Application of Normal Forms Analysis to SuperB

D. Quatraro
Thanks to G. Turchetti, A. Franchi, M. Giovannozzi

CERN, INFN Bologna

19 February 2009

Outline

The equation of motion

4-D phase space: $\mathbf{x} = (x, p_x, y, p_y)$. The Hamiltonian function reads

$$\mathcal{H}(\mathbf{x}; s) = \underbrace{\frac{{p_x}^2 + {p_y}^2}{2} + \left(\frac{1}{\rho(s)^2} - k_1(s)\right)\frac{x^2}{2} + k_1(s)\frac{y^2}{2}}_{\text{Linear Motion}}$$

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Detuning with amplitude

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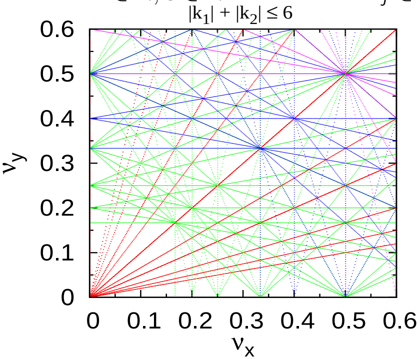

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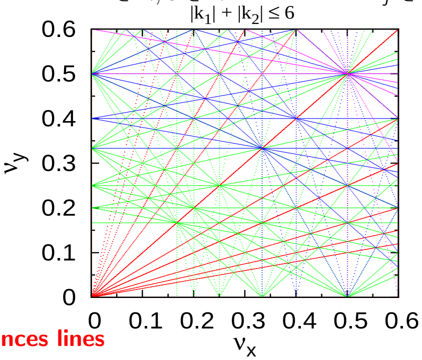

Intricate net of resonances lines

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we then gain the information to fulfil the resonant condition

$$\begin{cases} k_1\Omega_x\left(j_x, j_y\right) + k_2\Omega_y\left(j_x, j_y\right) = 2\pi l \\ |k_1| + |k_2| \leq N \end{cases}$$

with N is the order of the resonance and $l \in \mathbb{N}$

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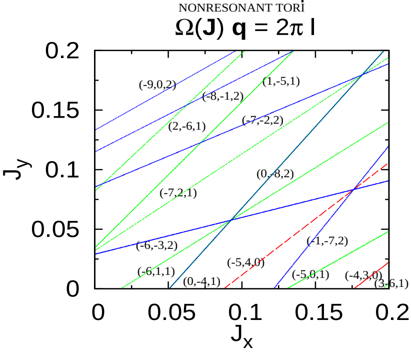

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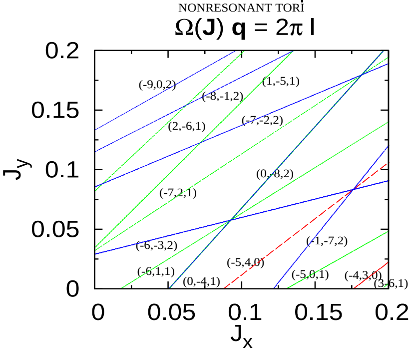

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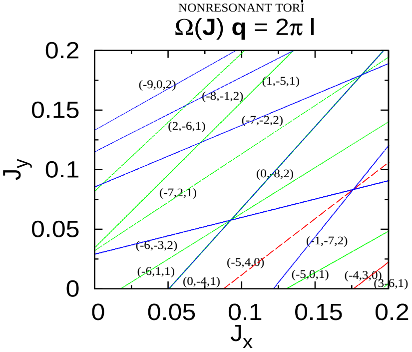

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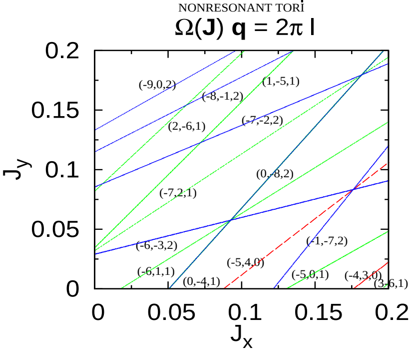

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$$\begin{cases} \Omega_x = \omega_x + 2\, h_{20}\, j_1 + h_{11}\, j_2 \\ \Omega_y = \omega_y + h_{11}\, j_2 + 2\, h_{02}\, j_2 \end{cases} \quad (3)$$

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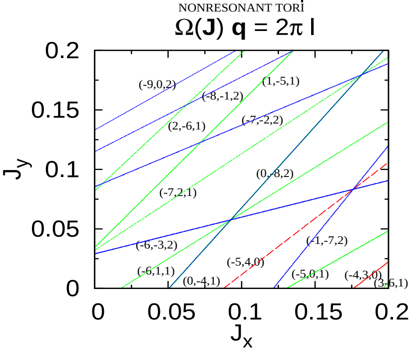

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Resonance conditions as function of emittances $\rightarrow \epsilon_i = 2\, j_i$

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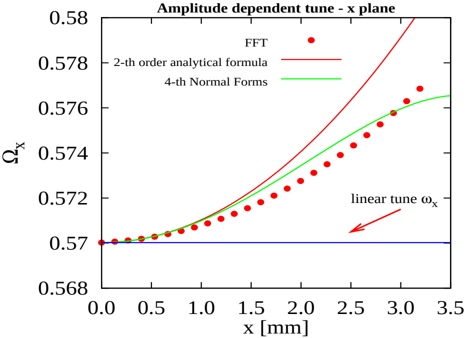

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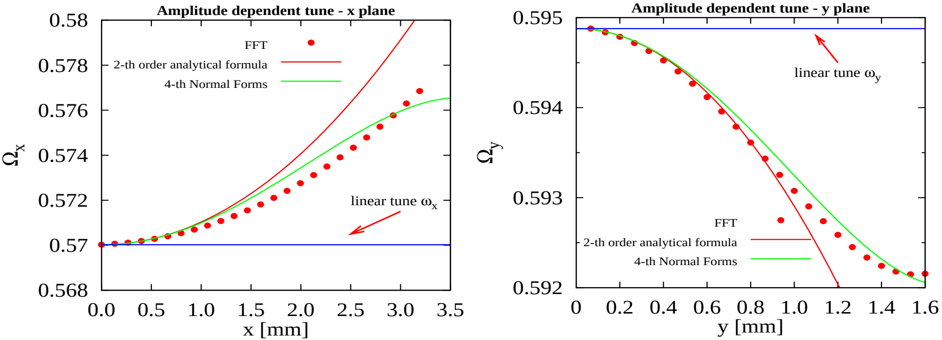

- FFT Routine courtesy of A. Bazzani (Bologna University)

Analytical formula: 2-th order Giovannozzi, Quatraro, Turchetti PRLSTAB (in press)

numerical 4-th order MAD-X, PTC

Detuning with amplitude II

Observation point: $s = 0$ with $\beta_x \simeq 21.7$ m, $\beta_y \simeq 13.4$ m ,
$\sigma_x \simeq 180\ \mu\mathrm{m}, \sigma_y \simeq 7.3\ \mu\mathrm{m}$ (CDR: http://arxiv.org/abs/0709.0451)and
$\Delta p/p_0 = 0$

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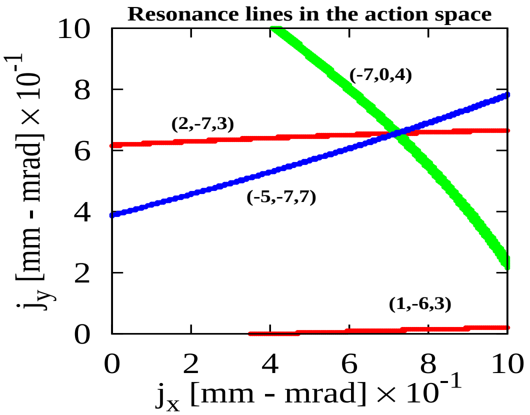

the lines of the (k_1, k_2, l)'s resonance in the action space (j_x, j_y) up to 12-th order

Outline

Dynamic aperture [1]

Initial condition in the sphere $\mathcal{B}(0, R)$, $R = 4.5$ mm for 1024 turns

$$\begin{cases} x = \rho cos(\varphi) \\ y = \rho sin(\varphi) \end{cases} \quad \text{with} \quad \rho \in [0 : R], \varphi \in [0 : \pi/2]$$

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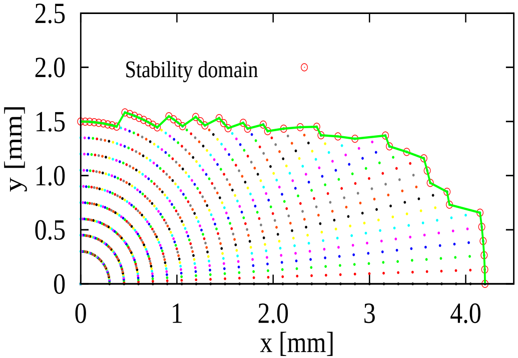

$\mathcal{DA}_x \simeq 0.004$ mm $\approx 23\ \sigma_x$
$\mathcal{DA}_y \simeq 0.0016$ mm $\approx 205\ \sigma_y$

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Frequency analysis

In the space (ω_x, ω_y) the tune footprint in the stable phase space
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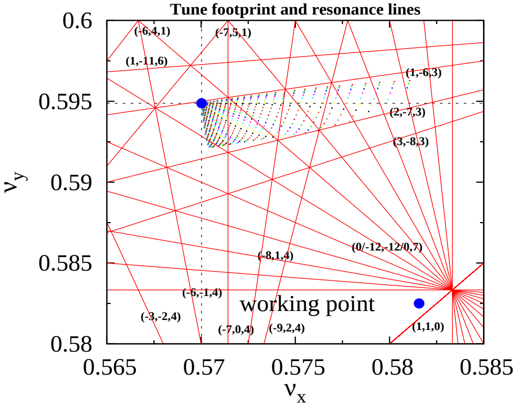

as we had already seen in the action space...

The non linear budget...

```
PCRAB: SEXTUPOLE, L:=LS2/4.0, K2:=KCRAB, APERTURE:=RS1 ;
MCRAB: SEXTUPOLE, L:=LS2/4.0, K2:=-KCRAB, APERTURE:=RS1 ;
SFM9: SEXTUPOLE, L:=LS1/2.0, K2:=KSFM9, APERTURE:=RS1 ;
SFX4: SEXTUPOLE, L:=LS1/2.0, K2:=KSFX0, APERTURE:=RS1 ;
SFX0: SEXTUPOLE, L:=LS1/2.0, K2:=KSFX0, APERTURE:=RS1 ;
SFM7: SEXTUPOLE, L:=LS1/2.0, K2:=KSFM7, APERTURE:=RS1 ;
SDY4: SEXTUPOLE, L:=LS2/4.0, K2:=KSDY0, APERTURE:=RS1 ;
SDY0: SEXTUPOLE, L:=LS2/4.0, K2:=KSDY0, APERTURE:=RS1 ;
SDM2: SEXTUPOLE, L:=LS1/2.0, K2:=KSDM2, APERTURE:=RS1 ;
SF1: SEXTUPOLE, L:=2.0*LSXT, K2:=KSF1 ;
SF2: SEXTUPOLE, L:=LSXT, K2:=KSF2 ;
SD1: SEXTUPOLE, L:=2.0*LSXT, K2:=KSD1 ;
SD2: SEXTUPOLE, L:=2.0*LSXT, K2:=KSD2 ;

OCTX0: OCTUPOLE, L:=LS1, K3:=0.0 ;
OCTY0: OCTUPOLE, L:=LS1, K3:=-20.0 ;
```

they were all on!! apart `OCTX0`...

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