \sum_{pn} \frac{\langle p \parallel \mathcal{O}_J \parallel n\rangle}{\sqrt{2J+1}} [c_p^+ \tilde{c}_n]_{JM}$$

$$e^{i\vec{q}\cdot\vec{r}} = 4\pi \sum_l i^l j_l(qr)(Y_{lm}(\Omega_r \cdot Y_{lm}(\Omega_q))$$

Decomposition of plane waves

$$\int e^{i\vec{q}\cdot\vec{r}_1} e^{-i\vec{q}\cdot\vec{r}_2} d\Omega_q =$$

$$(4\pi)^2 \sum_l (-1)^l \sqrt{2l+1}\, j_l(qr_1)\, j_l(qr_2)\, \{Y_{lm}(\Omega_{r_1}) \otimes Y_{lm}(\Omega_{r_2})\}_{00}$$

$$M_K = \sum_{J,\pi,k_i,k_f} \sum_{pnp'n'} (-)^J \frac{R}{g_A^2} \int_0^\infty \frac{\mathcal{P}^K_{pnp'n',J}(q)}{|q|(|q| + (\Omega^{k_i}_{J^\pi} + \Omega^{k_f}_{J^\pi})/2)} h_K(q^2) q^2 dq \times$$
$$\langle 0_f^+ ||\widetilde{[c_{p'}^+ \tilde{c}_{n'}]_J}|| J^\pi k_f\rangle \langle J^\pi k_f | J^\pi k_i\rangle \langle J^\pi k_i || [c_p^+ \tilde{c}_n]_J || 0_i^+\rangle$$

K=VV, MM, AA, PP, AP

Product of one-body matrix elements

$$\begin{aligned}
\mathcal{P}^{VV}_{pnp'n',J}(q) &= \langle p \parallel \mathcal{O}^{(1)}_J(q) \parallel n\rangle \langle p' \parallel \mathcal{O}^{(1)}_J(q) \parallel n'\rangle, \\
\mathcal{P}^{AA}_{pnp'n',J}(q) &= \sum_{L=J,J\pm1} (-)^{J+L+1} \times \\
&\quad \langle p \parallel \mathcal{O}^{(2)}_{LJ}(q) \parallel n\rangle \langle p' \parallel \mathcal{O}^{(2)}_{LJ}(q) \parallel n'\rangle, \\
\mathcal{P}^{PP}_{pnp'n',J}(q) &= \langle p \parallel \mathcal{O}^{(3)}_J(q) \parallel n\rangle \langle p' \parallel \mathcal{O}^{(3)}_J(q) \parallel n'\rangle, \\
\mathcal{P}^{AP}_{pnp'n',J}(q) &= \mathcal{P}^{PP}_{pnp'n',J}(q), \\
\mathcal{P}^{MM}_{pnp'n',J}(q) &= \mathcal{P}^{AA}_{pnp'n',J}(q) - \mathcal{P}^{PP}_{pnp'n',J}(q)\ .
\end{aligned}$$

One two-body operators

$$\langle p|O(1)|n\rangle\langle p'|O(2)|n'\rangle = \langle p,p'|O'(1,2)|n,n'\rangle$$

Integration over angular part of ν momentum

$$\int e^{i\vec{q}\cdot(\vec{r}_1-\vec{r}_2)}d\Omega_q = \int e^{i\vec{q}\cdot\vec{r}}d\Omega_q =$$

$$\sqrt{4\pi}\; 4\pi \sum_{lm} i^l j_l(qr)Y_{lm}(\Omega_r)\int Y^*_{lm}(\Omega_q)Y_{00}(\Omega_q)d\Omega_q = 4\pi j_0(qr)$$

Neutrino potential

$$\begin{aligned} O_F(r_{12},E^k_{J^\pi}) &= \tau^+(1)\tau^+(2)H_F(r_{12},E^k_{J^\pi})\,, \\ O_{GT}(r_{12},E^k_{J^\pi}) &= \tau^+(1)\tau^+(2)H_{GT}(r_{12},E^k_{J^\pi})\sigma_{12}\,, \\ O_T(r_{12},E^k_{J^\pi}) &= \tau^+(1)\tau^+(2)H_T(r_{12},E^k_{J^\pi})S_{12} \end{aligned}$$

$$H_K(r_{12},E^k_{J^\pi}) = \frac{2}{\pi g_A^2}R\int_0^\infty f_K(qr_{12})\,\frac{h_K(q^2)qdq}{q+E^k_{J^\pi}-(E_i+E_f)/2}$$

$$\begin{aligned} \sigma_{12} &= \vec{\sigma}_1\cdot\vec{\sigma}_2\,, \\ S_{12} &= 3(\vec{\sigma}_1\cdot\hat{r}_{12})(\vec{\sigma}_2\cdot\hat{r}_{12})-\sigma_{12} \end{aligned}$$

$$with\quad f_{F,GT}(qr_{12}) = j_0(qr_{12}),\qquad f_T(qr_{12}) = -j_2(qr_{12})$$

Nuclear matrix element

$$M^{0\nu} = -\frac{M_F}{g_A^2}+M_{GT}-M_T$$

$$M_K = \sum_{J^\pi,k_i,k_f,\mathcal{J}}\;\sum_{pnp'n'}(-1)^{j_n+j_{p'}+J+\mathcal{J}}\times$$

$$\sqrt{2\mathcal{J}+1}\begin{Bmatrix} j_p & j_n & J \\ j_{n'} & j_{p'} & \mathcal{J}\end{Bmatrix}\times$$

$$\langle p(1),p'(2);\mathcal{J}\parallel \bar{f}(r_{12})O_K\bar{f}(r_{12})\parallel n(1),n'(2);\mathcal{J}\rangle\times$$

$$\langle 0_f^+||[\widetilde{c_{p'}^+\tilde{c}_{n'}}]_J||J^\pi k_f\rangle\langle J^\pi k_f|J^\pi k_i\rangle\langle J^\pi k_f i||[c_p^+\tilde{c}_n]_J||0_i^+\rangle$$

Calculation of two-body matrix elements

From j-j to LS coupling

$$\mathcal{M}^{2body} = \langle a(1), b(2); J'|O(1,2)|c(1)d(2); J'\rangle$$

$$|n_c l_c j_c, n_d l_d j_d; J'M'\rangle = \sum_{SL} \hat{S}^2 \hat{L}^2 \hat{j}_c \hat{j}_d \left\{ \begin{array}{ccc} 1/2 & l_c & j_c \\ 1/2 & l_d & j_d \\ S & L & J' \end{array} \right\} |n_c l_c, n_d l_d, SL; J'M'\rangle$$

Moshinsky transformation to relative coordinates

$$|n_c l_c n_d l_d; LM_L\rangle = \sum_{\substack{nl \\ \mathcal{N}\mathcal{L}}} \langle nl, \mathcal{N}\mathcal{L}, L|n_c l_c, n_d l_d, L\rangle |nl, \mathcal{N}\mathcal{L}; LM_L\rangle$$

Two-body m.e.

$$\begin{aligned}\mathcal{M}^{2body}_{F,GT} &= \hat{J}' \sum_{SL} \hat{S}\hat{L}\hat{j}_a\hat{j}_b\hat{j}_c\hat{j}_d \left\{ \begin{array}{ccc} 1/2 & l_c & j_c \\ 1/2 & l_d & j_d \\ S & L & J' \end{array} \right\} \left\{ \begin{array}{ccc} 1/2 & l_a & j_a \\ 1/2 & l_b & j_b \\ S & L & J' \end{array} \right\} \\ &\times \sum_{\substack{nl \\ \mathcal{N}\mathcal{L}}} \sum_{\substack{n'l' \\ \mathcal{N}'\mathcal{L}'}} \langle nl, \mathcal{N}\mathcal{L}, L|n_c l_c, n_d l_d, L\rangle \langle n'l', \mathcal{N}'\mathcal{L}', L|n_a l_a, n_b l_b, L\rangle \\ &\times \langle n'l', \mathcal{N}'\mathcal{L}'; L||j_0(q|\vec{r}_{i,j})|||nl, \mathcal{N}\mathcal{L}; L\rangle \langle s_a s_b; S|| \left(\begin{array}{c} 1 \\ \vec{\sigma}_1 \cdot \vec{\sigma}_2 \end{array} \right) ||s_c s_d; S\rangle\end{aligned}$$

$$\langle n'l', \mathcal{N}'\mathcal{L}'; L||j_0(q|\vec{r}_{i,j}|)||nl, \mathcal{N}\mathcal{L}; L\rangle = \delta_{ll'}\delta_{\mathcal{N}\mathcal{N}'}\delta_{\mathcal{L}\mathcal{L}'}\langle n'l|j_0(q|\vec{r}_{i,j}|)|nl\rangle$$

$$\begin{aligned}\langle s_a s_b; S||\vec{\sigma}_1 \cdot \vec{\sigma}_2||s_c s_d; S\rangle &= \hat{S}(\delta_{S1} - 3\delta_{S0}), \\ \langle s_a s_b; S||1||s_c s_d; S\rangle &= \hat{S}(\delta_{S1} + \delta_{S0})\end{aligned}$$

Many-body wave functions

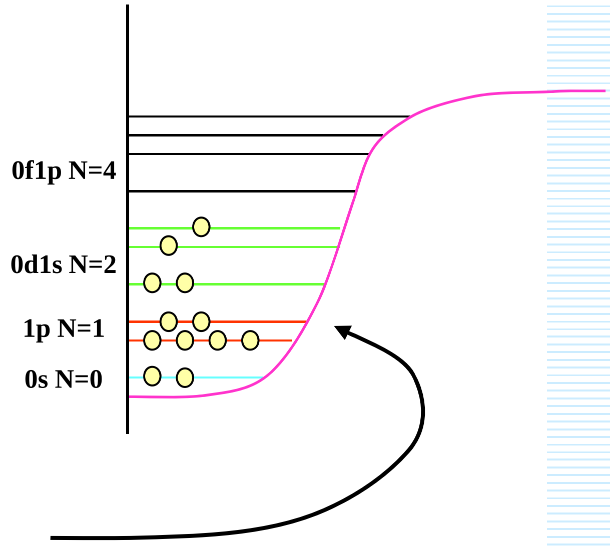
Many-body Hamiltonian

- **Start with the many-body Hamiltonian**

$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<j} V_{NN}\left(\vec{r}_i - \vec{r}_j\right)$$

- **Introduce a mean-field *U* to yield basis**

$$H = \sum_i \left(\frac{\vec{p}_i^{\,2}}{2m} + U(r_i) \right) + \underbrace{\sum_{i<j} V_{NN}\left(\vec{r}_i - \vec{r}_j\right) - \sum_i U(r_i)}_{\text{Residual interaction}}$$



The success of any nuclear structure calculation depends on the choice of the mean-field basis and the residual interaction!

- **The mean field determines the shell structure**
- **In effect, nuclear-structure calculations rely on perturbation theory**

Shell structure of spherical nucleus

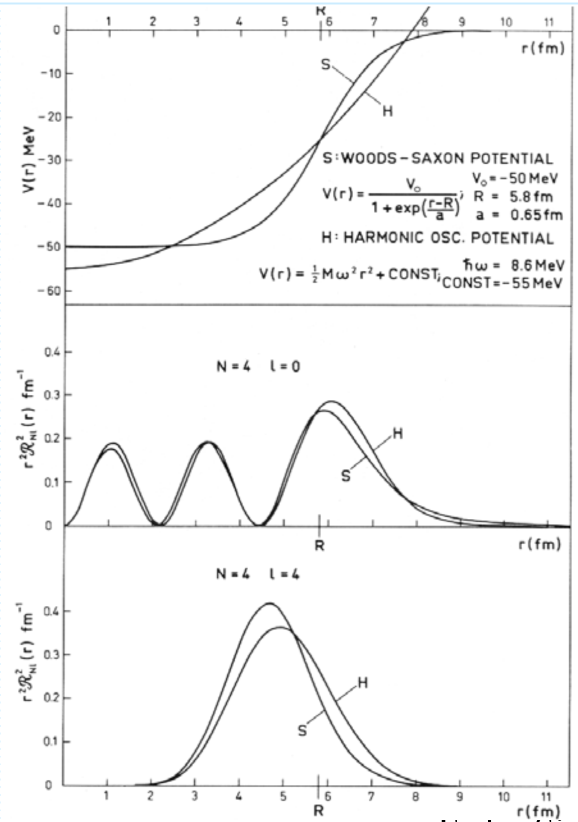
Goeppert-Mayer and Haxel, Jensen, and Suess proposed the independent-particle shell model to explain the magic numbers

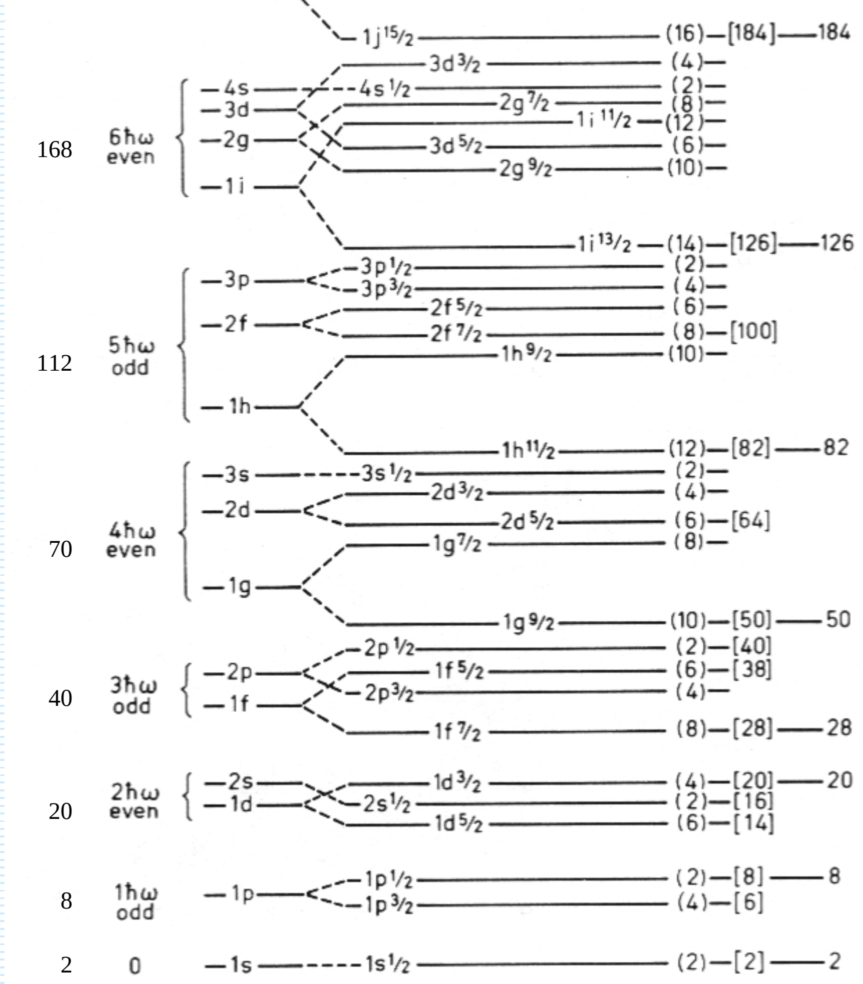
2, 8, 20, 28, 50, 82, 126, 184

s, p, d, f, g, h, i

l = 0, 1, 2, 3, 4, 5, 6

Harmonic oscillator with spin-orbit is a reasonable approximation to the nuclear mean field





M.G. Mayer and J.H.D. Jensen, *Elementary Theory of Nuclear Shell Structure*, p. 58, Wiley, New York, 1955

Nuclear many-body wave function

Single nucleon state

$$(H(r,\theta,\phi) - E_a)|a> = 0$$
$$H(r,\theta,\phi) = -\frac{\hbar^2}{2m} + U(r)$$

$$|a> = |\tau_a n_a l_a j_a m_a>$$

ℓ	orbital angular momentum
j	total angular momentum $j = \ell + s$
m	z component of j
n	nordal quantum number

$$\phi_a(r,\theta,\phi) = R_{\tau_a n_a l_a j_a} \sum_{l_a m_a} C^{j_a m_a}_{l_a m_a \; 1/2\sigma} \, Y_{l_a m_a} \, \chi_{1/2\sigma} \, \eta_{1/2\tau}$$

Two-nucleon state (nucleons are fermions)

$$\Psi_{ab}(1,2) = \frac{1}{\sqrt{2}}(\Psi_a(1)\Psi_b(2) - \Psi_a(2)\Psi_b(1))$$
$$= \frac{1}{\sqrt{2}}\begin{vmatrix} \Psi_a(1) & \Psi_a(2) \\ \Psi_b(1) & \Psi_b(1) \end{vmatrix}$$

Slater determinat

Ground state of A-nucleons system

$$\Psi_{a_1 a_2 \cdots a_A}(1,2,\cdots,A)) = \frac{1}{\sqrt{A!}}\begin{vmatrix} \Psi_1(1) & \cdots & \Psi_1(A) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \Psi_A(1) & \cdots & \Psi_A(A) \end{vmatrix}$$

Second quantization

Creation and annihilation operators

$$c_a^\dagger|0> = |a> \qquad c_a|0> = 0$$

Anticommutators

$$\{c_a, c_b^\dagger\} = c_a c_b^\dagger + c_b^\dagger c_a = \delta(a,b)$$

$$\{c_a, c_b\} = \{c_a^\dagger, c_b^\dagger\} = 0$$

$$c_a^\dagger|a> = 0$$
$$<a|c_a = 0$$

Ground state of A-nucleons system

$$|a_1, a_2, \cdots, a_A> = \left(\Pi_{i=1}^A c_{a_i}^\dagger\right) |0>$$

$$c_a^\dagger|a> = c_a^\dagger\ c_a^\dagger\ |0> = -\ c_a^\dagger\ c_a^\dagger\ |0>$$

One-body operator

$$\hat{O} = \sum_i^A \hat{O}(i) = \sum_{a,b} <a|O|b>\ c_b^\dagger c_a$$

Nuclear Hamiltonian

$$\hat{H} = \sum_i^A \frac{\hat{p}^2}{2m_i} + \sum_i^A \hat{V}(i,j)$$

$$\hat{H} = \sum_{ab} t(a,b) c_a^\dagger c_b + \frac{1}{4}\sum_{abcd} V_{abcd} c_a^\dagger c_b^\dagger c_d c_c$$

Two -body operator

$$\hat{V} = \sum_{i<j}^A \hat{V}(i,j) = \frac{1}{4}\sum_{abcd} V_{abcd}\ c_a^\dagger c_b^\dagger c_d c_c$$

$$\hat{H} = \hat{H}_0 + \hat{V}_{res}$$

$$\hat{H} = \sum_a E_a c_a^\dagger c_a + \frac{1}{4}\sum_{abcd} V_{abcd}^{res} c_a^\dagger c_b^\dagger c_d c_c$$

BCS approximation

Pairing interaction is strongly attractive. This force acts between like nucleons in the same single particle orbit. It induces the two nucleons to couple with J=0

Two-particle state:

$$|\, j^2 \,,\, J=0\,\rangle = A^\dagger(j^2 00)|0>$$

$$A^\dagger(j^2 00) = \frac{1}{\sqrt{2}} \sum_m \langle\, jmj\, -m \,|\, 00\,\rangle\, c^\dagger_{jm} c^\dagger_{j-m} = \frac{1}{\sqrt{2(2j+1)}} \sum_m (-1)^{j-m} c^\dagger_{jm} c^\dagger_{j-m}$$

Ansatz for ground state with pairing correlations

$$|\,\mathrm{BCS}\,\rangle = \prod_j \prod_{m>0} \left(u_j + v_j\, c^\dagger_{jm} c^\dagger_{j-m}\right)_{\mathrm{BCS}} |\,0\,\rangle$$

The particle number is not conserved

The BCS g.s. is a superposition of states with different numbers of nucleons

$$|\,\mathrm{BCS}\,\rangle \sim |\,-\rangle + \sum_{j,m>0} \frac{v_j}{u_j} c^\dagger_{jm} c^\dagger_{j-m} \,|\,-\rangle + \frac{1}{2} \sum_{j,m>0} \frac{v_j}{u_j} c^\dagger_{jm} c^\dagger_{j-m} \sum_{j',m'>0} \frac{v_{j'}}{u_{j'}} c^\dagger_{j'm'} c^\dagger_{j'-m'} \,|\,-\rangle + \cdots$$

The normalization of the BCS ground state to unity requires:

$$u_j^2 + v_j^2 = 1, \qquad u_j,\, v_j \geq 0$$

Occupation probability

$$\frac{1}{2j+1} \langle \mathrm{BCS} \,|\, \sum_m c^\dagger_{jm} c_{jm} \,|\, \mathrm{BCS} \rangle \;=\; v_j^2$$

The number operator in a single particle orbit j

$$\begin{aligned} \widehat{N}_j &= \sum_m c^\dagger_{jm} c_{jm} \\ &= \sum_{m>0} \left(c^\dagger_{jm} c_{jm} + c^\dagger_{j-m} c_{j-m} \right) \end{aligned}$$

The number of particles in the single particle orbit

$$\begin{aligned} N_j &= \langle \mathrm{BCS} \,|\, \widehat{N}_j \,|\, \mathrm{BCS} \rangle \\ &= \sum_j (2j+1)\, v_j^2 \end{aligned}$$

BCS equation

When the Hamiltonian of system is provided, the occupation amplitudes are determined by energy variation

$$\delta \langle \mathrm{BCS} \,|\, H \,|\, \mathrm{BCS} \rangle = 0$$

The BCS ground state is expressed by considering p and n degrees Of freedom

$$| \mathrm{BCS} \rangle = \prod_{j_p} \prod_{m_p>0} \left(u_{j_p} + v_{j_p} \, c^{\dagger}_{j_p m_p} c^{\dagger}_{j_p -m_p} \right) \times \prod_{j_n} \prod_{m_n>0} \left(u_{j_n} + v_{j_n} \, c^{\dagger}_{j_n m_n} c^{\dagger}_{j_n -m_n} \right) | 0 \rangle$$

Variation with a constraint that expectation value of number operators should be equal to particle number

$$\langle \mathrm{BCS} \,|\, \widehat{N}_p \,|\, \mathrm{BCS} \rangle = N_p$$
$$\langle \mathrm{BCS} \,|\, \widehat{N}_n \,|\, \mathrm{BCS} \rangle = N_n$$

The modified Hamiltonian for variation

$$H' = H - \lambda_p \widehat{N}_p - \lambda_n \widehat{N}_n$$

u and v are not independent. they are constrained by a normalization condition

$$u_{j_p}^2 + v_{j_p}^2 = 1$$
$$\Rightarrow \frac{\partial u_{j_p}}{\partial v_{j_p}} = -\frac{v_{j_p}}{u_{j_p}}$$

Variation is explicitly written as

$$\left(\frac{\partial}{\partial v_{j_p}} + \frac{\partial u_{j_p}}{\partial v_{j_p}} \frac{\partial}{\partial u_{j_p}}\right) \langle \mathrm{BCS} \,|\, H' \,|\, \mathrm{BCS} \rangle = 0$$

The nuclear Hamiltonian

$$\begin{aligned} H &= H_0 + V \\ H_0 &= \sum_{j_p m_p} \varepsilon_{j_p}\, c^\dagger_{j_p m_p}\, c_{j_p m_p} + \sum_{j_n m_n} \varepsilon_{j_n}\, c^\dagger_{j_n m_n}\, c_{j_n m_n} \\ V &= V_{pp} + V_{nn} + V_{pn} \end{aligned}$$

BCS expectation value

$$\begin{aligned} &\langle \mathrm{BCS} \,|\, H' \,|\, \mathrm{BCS} \rangle \\ &= -\frac{1}{2} \sum_{j_p} (2j_p+1) \left(u_{j_p} v_{j_p} \Delta_{j_p} - v_{j_p}{}^2\, 2(\varepsilon_{j_p} - \lambda_p)\right) \\ &\quad -\frac{1}{2} \sum_{j_n} (2j_n+1) \left(u_{j_n} v_{j_n} \Delta_{j_n} - v_{j_n}{}^2\, 2(\varepsilon_{j_n} + -\lambda_n)\right) \end{aligned}$$

$$\begin{aligned} \Delta_{j_p} &= -\sum_{j'_p} (2j'_p+1)\, u_{j'_p} v_{j'_p} V_\Delta(j'_p, j_p) \\ \Delta_{j_n} &= -\sum_{j'_n} (2j'_n+1)\, u_{j'_n} v_{j'_n} V_\Delta(j'_n, j_n), \end{aligned}$$

$$V_\Delta(j, j') = \frac{1}{\sqrt{(2j+1)(2j'+1)}} \langle j^2 \,|\, V_{pp/nn} \,|\, j'^2 \rangle_{J=0}$$

Performing the variation

$$\left(\frac{\partial}{\partial v_{j_p}} + \frac{\partial u_{j_p}}{\partial v_{j_p}} \frac{\partial}{\partial u_{j_p}}\right) \langle \mathrm{BCS} \,|\, H' \,|\, \mathrm{BCS} \rangle = 0$$

We obtain a set of BCS equation

$$\begin{aligned} 2\, u_{j_p} v_{j_p} (\varepsilon_{j_p} - \lambda_p) + (v_{j_p}^2 - u_{j_p}^2)\, \Delta_{j_p} &= 0 \\ 2\, u_{j_n} v_{j_n} (\varepsilon_{j_n} - \lambda_n) + (v_{j_n}^2 - u_{j_n}^2)\, \Delta_{j_n} &= 0 \end{aligned}$$

We get quadratic equation for v^2

$$\begin{aligned} 4\,(\eta_j^2 + \Delta_j^2)\, v_j^4 - 4\,(\eta_j^2 + \Delta_j^2)\, v_j^2 + \Delta_j^2 &= 0 \\ \eta_j &= \varepsilon_j - \lambda \end{aligned}$$

Solution for BCS amplitudes:

$$\begin{aligned} v_j^2 &= \frac{1}{2}\left[1 - \frac{\varepsilon_j - \lambda}{\sqrt{(\varepsilon_j - \lambda)^2 + \Delta_j^2}}\right], \\ u_j^2 &= \frac{1}{2}\left[1 + \frac{\varepsilon_j - \lambda}{\sqrt{(\varepsilon_j - \lambda)^2 + \Delta_j^2}}\right] \end{aligned}$$

BCS algoritmus

Step 0: **Give single-particle energies for p and n, two-body matrix elements in the nucleon basis**

$$\varepsilon_{j_p}, \quad , \varepsilon_{j_n}, \quad \langle j^2 \,|\, V_{pp/nn} \,|\, j'^2 \rangle_{J^\pi=0^+}$$

Calculate:

$$V_\Delta(j, j') \;=\; \frac{1}{\sqrt{(2j+1)(2j'+1)}} \langle j^2 \,|\, V_{pp/nn} \,|\, j'^2 \rangle_{J^\pi=0^+},$$

Give also trial values of occupation amplitudes and chemical potentials

$$\text{trial values} \;:\; \begin{array}{llll} \lambda_{j_p}, & p, & p, & (u_{j_p}^2 + v_{j_p}^2 = 1), \\ \lambda_{j_n}, & n, & n, & (u_{j_n}^2 + v_{j_n}^2 = 1). \end{array}$$

Step 1: **Calculate proton and neutron pairing gaps:**

$$\Delta_{j_p} \;=\; -\sum_{j'_p} (2j'_p+1)\, u_{j'_p}\, v_{j'_p}\, V_\Delta(j_p, j'_p),$$

$$\Delta_{j_n} \;=\; -\sum_{j'_n} (2j'_n+1)\, u_{j'_n}\, v_{j'_n}\, V_\Delta(j_n, j'_n),$$

Step 2:

Calculate occupation Amplitudes:

$$v^2_{j_{p,n}} = \frac{1}{2}\left[1 - \frac{\varepsilon_{j_{p,n}} - \lambda}{\sqrt{(\varepsilon_{j_{p,n}} - \lambda)^2 + \Delta^2_{j_{p,n}}}}\right]$$

$$u^2_{j_{p,n}} = \frac{1}{2}\left[1 + \frac{\varepsilon_{j_{p,n}} - \lambda}{\sqrt{(\varepsilon_{j_{p,n}} - \lambda)^2 + \Delta^2_{j_{p,n}}}}\right]$$

Step 3:

Calculate expectation values of the number of protons and neutrons with occupation amplitudes, which are evaluated in previous step

$$N_p = \sum_{j_p} (2j_p + 1)\, v_{j_p}^{\ 2}$$

$$N_n = \sum_{j_n} (2j_n + 1)\, v_{j_n}^{\ 2}$$

Step 4:

Compare the proton and neutron numbers in the previous step and those of the nuclear under consideration. If they agree within a certain small number, the iterative calculation is converged. if not, the chemical potentials are slightly changed. A larger chemical potential results in a larger nucleon number. We then go back to Step 1 and the iterative processes from Step 1 through Step 5 are repeated again.

Qusiparticles – Bogoliubov transforamtion

Creation and annihilation operators of quasiparticles:

$$\begin{aligned} a^\dagger_{jm} &= u_j c^\dagger_{jm} - v_j (-1)^{j-m} c_{j-m} \\ a^\dagger_{j-m} &= u_j c^\dagger_{j-m} + v_j (-1)^{j-m} c_{jm} \end{aligned}$$

$$\{ a^\dagger_{jm},\, a_{j'm'} \} = \delta_{j,j'}\, \delta_{m,m'}\, (u_j{}^2 + v_j{}^2),$$

$$\{ a^\dagger_{jm},\, a^\dagger_{j'm'} \} = \{ a_{jm},\, a_{j'm'} \} = 0.$$

Anticommutation relations for q.p. operators calculated by using those of particle operators

$$\{ a^\dagger_{jm},\, a_{j'm'} \} = \delta_{j,j'}\, \delta_{m,m'}, \qquad \{ a^\dagger_{jm},\, a^\dagger_{j'm'} \} = \{ a_{jm},\, a_{j'm'} \} = 0$$

Notation:

$$\tilde{a}_{jm} = (-1)^{j+m}\, a_{j-m}$$

Unitary transformations between particles and quasiparticles

$$\begin{aligned} a^\dagger_{jm} &= u_j\, c^\dagger_{jm} - v_j\, \tilde{c}_{jm} \\ \tilde{a}_{jm} &= u_j\, \tilde{c}_{jm} + v_j\, c^\dagger_{jm} \end{aligned}$$

$$\begin{aligned} c^\dagger_{jm} &= u_j\, a^\dagger_{jm} + v_j\, \tilde{a}_{jm} \\ \tilde{c}_{jm} &= u_j\, \tilde{a}_{jm} - v_j\, a^\dagger_{jm} \end{aligned}$$

BCS equation (from equation of motion)

BCS equation:

$$\begin{pmatrix} e_\tau - \lambda_\tau & \Delta_\tau \\ \Delta_\tau & -(e_\tau - \lambda_\tau) \end{pmatrix} \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = E_\tau \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix}$$

Quasiparticle Energy:

$$\begin{vmatrix} e_\tau - E_\tau & \Delta_\tau \\ \Delta_\tau & -e_\tau - E_\tau \end{vmatrix} = 0$$

$$E_\tau^2 = e_\tau^2 + \Delta_\tau^2$$

Pairing gap (renormalization of pairing int.):

$$\Delta_\tau = \frac{1}{\sqrt{2j_\tau + 1}} \sum_{\tau'} \sqrt{2j_{\tau'} + 1}\; d_{\tau'\tau'} G(\tau'\tau', \tau\tau, J=0)\; u_{\tau'} v_{\tau'}$$

Diagonalization:

$$\begin{pmatrix} e_\tau & \Delta_\tau \\ \Delta_\tau & -e_\tau \end{pmatrix} = \begin{pmatrix} u_\tau & -v_\tau \\ v_\tau & u_\tau \end{pmatrix} \begin{pmatrix} E_\tau & 0 \\ 0 & -E_\tau \end{pmatrix} \begin{pmatrix} u_\tau & v_\tau \\ -v_\tau & u_\tau \end{pmatrix}$$

$$= \begin{pmatrix} E_\tau(u_\tau^2 - v_\tau^2) & 2u_\tau v_\tau E_\tau \\ 2u_\tau v_\tau E_\tau & -E_\tau(u_\tau^2 - v_\tau^2) \end{pmatrix}$$

Solution:

$$E_\tau = \sqrt{e_\tau^2 + \Delta_\tau^2}$$

$$u_\tau^2 = \frac{1}{2}\left(1 + \frac{e_\tau}{\sqrt{e_\tau^2 + \Delta_\tau^2}}\right)$$

$$v_\tau^2 = \frac{1}{2}\left(1 - \frac{e_\tau}{\sqrt{e_\tau^2 + \Delta_\tau^2}}\right)$$

Experimental pairing gaps:

$$M(Z,N)_{odd-proton} = \mathcal{M}(Z,N) + \Delta_p^{emp.}$$

$$M(Z,N)_{odd-neutron} = \mathcal{M}(Z,N) + \Delta_n^{emp.}$$

Mass formula:

$$\Delta_p^{emp.} = -\frac{1}{8}[M(Z+2,N) - 4M(Z+1,N) + 6M(Z,N) - 4M(Z-1,N) + M(Z-2,N)],$$

$$\Delta_n^{emp.} = -\frac{1}{8}[M(Z,N+2) - 4M(Z,N+1) + 6M(Z,N) - 4M(Z,N-1) + M(Z,N-2)]$$

Qusiparticle Random Phase Approximation (General formalism)

Quasiparticle RPA Models
Equation of motion

The correlated ground state: |rpa>

$$H\,|\,rpa\,\rangle = E_0\,|\,rpa\,\rangle$$

Excited states are constructed on the RPA g.s. by a set of operators:

$$Q^\dagger_\omega\,|\,rpa\,\rangle, \quad H\,|\,\omega\,\rangle = E_\omega\,|\,\omega\,\rangle$$

$$\begin{aligned}
[H,\, Q^\dagger_\omega]\,|\,rpa\,\rangle &= HQ^\dagger_\omega\,|\,rpa\,\rangle - Q^\dagger_\omega H\,|\,rpa\,\rangle \\
&= H\,|\,\omega\,\rangle - Q^\dagger_\omega E_0\,|\,rpa\,\rangle \\
&= (\,E_\omega - E_0\,)\,|\,rpa\,\rangle,
\end{aligned}$$

Equation of motion for operator Q$^+$

$$\begin{aligned}
[\,H,\, Q^\dagger_\omega\,]\,|\,rpa\,\rangle &= \omega\, Q^\dagger_\omega\,|\,rpa\,\rangle, \\
\omega &= E_\omega - E_0
\end{aligned}$$

The RPA ground state is the vacuum for the set of operators

$$Q_\omega\,|\,rpa\,\rangle = 0$$

Equation of motion

Instead of operator equation, we consider expectation value equation

$$\langle rpa \,|\, \delta Q \,[\, H,\, Q^\dagger_\omega \,]\,|\, rpa \,\rangle \;=\; \omega \,\langle rpa \,|\, \delta Q \, Q^\dagger_\omega \,|\, rpa \,\rangle$$

Rewritten with commutators

$$\begin{aligned} \delta Q \,[\, H,\, Q^\dagger_\omega \,] &= [\, \delta Q,\, [\, H,\, Q^\dagger_\omega \,]\,] + [\, H,\, Q^\dagger_\omega \,]\, \delta Q, \\ \delta Q \, Q^\dagger_\omega &= [\, \delta Q,\, Q^\dagger_\omega \,] + Q^\dagger_\omega \, \delta Q. \end{aligned}$$

Vanishing of matrix elements

$$\langle 0 \,|\, Q^\dagger_\omega \;=\; 0 \quad \Rightarrow \quad \langle rpa \,|\, Q^\dagger_\omega \, \delta Q \,|\, rpa \,\rangle \;=\; 0$$

$$\begin{aligned} \langle rpa \,|\, [\, H,\, Q^\dagger_\omega \,] &= \langle rpa \,|\, H \, Q^\dagger_\omega - \langle rpa \,|\, Q^\dagger_\omega \, H \\ &= \langle rpa \,|\, Q^\dagger_\omega \, (E_0 - H) \\ &= 0 \end{aligned}$$

The equation of motion has been derived without any approximation, from only the eigenvalue equation

$$\langle rpa \,|\, [\, \delta Q,\, [\, H,\, Q^\dagger_\omega \,]\,] \,|\, rpa \,\rangle \;=\; \omega \,\langle rpa \,|\, [\, \delta Q,\, Q^\dagger_\omega \,] \,|\, rpa \,\rangle$$

Assumption on excitation operators

We assume that the excitation operators are expressed by a sum of two-quasiparticle creation and annihilation operators

$$Q^{\dagger}_{\omega,JM} = \sum_{pn} \left(X^{pn}_{\omega,J} A^{\dagger}_{JM}(pn) + Y^{pn}_{\omega,J} \widetilde{A}_{JM}(pn) \right)$$

creates a proton-neutron quasiparticle pair

$$A^{\dagger}_{JM}(pn) = \sum_{m_p,m_n} \langle\, j_p m_p j_n m_n \,|\, JM \,\rangle\; a^{\dagger}_{pm_p} a^{\dagger}_{nm_n}$$

corresponding annihilation operator

$$\widetilde{A}_{JM}(pn) = (-1)^{1+J-M} A_{J\text{-}M}(pn)$$

The Hermitian conjugate are

$$(\, A^{\dagger}_{JM}(pn)\,)^{\dagger} = A_{JM}(pn) = (-1)^{1+J+M} \widetilde{A}_{J\text{-}M}(pn)$$

$$(\, \widetilde{A}_{JM}(pn)\,)^{\dagger} = (-1)^{1+J-M} (\, A_{J\text{-}M}(pn)\,)^{\dagger} = (-1)^{1+J-M} A^{\dagger}_{J\text{-}M}(pn)$$

$$\widetilde{Q}_{\omega,JM} = (-1)^{1+J-M} \left(Q^{\dagger}_{\omega,J\text{-}M} \right)^{\dagger} = \sum_{pn} \left(Y^{pn}_{\omega,J} A^{\dagger}_{JM}(pn) + X^{pn}_{\omega,J} \widetilde{A}_{JM}(pn) \right)$$

This is a spherical tensor of rank-J and its projection M

ic

QRPA equations

$$\langle rpa | [\delta Q, [H, Q^\dagger_\omega]] | rpa \rangle = \omega \langle rpa | [\delta Q, Q^\dagger_\omega] | rpa \rangle$$

Equation of motion is described in a basis of proton-neutron pairs pn

Operator between RPA ground state has to be a sclar

substituing $\delta Q = \tilde{A}_{J\text{-}M}(p'n'), \quad \delta Q = A^\dagger_{J\text{-}M}(p'n')$ **we get**

$$\langle rpa | [\tilde{A}_{J\text{-}M}(p'n'), [H, Q^\dagger_{\omega,JM}]] | rpa \rangle = \omega \langle rpa | [\tilde{A}_{J\text{-}M}(p'n'), Q^\dagger_{\omega,JM}] | rpa \rangle,$$

$$\langle rpa | [A^\dagger_{J\text{-}M}(p'n'), [H, Q^\dagger_{\omega,JM}]] | rpa \rangle = \omega \langle rpa | [A^\dagger_{J\text{-}M}(p'n'), Q^\dagger_{\omega,JM}] | rpa \rangle,$$

More explicitely

$$\sum_{pn} X^{pn}_{\omega,J} \langle 0 | [\tilde{A}_{J\text{-}M}(p'n'), [H, A^\dagger_{JM}(pn)]] | 0 \rangle + \sum_{pn} Y^{pn}_{\omega,J} \langle 0 | [\tilde{A}_{J\text{-}M}(p'n'), [H, \tilde{A}_{JM}(pn)]] | 0 \rangle = \omega \sum_{pn} X^{pn}_{\omega,J} \langle 0 | [\tilde{A}_{J\text{-}M}(p'n'), A^\dagger_{JM}(pn)] | 0 \rangle,$$

$$\sum_{pn} X^{pn}_{\omega,J} \langle 0 | [A^\dagger_{J\text{-}M}(p'n'), [H, A^\dagger_{JM}(pn)]] | 0 \rangle + \sum_{pn} Y^{pn}_{\omega,J} \langle 0 | [A^\dagger_{J\text{-}M}(p'n'), [H, \tilde{A}_{JM}(pn)]] | 0 \rangle = \omega \sum_{pn} Y^{pn}_{\omega,J} \langle 0 | [A^\dagger_{J\text{-}M}(p'n'), \tilde{A}_{JM}(pn)] | 0 \rangle.$$

we take the advantage of relations

$$\begin{aligned}
[A^{\dagger}_{J\text{-}M}(p'n'),\,[H,\,A^{\dagger}_{JM}(pn)]]^{\dagger} &= [\tilde{A}_{JM}(p'n'),\,[H,\,\tilde{A}_{J\text{-}M}(pn)]] \\
[A^{\dagger}_{J\text{-}M}(p'n'),\,[H,\,\tilde{A}_{JM}(pn)]]^{\dagger} &= [\tilde{A}_{JM}(p'n'),\,[H,\,A^{\dagger}_{J\text{-}M}(pn)]] \\
[A^{\dagger}_{J\text{-}M}(p'n'),\,\tilde{A}_{JM}(pn)]^{\dagger} &= -[\tilde{A}_{JM}(p'n'),\,A^{\dagger}_{J\text{-}M}(pn)]
\end{aligned}$$

QRPA equations in matrix form

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \omega \begin{bmatrix} C & 0 \\ 0 & -C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Elements of submatrices A, B and C are

$$\begin{aligned}
A_{p'n',pn} &= \langle rpa|\,[A_{JM}(p'n'),\,[H,\,A^{\dagger}_{JM}(pn)]]\,|rpa\rangle \\
B_{p'n',pn} &= \langle rpa|\,[A_{JM}(p'n'),\,[H,\,\tilde{A}_{JM}(pn)]]\,|rpa\rangle \\
C_{p'n',pn} &= \langle rpa|\,[A_{JM}(p'n'),\,A^{\dagger}_{JM}(pn)]\,|rpa\rangle
\end{aligned}$$

Rows of the matrices X and Y are labeled by pn-pairs, while columns of them by eigenvalues **ω**. That is X_{ij} represents the i-th component of the forward-going amplitude of j-th eigenstate. Excitations energies ω and amplitudes are obtained by solving QRPA equations. When number of pn-pairs is **n** in the model space for a given angular momentum **J**, submatrices A, B and C are **nxn** matrices

Qrthonormality of excited states

The excited states, which are eigenstates of Hamiltonian H, are orthogonal To one another, and are normalized to unity:

$$\begin{aligned}\langle \omega, JM \,|\, \omega', J'M' \rangle &= \delta_{\omega,\omega'}\, \delta_{J,J'}\, \delta_{M,M'} \\ &= \delta_{J,J'}\, \delta_{M,M'} \langle rpa \,|\, [\, Q_{\omega,JM},\, Q^{\dagger}_{\omega',JM} \,] \,|\, rpa \rangle\end{aligned}$$

With explicit form

$$\begin{aligned}Q_{\omega,JM} &= \sum_{pn} (-1)^{1+J+M} \left(X^{pn}_{\omega,J} \widetilde{B}_{J\text{-}M}(pn) + Y^{pn}_{\omega,J} B^{\dagger}_{J\text{-}M}(pn) \right) \\ Q^{\dagger}_{\omega',JM} &= \sum_{p'n'} \left(X^{p'n'}_{\omega',J} B^{\dagger}_{JM}(p'n') + Y^{p'n'}_{\omega',J} \widetilde{B}_{JM}(pn) \right)\end{aligned}$$

the commutator is evaluated as

$$\begin{aligned}&\langle rpa \,|\, [\, Q_{\omega,JM},\, Q^{\dagger}_{\omega',JM} \,] \,|\, rpa \rangle \\ &= \sum_{pn} \sum_{p'n'} (-1)^{1+J+M} X^{pn}_{\omega,J} X^{p'n'}_{\omega',J} \langle rpa \,|\, [\, \widetilde{A}_{J\text{-}M}(pn),\, A^{\dagger}_{JM}(p'n') \,] \,|\, rpa \rangle \\ &\qquad + \sum_{pn} \sum_{p'n'} (-1)^{1+J+M} Y^{pn}_{\omega,J} Y^{p'n'}_{\omega',J} \langle rpa \,|\, [\, A^{\dagger}_{J\text{-}M}(pn),\, \widetilde{A}_{JM}(p'n') \,] \,|\, rpa \rangle \\ &= \sum_{pn} \sum_{p'n'} X^{pn}_{\omega,J} X^{p'n'}_{\omega',J} \langle rpa \,|\, [\, A_{JM}(pn),\, A^{\dagger}_{JM}(p'n') \,] \,|\, rpa \rangle \\ &\qquad + \sum_{pn} \sum_{p'n'} Y^{pn}_{\omega,J} Y^{p'n'}_{\omega',J} \langle rpa \,|\, [\, A^{\dagger}_{J\text{-}M}(pn),\, A_{J\text{-}M}(p'n') \,] \,|\, rpa \rangle\end{aligned}$$

with help of

$$\langle rpa \,|\, [\, A_{JM}(pn),\, A^{\dagger}_{JM}(p'n') \,] \,|\, rpa \,\rangle \;=\; C_{pn,p'n'}$$
$$\langle rpa \,|\, [\, B^{\dagger}_{J\text{-}M}(pn),\, B_{J\text{-}M}(p'n') \,] \,|\, rpa \,\rangle \;=\; -\,C_{pn,p'n'}$$

The orthonormality condition is expressed as

$$\sum_{pn} \sum_{p'n'} \left(X^{pn}_{\omega,J}\, C_{pn,p'n'}\, X^{p'n'}_{\omega',J} - Y^{pn}_{\omega,J}\, C_{pn,p'n'}\, Y^{p'n'}_{\omega',J} \right) = \delta_{\omega,\omega'}$$

In matrix form

$$\begin{bmatrix} X^T_{\omega} & Y^T_{\omega} \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & -C \end{bmatrix} \begin{bmatrix} X_{\omega'} \\ Y_{\omega'} \end{bmatrix} = \delta_{\omega,\omega'}\, I$$

Eigenstates are thus orthonormalized not by amplitudes themselves, but with overlap matrix C

Overlap matrix

Elements of overlap matrix are defined by expectation values of the commutator

$$C_{p'n',pn} = \langle rpa \,|\, [\, A_{JM}(p'n'),\, A^{\dagger}_{JM}(pn) \,] \,|\, rpa \,\rangle$$

commutator is written as

$$[\, A_{JM}(p'n'),\, A^{\dagger}_{JM}(pn) \,]$$
$$= \sum_{m'_p, m'_n} \sum_{m_p, m_n} \langle j'_p m'_p j'_n m'_n \,|\, JM \,\rangle \langle j_p m_p j_n m_n \,|\, JM \,\rangle \, [\, a_{j'_n m'_n} a_{j'_p m'_p},\, a^{\dagger}_{j_p m_p} a^{\dagger}_{j_n m_n} \,]$$

The commutator for creation and annihilation operators is

$$\begin{aligned}[\, a_{j'_n m'_n} a_{j'_p m'_p},\, a^{\dagger}_{j_p m_p} a^{\dagger}_{j_n m_n} \,] &= \delta_{j_p, j'_p} \delta_{m_p, m'_p} \delta_{j_n, j'_n} \delta_{m_n, m'_n} \\ &- \delta_{j_n, j'_n} \delta_{m_n, m'_n} a^{\dagger}_{j_p m_p} a_{j'_p m'_p} \\ &- \delta_{j_p, j'_p} \delta_{m_p, m'_p} a^{\dagger}_{j_n m_n} a_{j'_n m'_n}\end{aligned}$$

$$a^{\dagger}_{j_p m_p} a_{j'_p m'_p} = \sum_{kq} \langle\, j_p m_p j'_p\text{-}m'_p \,|\, kq \,\rangle\, (-1)^{j'_p - m'_p} [\, a^{\dagger}_{j_p} \tilde{a}_{j'_p} \,]^{(k)}_q$$

$$\begin{aligned}
&\delta_{j_n, j'_n} \sum_{m_p, m'_p, m_n} \langle\, j'_p m'_p j_n m_n \,|\, JM \,\rangle \langle\, j_p m_p j_n m_n \,|\, JM \,\rangle \langle\, rpa \,|\, a^{\dagger}_{j_p m_p} a_{j'_p m'_p} \,|\, rpa \,\rangle \\
&= \delta_{j_p, j'_p} \delta_{j_n, j'_n} \sum_{m_p, m_n} \langle\, j_p m_p j_n m_n \,|\, JM \,\rangle \langle\, j_p m_p j_n m_n \,|\, JM \,\rangle \\
&\quad \times \langle\, j_p m_p j_p\text{-}m_p \,|\, 00 \,\rangle\, (-1)^{j_p - m_p} \langle\, rpa \,|\, [\, a^{\dagger}_{j_p} \tilde{a}_{j_p} \,]^{(0)}_0 \,|\, rpa \,\rangle \\
&= \delta_{j_p, j'_p} \delta_{j_n, j'_n} \sum_{m_p, m_n} \langle\, j_p m_p j_n m_n \,|\, JM \,\rangle^2 \frac{1}{\sqrt{2j_p + 1}} \langle\, rpa \,|\, [\, a^{\dagger}_{j_p} \tilde{a}_{j_p} \,]^{(0)}_0 \,|\, rpa \,\rangle \\
&= \delta_{j_p, j'_p} \delta_{j_n, j'_n} \frac{1}{2j_p + 1} \langle\, rpa \,|\, \hat{n}_{j_p} \,|\, rpa \,\rangle.
\end{aligned}$$

$$\hat{n}_{j_p} = \sum_{m} a^{\dagger}_{j_p m_p} a_{j_p m_p}$$

is number operator for the quasiparticle j_p orbit

Matrix elements of the overlap matrix are

$$C_{p'n', pn} = \delta_{p'p} \delta_{n'n} \left(1 - \rho_p - \rho_n \right)$$

$$\rho_j = \frac{1}{2j + 1} \langle\, rpa \,|\, \hat{n}_j \,|\, rpa \,\rangle$$

ρ_j is the occupation probability of the quasiparticle j-orbit

Hamiltonian in particle representation

Quasiparticle Hamiltonian

$$H' = \sum_{\tau m_\tau} e_\tau c^\dagger_{\tau m_\tau} c_{\tau m_\tau} - \lambda_p \hat{N}_p - \lambda_n \hat{N}_n$$
$$+\frac{1}{4} \sum_{p m_p n m_n n' m_{n'} p' m_{p'}} < p m_p n m_n | V^{res} | p' m_{p'} n' m_{n'} > c^\dagger_{p m_p} c^\dagger_{n m_n} c_{n' m_{n'}} c_{p' m_{p'}}$$

BCS transformation

$$\begin{pmatrix} a^\dagger_{\tau m_\tau} \\ \tilde{a}_{\tau m_\tau} \end{pmatrix} = \begin{pmatrix} u_\tau & -v_\tau \\ v_\tau & u_\tau \end{pmatrix} \begin{pmatrix} c^\dagger_{\tau m_\tau} \\ \tilde{c}_{\tau m_\tau} \end{pmatrix}$$

Normal product

$$N(c^\dagger_p c^\dagger_n c_{n'} c_{p'}) =$$
$$+u_p u_n v_{n'} v_{p'} \; a^\dagger_p a^\dagger_n a^\dagger_{n'} a^\dagger_{p'} + v_p v_n u_{n'} u_{p'} \; a_p a_n a_{n'} a_{p'} \quad (40, 04)$$
$$-u_p u_n u_{n'} v_{p'} \; a^\dagger_p a^\dagger_n \tilde{a}^\dagger_{p'} a_{n'} + v_p v_n u_{n'} u_{p'} \; a^\dagger_p \tilde{a}_n a_{n'} a_{p'} \quad (31, 13)$$
$$+u_p u_n v_{n'} u_{p'} \; a^\dagger_p a^\dagger_n \tilde{a}^\dagger_{n'} a_{p'} - v_p u_n u_{n'} u_{p'} \; a^\dagger_n \tilde{a}_p a_{n'} a_{p'} \quad (31, 13)$$
$$+u_p v_n v_{n'} v_{p'} \; a^\dagger_p \tilde{a}^\dagger_{n'} \tilde{a}^\dagger_{p'} \tilde{a}_n - v_p v_n u_{n'} v_{p'} \; \tilde{a}^\dagger_{p'} \tilde{a}_p \tilde{a}_n a_{n'} \quad (31, 13)$$
$$-v_p u_n v_{n'} v_{p'} \; a^\dagger_n \tilde{a}^\dagger_{n'} \tilde{a}^\dagger_{p'} \tilde{a}_p + v_p v_n v_{n'} u_{p'} \; \tilde{a}^\dagger_{n'} \tilde{a}_p \tilde{a}_n a_{p'} \quad (31, 13)$$
$$+u_p u_n u_{n'} u_{p'} \; a^\dagger_p a^\dagger_n a_{n'} a_{p'} + v_p v_n v_{n'} v_{p'} \; \tilde{a}^\dagger_{n'} \tilde{a}^\dagger_n \tilde{a}_p \tilde{a}_n \quad (22)$$
$$+u_p v_n u_{n'} v_{p'} \; a^\dagger_p \tilde{a}^\dagger_{p'} \tilde{a}_n a_{n'} + v_p u_n v_{n'} u_{p'} \; a^\dagger_n \tilde{a}^\dagger_{n'} \tilde{a}_p a_{p'} \quad (22)$$
$$-u_p v_n v_{n'} u_{p'} \; a^\dagger_p \tilde{a}^\dagger_{n'} \tilde{a}_n a_{p'} - v_p u_n u_{n'} v_{p'} \; a^\dagger_n \tilde{a}^\dagger_{p'} \tilde{a}_p a_{n'} \quad (22)$$

Hamiltonian in quasi-particle representation

$$H'_{q.p.} = \sum_{\tau m_\tau} E_\tau a^\dagger_{\tau m_\tau} a_{\tau m_\tau} + H_{22} + H_{40} + H_{04} + H_{31} + H_{13}$$

$$\begin{aligned}
H_{22} &= -\frac{1}{2} \sum_{pn,p'n',JM} \Big(G_{pn,p'n',J} (u_p u_n u_{p'} u_{n'} + v_{p'} v_n v_{p'} v_{n'}) \\
&+ 4\ F_{pn,p'n',J} u_p v_n u_{p'} v_{n'} \Big) A^\dagger_{pn,JM} A_{p'n',JM}, \\
H_{40} &= \frac{1}{2} \sum_{pn,p'n',JM} G_{pn,p'n',J}\ u_p u_n v_{p'} v_{n'}\ A^\dagger_{pn,JM} \tilde{A}^\dagger_{p'n',JM}, \\
H_{31} &= \sum_{pn,p'n',JM} G_{pn,p'n',J} (u_p u_n u_{p'} v_{n'} + v_{p'} v_n v_{p'} u_{n'}) A^\dagger_{pn,JM} \tilde{B}_{p'n',JM}, \\
H_{04} &= (H_{40})^\dagger, \quad H_{13} = (H_{31})^\dagger.
\end{aligned}$$

two-quasiparticle operators

$$\begin{aligned}
A^\dagger_{pn,JM} &= [a^\dagger_p a^\dagger_n]^{JM} = \sum_{m_p,m_n} C^{JM}_{j_p m_p j_n m_n} a^\dagger_{pm_p} a^\dagger_{nm_n}, \\
A_{pn,JM} &= (A^\dagger_{pn,JM})^\dagger, \\
\tilde{A}_{pn,JM} &= -[\tilde{a}_p \tilde{a}_n]^{JM} = (-1)^{J-M} A_{pn,J-M} \\
&= (-1)^{J-M} \sum_{m_p,m_n} C^{J-M}_{j_p m_p j_n m_n} \alpha_{pm_p} \alpha_{nm_n}, \\
B_{pn,JM} &= [a^\dagger_p \tilde{a}_n]^{JM},
\end{aligned}$$

QRPA matrices

Definition:

$$\mathcal{A}_{pn,p'n',JM} = \langle \mathrm{rpa}|[A_{pn,JM}, H, A^{\dagger}_{p'n',JM}]|\mathrm{rpa}\rangle,$$
$$\mathcal{B}_{pn,p'n',JM} = -\langle \mathrm{rpa}|[A_{pn,JM}, H, \tilde{A}_{p'n',JM}]|\mathrm{rpa}\rangle,$$
$$\mathcal{C}_{pn,p'n',JM} = \langle \mathrm{rpa}|[A_{pn,JM}, A^{\dagger}_{p'n',JM}]|\mathrm{rpa}\rangle$$

Calculated:

$$\begin{aligned}
\mathcal{A}_{pn,p'n',J} &= \delta_{pp'}\delta_{nn'}(E_p + E_n)(1-\rho_p-\rho_n) \\
&\quad - 2\Big(G_{pn,p'n',J}(u_p u_n u_{p'} u_{n'} + v_p v_n v_{p'} v_{n'}) \\
&\quad + F_{pn,p'n',J}(u_p v_n u_{p'} v_{n'} + v_p u_n v_{p'} u_{n'})\Big) \times \\
&\quad (1-\rho_p-\rho_n)(1-\rho_{p'}-\rho_{n'}), \\
\mathcal{B}_{pn,p'n',J} &= 2\Big(G_{pn,p'n',J}(u_p u_n v_{p'} v_{n'} + v_p v_n u_{p'} u_{n'}) \\
&\quad - F_{pn,p'n',J}(u_p v_n v_{p'} u_{n'} + v_p u_n u_{p'} v_{n'})\Big) \times \\
&\quad (1-\rho_p-\rho_n)(1-\rho_{p'}-\rho_{n'}) \\
\mathcal{C}_{pn,p'n',J} &= \delta_{pp'}\delta_{nn'}(1-\rho_p-\rho_n)
\end{aligned}$$

For numerical calculation we need:

Solution of BCS equation: E_p, E_n, u_p, v_p, u_n, v_n
G-matrix elements of realistic NN interaction
ρ_p, ρ_n not yet determined

Standard QRPA
Renormalized QRPA
Selfconsistent Renormalized QRPA

Standard QRPA

Quasiboson approximation:

$$[A_{pn,JM}, A^{\dagger}_{p'n',JM}] \approx \langle \mathrm{BCS}|[A_{pn,JM}, A^{\dagger}_{p'n',JM}]|\mathrm{BCS}\rangle = \delta_{pp'}\delta_{nn'}$$

QRPA equation:

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} X^m \\ Y^m \end{pmatrix} = \omega_{\mathrm{RPA}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X^m \\ Y^m \end{pmatrix}$$

RPA matrices:

$$\begin{aligned}
\mathcal{A}_{pn,p'n',J} &= \delta_{pp'}\delta_{nn'}(E_p + E_n) \\
&\quad - 2\left(g_{pp} G_{pn,p'n',J}(u_p u_n u_{p'} u_{n'} + v_p v_n v_{p'} v_{n'}) \right. \\
&\quad \left. + g_{ph} F_{pn,p'n',J}(u_p v_n u_{p'} v_{n'} + v_p u_n v_{p'} u_{n'})\right) \\
\mathcal{B}_{pn,p'n',J} &= 2\left(g_{pp} G_{pn,p'n',J}(u_p u_n v_{p'} v_{n'} + v_p v_n u_{p'} u_{n'}) \right. \\
&\quad \left. - g_{ph} F_{pn,p'n',J}(u_p v_n v_{p'} u_{n'} + v_p u_n u_{p'} v_{n'})\right)
\end{aligned}$$

g_{pp} – renormalization of particle-particle NN interaction
g_{ph} – renormalization of particle-hole NN interaction

Operator changing neutron to proton:

$$\hat{\mathcal{O}}_{JM} = \frac{1}{\sqrt{2J+1}} \sum_{p,n} < p||\mathcal{O}||n > C_{pn,JM}$$

proton particle – neutron hole operator rewritten with quasiparticles

$$[c_p^\dagger \tilde{c}_n]_{JM} = u_p v_n A^\dagger_{pn,JM} + v_p u_n \tilde{A}_{pn,JM} + u_p u_n B^\dagger_{pn,JM} - v_p v_n \tilde{B}_{pn,JM}$$

Transition amplitude to excited state

$$< m, JM|\hat{\mathcal{O}}_{JM}|rpa >= \frac{1}{2J+1} \sum_{p,n} < p||\mathcal{O}||n >< rpa|[Q^m_{JM}, C_{pn,JM}]|rpa >$$

One-body density within the QRPA:

$$< rpa|[Q^m_{JM}, C^\dagger_{pn,JM}]|rpa >= u_p v_n X_{pn,J} + v_p u_n Y_{pn,J}$$

Diagonalization of the QRPA equation

QRPA equation:

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ -Y \end{bmatrix} \omega$$

$$\begin{array}{ll} AX + BY = X\omega, & (+): \quad (A+B)(X+Y) = (X-Y)\omega \\ BX + AY = -Y\omega, & (-): \quad (A-B)(X-Y) = (X+Y)\omega \end{array}$$

$$(A+B)(A-B)(X-Y) = (X-Y)\omega^2$$

Cholesky decomposition: $A - B = L\, L^T$

Standard eigenvalue problem:

$$\begin{array}{c} L^T(A+B)L\; L^T(X-Y) \;=\; L^T(X-Y)\omega^2 \\ M\; g = g\; \omega^2 \end{array}$$

QRPA amplitudes

$$\begin{aligned} X &= \frac{1}{2}\left(Lg\; \omega^{-1/2} + (L^{-1})^T g\; \omega^{1/2}\right) \\ Y &= \frac{1}{2}\left(Lg\; \omega^{-1/2} - (L^{-1})^T g\; \omega^{1/2}\right) \end{aligned}$$

Renormalized QRPA

Renormalized QBA:

$$\begin{aligned}[A_{pn,JM}, A^{\dagger}_{p'n',JM}] &\approx \langle \mathrm{rpa}|[A_{pn,JM}, A^{\dagger}_{p'n',JM}]|\mathrm{rpa}\rangle \\ &= \delta_{pp'}\delta_{nn'}(1-\rho_p-\rho_n) \\ &= \delta_{pp'}\delta_{nn'}{D_{pn}}^2\end{aligned}$$

Diagonal elements of the overlap matrix can be used to scale various quantities

$$D^2 = C$$

$$\begin{aligned}D_{p'n',pn} &= \delta_{p'p}\,\delta_{n'n}\,D_{pn} \\ D_{pn} &= \sqrt{1-\rho_p-\rho_n}\end{aligned}$$

Using the diagonal matrix D , the RPA equation can be written as

$$\begin{aligned}&\begin{bmatrix} D^{-1} & 0 \\ 0 & D^{-1}\end{bmatrix}\begin{bmatrix} A & B \\ B & A\end{bmatrix}\begin{bmatrix} D^{-1} & 0 \\ 0 & D^{-1}\end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & D\end{bmatrix}\begin{bmatrix} X \\ Y\end{bmatrix} \\ = \;\; \omega &\begin{bmatrix} D^{-1} & 0 \\ 0 & D^{-1}\end{bmatrix}\begin{bmatrix} C & 0 \\ 0 & -C\end{bmatrix}\begin{bmatrix} D^{-1} & 0 \\ 0 & D^{-1}\end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & D\end{bmatrix}\begin{bmatrix} X \\ Y\end{bmatrix}.\end{aligned}$$

Defining scaled submatrices and scaled amplitudes

$$\begin{aligned}\mathcal{A} &= D^{-1}\,A\,D^{-1}, \qquad \mathcal{B} = D^{-1}\,B\,D^{-1} \\ \mathcal{X} &= D\,X, \qquad \mathcal{Y} = D\,Y\end{aligned}$$

The well-known form of RPA matrix equation

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & \mathcal{A} \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} = \omega \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

The excitation operators and their hermitian conjugates:

$$Q^{\dagger}_{\omega,JM} = \sum_{pn} \frac{1}{D_{pn}} \left(\mathcal{X}^{pn}_{\omega,J} A^{\dagger}_{JM}(pn) + \mathcal{Y}^{pn}_{\omega,J} \tilde{A}_{JM}(pn) \right),$$

$$\tilde{Q}_{\omega,JM} = \sum_{pn} \frac{1}{D_{pn}} \left(\mathcal{Y}^{pn}_{\omega,J} A^{\dagger}_{JM}(pn) + \mathcal{X}^{pn}_{\omega,J} \tilde{A}_{JM}(pn) \right)$$

Inverse relations:

$$A^{\dagger}_{JM}(pn) = \sum_{\omega} D_{pn} \left(\mathcal{X}^{pn}_{\omega,J} Q^{\dagger}_{\omega,J} - \mathcal{Y}^{pn}_{\omega,J} \tilde{Q}_{\omega,J} \right),$$

$$\tilde{A}_{JM}(pn) = \sum_{\omega} D_{pn} \left(\mathcal{X}^{pn}_{\omega,J} \tilde{Q}_{\omega,J} - \mathcal{Y}^{pn}_{\omega,J} Q^{\dagger}_{\omega,J} \right)$$

Gamow-Teller transition amplitudes written with scaled amplitudes and scaling factors:

$$\langle \omega \,\|\, t_{-}\boldsymbol{\sigma} \,\|\, - \rangle = \sum_{pn} \langle p \,\|\, t_{-}\boldsymbol{\sigma} \,\|\, n \rangle \, D_{pn} \left(u_p v_n \mathcal{X}^{\omega}_{J}(pn) + v_p u_n \mathcal{Y}^{\omega}_{J}(pn) \right)$$

$$\langle \omega \,\|\, t_{+}\boldsymbol{\sigma} \,\|\, - \rangle = \sum_{pn} \langle n \,\|\, t_{+}\boldsymbol{\sigma} \,\|\, p \rangle \, D_{pn} \left(v_p u_n \mathcal{X}^{\omega}_{J}(pn) + u_p v_n \mathcal{Y}^{\omega}_{J}(pn) \right)$$

QRPA ground state wave function

The RPA g.s. wave function is connected by a canonical transformation to quasiparticle ground state

$$|rpa >= U|BCS >$$

$$U = Ne^S, \qquad S = \frac{1}{2} \sum_{JM,p'n'pn} Z^J_{p'n',pn} (-1)^{J-M} A^\dagger_{JM}(p'n') A^\dagger_{J\text{-}M}(pn)$$

$$Z^J_{p'n',pn} = Z^J_{pn,p'n'}$$

Coefficients Z are evaluated by using the fact that the ground state is the vaccum for excitations operators

$$Q_\omega \,|\, rpa \,\rangle \;=\; 0 \qquad \sum_{pn} \left(Y^{pn}_{\omega,J} A^\dagger_{JM}(pn) + X^{pn}_{\omega,J} U^{-1} \tilde{A}_{JM}(pn) U \right) |\, BCS \,\rangle = 0$$

$$\mathrm{e}^{-A} \, B \, \mathrm{e}^{A} = B - [A,B] + \frac{1}{2} [A,[A,B]] - \cdots$$

$$\begin{aligned} Y &= ZX \\ Z &= D^{-1} \mathcal{Y} \mathcal{X}^{-1} D \end{aligned}$$

Occupation probabilities of quasiparticle states

Occupation probability
(matrix element of number op.)

$$\rho_p = \langle rpa \,|\, \hat{n}_p \,|\, rpa \rangle \;=\; N \langle rpa \,|\, \hat{n}_p \mathrm{e}^S \,|\, BCS \rangle \;=\; N \langle rpa \,|\, [\hat{n}_p, \mathrm{e}^S] \,|\, BCS \rangle$$

$$\begin{aligned} \mathrm{e}^S \hat{n}_p \mathrm{e}^{-S} &= \hat{n}_p + [S, \hat{n}_p] \\ \mathrm{e}^S \hat{n}_p &= \hat{n}_p \mathrm{e}^S + [S, \hat{n}_p] \mathrm{e}^S \\ \left[\hat{n}_p, \mathrm{e}^S\right] &= [\hat{n}_p, S]\, \mathrm{e}^S \end{aligned}$$

$$\langle rpa \,|\, \hat{n}_p \,|\, rpa \rangle = \langle rpa \,|\, [\hat{n}_p, S] \,|\, rpa \rangle$$

$$[\hat{n}_p, S] = \sum_{JM, p'n'n} (-1)^{J-M} Z^J_{pn,p'n'} A^\dagger_{JM}(pn) A^\dagger_{J\text{-}M}(p'n')$$

Occupation probabilities
of proton and neutron quasiparticle orbits

$$\rho_p = \sum_J \frac{2J+1}{2j_p+1} \sum_{\omega,n} \left(D_{pn} \mathcal{Y}^{pn}_{\omega,J} \right)^2, \qquad \rho_n = \sum_J \frac{2J+1}{2j_n+1} \sum_{\omega,p} \left(D_{pn} \mathcal{Y}^{pn}_{\omega,J} \right)^2$$

QRPA-like aproaches
(QRPA, RQRPA, SRQRPA)

Particle number condition

i) Uncorrelated BCS ground state

Z=<BCS|Z|BCS>

N=<BCS|N|BCS>

QRPA, RQRPA

ii) Correlated RPA ground state

Z=<RPA|Z|RPA>
N=<BCS|N|BCS>
SRQRPA

Complex numerical procedure
BCS and QRPA equations are coupled

Pauli exclusion principle

i) violated (QBA)

$[A,A^+] = <BCS|[A,A^+]|BCS>$

QRPA

ii) Partially restored (RQBA)

$[A,A^+] = <RPA|[A,A^+]|RPA>$

RQRPA, SRQRPA

$$N = (2j+1)[v_j^2 + (u_j^2 - v_j^2)\frac{< rpa|[a_j^\dagger \tilde{a}_j]_{00}|rpa >}{\sqrt{2j+1}]}$$

Overlap factor – Two QRPA diagonalizations

$$\mathcal{O}_{m_f m_i}(JM) = < JM, m_f | JM, m_i >$$

Two sets of states

$$|JM, m_i> \; = \; Q^{\dagger}_{m_i}(JM)|RPA>_i \quad initial$$

$$|JM, m_f> \; = \; Q^{\dagger}_{m_f}(JM)|RPA>_f \quad final$$

Phonon operators

$$Q^{\dagger}_{m_i}(JM) = \sum_{pn}\left[X^{m_i}_{(pn)J} A^{\dagger}(pnJM) - Y^{m_i}_{(pn)J}\widetilde{A}(pnJM)\right]$$

$$Q^{\dagger}_{m_f}(JM) = \sum_{pn}\left[X^{m_f}_{(pn)J} B^{\dagger}(pnJM) - Y^{m_f}_{(pn)J}\widetilde{B}(pnJM)\right]$$

$$\begin{pmatrix} a^{\dagger}_{pm_\tau} \\ a_{\widetilde{pm_\tau}} \end{pmatrix} = \begin{pmatrix} \tilde{u}_\tau & \tilde{v}_\tau \\ -\tilde{v}_\tau & \tilde{u}_\tau \end{pmatrix} \begin{pmatrix} b^{\dagger}_{pm_\tau} \\ b_{\widetilde{pm_\tau}} \end{pmatrix}, \quad \begin{pmatrix} b^{\dagger}_{pm_\tau} \\ b_{\widetilde{pm_\tau}} \end{pmatrix} = \begin{pmatrix} \tilde{u}_\tau & -\tilde{v}_\tau \\ \tilde{v}_\tau & \tilde{u}_\tau \end{pmatrix} \begin{pmatrix} a^{\dagger}_{pm_\tau} \\ a_{\widetilde{pm_\tau}} \end{pmatrix}$$

Unitary transformation between quasiparticles

$$\tilde{u}_\tau = u^i{}_\tau u^f{}_\tau + v^i{}_\tau v^f{}_\tau \qquad \tilde{v}_\tau = u^i{}_\tau v^f{}_\tau - v^i{}_\tau u^f{}_\tau,$$

$$\tilde{u}^2_\tau + \tilde{v}^2_\tau = 1$$

By neglecting scattering terms

$$Q^{\dagger}_{m_i}(JM) = \sum_{m_f}\left[a_{m_i m_f} Q^{\dagger}_{m_f}(JM) + b_{m_i m_f}\tilde{Q}_{m_f}(JM)\right]$$

$$Q^{\dagger}_{m_f}(JM) = \sum_{m_i}\left[a_{m_f m_i} Q^{\dagger}_{m_i}(JM) + b_{m_f m_i}\tilde{Q}_{m_i}(JM)\right]$$

With help of above expresion

$$\begin{aligned} < JM, m_f | JM, m_i > &= {}_f<RPA|Q_{m_f}(JM) Q^{\dagger}_{m_i}(JM)|RPA>_i \\ &= \sum_{m'_f}\left[a_{m_i m'_f}\ \delta_{m_i m'_f}\ {}_f<RPA|RPA>_i \right. \\ &\quad \left. + b_{m_i m'_f f}\ <RPA|Q_{m_f}(JM)\ \tilde{Q}_{m_f}(JM)|RPA>_i\right] \end{aligned}$$

Frequently considered overlap factor of two sets of states

$$\begin{aligned} < JM, m_f | JM, m_i > &\approx a_{m_i m_f} \\ &\approx \sum_{pn}\left[X^{m_i}_{(pn)J} X^{m_f}_{(pn)J} - Y^{m_i}_{(pn)J} Y^{m_f}_{(pn)J}\right] \end{aligned}$$

second term neglected

$$< JM, m_f | JM, m_i >= \left[X^{m_i}_{(pn)J} X^{m_f}_{(pn)J} - Y^{m_i}_{(pn)J} Y^{m_f}_{(pn)J} \right] \tilde{u}_p \tilde{u}_{n_f} <BCS|BCS>_i$$

BCS overlap of intial and final states

$$\begin{aligned} {}_f<RPA|RPA>_i &\approx {}_f<BCS|BCS>_i \\ &= \Pi_p (u^f{}_p u^i{}_p + v^f{}_p v^i{}_p)\ \Pi_n (u^f{}_n u^i{}_n + v^f{}_n v^i{}_n) \end{aligned}$$

For spherical nuclei BCS overlap about 0.8
For deformed nuclei might be smaller
=> Can not be neglected!

Instead of Conclusions

We are at
the beginning
of the Road…

The future of neutrino physics is bright.

remaining

The World Neutrino Experimental Program

- **Parameter Measurement**
  - θ_{23} Octant (>, < 45°)
  - Mass hierarchy
  - Mass scale
  - CP violation δ
  - Dirac or Majorana?
  - More accuracy for θ_{12}, θ_{23}, θ_{13}, Δm^2_{32}, Δm^2_{21}

Questions with answers

- **Paradigm testing**
  - Sterile neutrinos?
  - Non standard Interactions?
  - Lorentz violation?
  - CPT violation?
  - Non-Unitarity of PMNS matrix?

Questions which might or might not have answers

Like most people,
physicists enjoy a good mystery.

When you start investigating a mystery
you rarely know where it is going

Mathematics is Egyptian

Neutrino physics is Babylonian

The truth is covered in ν–experiments.

Thanks to neutrinos we understand Sun, Supernova, Earth (nuclear reactions)