A Probe into the Schrodinger-Newton Equation

Mohammad Bahrami

Physics Department, University of Trieste, Trieste, Italy

To “quantize” or “not quantize” the gravity: that is the question

Despite alluring temptation of quantizing gravity, the ultimate necessity of this quantization can only be found on experience.

- C. Moller. Les Theories Relativistes de la Gravitation, “Colloques Internationaux CNRX 91” A. Lichnerowicz and M-A Tonnelat (eds.) (Paris: CNRS) (1962).
- L. Rosenfeld, Nucl. Phys. 40 ,353 (1963).
- T. W. B. Kibble, Is a Semi-classical Theory of Gravity Viable? In: “Quantum Gravity 2; A Second Oxford Symposium”, Edited by C. J. Isham, R. Penrose and D. W. Sciama (Clarendon Press, Oxfrd, England, 1981).
- F. J. Dyson, Int. J. Modern Phys. A 28, 1330041 (2013).
- S. L. Adler, arXiv:1401.0353v3 (2014).

Semi-classical gravity: a possible combination of classical gravity with quantum matter

- The source of gravitation field is determined by the renormalized expectation value of the energy-momentum tensor (Moller-Rosenfeld Equation or Semi-Classical Einstein Equation):

$$G_{\mu\nu}(g) = 8\pi G \langle \psi(\hat{\phi}) | \hat{T}_{\mu\nu}(\hat{\phi}, g) | \psi(\hat{\phi}) \rangle$$

- The right-hand side is defined in “Quantum Field Theory in Curved Space-Time”.

Semi-classical gravity is in curved spacetime

- Semi-classical gravity suffers from all problems that we have in QFT in curved spacetime, e.g., the renormalization.

R. M. Wald, "Quantum field theory in curved spacetime and black hole thermodynamics" (Univ. Chicago Press, 1994), pp 87-100.

- Nevertheless, a consistent semi-classical theory of gravity is *viable*.
- A straightforward approach to investigate the implications of semi-classical gravity with the covariant perturbation method (linearized GR)

Weak-field limit of classical gravity

- In the weak-field limit, one can represent classical GR by a linearized theory of gravity.
- In the linearied theory of gravity, the definition of metric is:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

with

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

which is a Lorenzt invariant, massless, spin-2 field in the flat spacetime.

$h \equiv h_\mu{}^\mu = \eta_{\mu\nu}h^{\mu\nu}$ is the trace of $h_{\mu\nu}$,

Weak-field limit of classical gravity

- The field equations are

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\,\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial^\alpha\partial_\nu\bar{h}_{\mu\alpha} - \partial^\alpha\partial_\mu\bar{h}_{\nu\alpha} = -16\pi G\,T_{\mu\nu}$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the matter fields.

- The first term on the left-hand side is the usual flat-space d'Alembertian and the other terms serve merely to keep the equations "gauge-invariant". Without loss of generality, one can impose the Lorentz gauge:

$$\partial^\nu\bar{h}_{\mu\nu} = 0$$

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

Weak-field limit of classical gravity

Accordingly, the field equations become:

$$\Box \bar{h}_{\mu\nu} = -16\pi G \, T_{\mu\nu}$$

which implies that $T^{\mu\nu}$ obeys the flat-space conservation law:

$$\partial^\mu T_{\mu\nu} = 0.$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\partial^\nu \bar{h}_{\mu\nu} = 0$$

Weak-field limit of semi-classical gravity

$$\partial^\nu \bar{h}_{\mu\nu} = 0$$

$$\Box \bar{h}_{\mu\nu} = -16\pi G \left\langle \psi(\hat{\phi}) \left| T_{\mu\nu}(\hat{\phi}) \right| \psi(\hat{\phi}) \right\rangle$$

The general solution is given by:

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4G \int d\mathbf{y} \, \frac{\left\langle \hat{T}_{\mu\nu}(t - |\mathbf{x}-\mathbf{y}|, \mathbf{x}) \right\rangle}{|\mathbf{x}-\mathbf{y}|}$$

which is a retarded integral form.

Weak-field and nonrelativistic limit of semi-classical gravity

For a Newtonian case where

$$\left\langle \hat{T}_{00} \right\rangle \gg \left| \left\langle \hat{T}_{0j} \right\rangle \right| \text{ and } \left\langle \hat{T}_{00} \right\rangle \gg \left| \left\langle \hat{T}_{jk} \right\rangle \right| \text{ with } \quad j,k = 1,2,3,$$

and velocities slow enough to neglect the retardation, then one gets:

$$\bar{h}_{00} \approx -4G\Phi; \quad \bar{h}_{0j} = \bar{h}_{jk} = 0,$$

where Φ is Newtonian potential:

$$\Phi(t,\mathbf{x}) = -G \int d\mathbf{y} \frac{\left\langle \hat{T}_{00}(t,\mathbf{y}) \right\rangle}{|\mathbf{x}-\mathbf{y}|}.$$

Weak-field and nonrelativistic limit of semi-classical gravity

- In this limit, the semi-classical gravity contributes a new term into the Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi_t\rangle = \left(\hat{H} - G\int d\mathbf{y}\,\frac{\langle\psi_t)|\,\hat{\varrho}\,|\psi_t)\rangle}{|\mathbf{x}-\mathbf{y}|}\right)|\psi_t\rangle\,,$$

with $\hat{\varrho}$ the density operator:

$$\hat{\varrho} = \sum_{j=1}^{N} m_j\,\delta(\mathbf{x}-\hat{\mathbf{x}}_j)$$

This is Schrodinger-Newton equation, first proposed by L. Diosi:

L.Diosi,Phys.Lett. 105A, 199-202 (1984)

Schrodinger-Newton equation for the center-of-mass of a many-body system

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \left(-\frac{\hbar^2}{2M}\nabla^2_{\mathbf{x}} + V(\mathbf{x}) + \int d\mathbf{x}'\,|\psi(\mathbf{x}',t)|^2\,U(\mathbf{x}-\mathbf{x}')\right)\psi(\mathbf{x},t)$$

with $\mathbf{x}$ the center-of-mass position, $V(\mathbf{x})$ the external potential on the center-of-mass, and $U(\mathbf{x}-\mathbf{x}')$ the Newtonian potential of two bulky systems at $\mathbf{x}$ and $\mathbf{x}'$:

$$U(\mathbf{x}-\mathbf{x}') = -G\int \frac{d\mathbf{r}\,d\mathbf{r}'\,\varrho(\mathbf{r})\,\varrho(\mathbf{r}')}{|\mathbf{x}-\mathbf{x}'+\mathbf{r}-\mathbf{r}'|}$$

where $\varrho(\mathbf{r})$ is the internal density of the system

Giulini & Grossardt arXiv:1404.0624 (2014)

Schrodinger Newton (SN) equation is a deterministic nonlinear equation where Newtonian self-gravity is introduced as an attractive nonlinearity.

It is mathematically equivalent to the motion of solitons in nonlinear nonlocal media with a ***focusing*** nonlinearity and the response function *U(x)*.

Effect of nonlinearity in SN equation

The nonlinear Newtonian term in SN equation implies two effects:

- **self-focusing**
- **gravitational lensing**

We elaborate these effects with an example. Consider a system prepared in a superposition of two wavepackets at different locations:

$$\psi(z) = \frac{1}{\sqrt{2}}(\psi_1(z) + \psi_2(z))$$

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

Effect of nonlinearity in SN equation

For the time scales where wavepackets remain separated, the SN equation reads approximately:

$$i\hbar\frac{\partial}{\partial t}\psi_1(z,t) \approx \left(\hat{H} - \frac{GM^2}{2}\int dz'\,\frac{|\psi_1(z',t)|^2}{|z-z'|} - \frac{GM^2}{2}\int dz'\,\frac{|\psi_2(z',t)|^2}{|z-z'|}\right)\psi_1(z,t),$$

$$i\hbar\frac{\partial}{\partial t}\psi_2(z,t) \approx \left(\hat{H} - \frac{GM^2}{2}\int dz'\,\frac{|\psi_2(z',t)|^2}{|z-z'|} - \frac{GM^2}{2}\int dz'\,\frac{|\psi_1(z',t)|^2}{|z-z'|}\right)\psi_2(z,t),$$

Self-focusing

Gravitational lensing

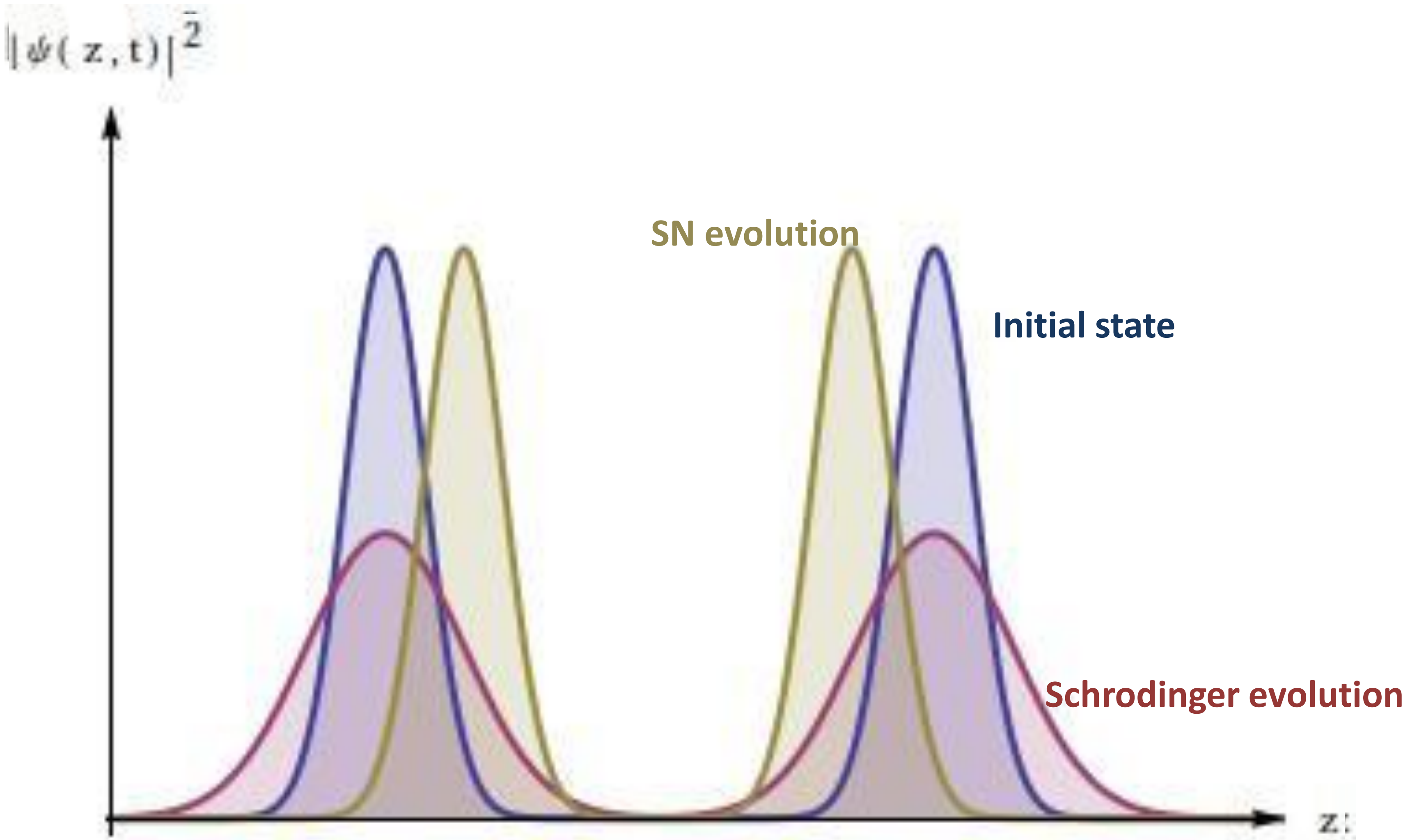
Mohammad Bahrami, Department of Physics, University of Trieste, Italy

Effect of nonlinearity in SN equation

- By **self-focusing** we mean the effect inhibiting the free-spreading of wavepackets, thus resulting in a self-focusing (or say, **shrinking**) of wavepackets at their locations. The self-focusing results to smaller width compared the one given by Schrodinger equation

- By **gravitational lensing** we mean the **attractive force** between two wavepackets, resulting wavepackets move toward each other. The gravitational lensing results to smaller separation between the localtions of wavepackets.

Effect of nonlinearity in SN equation



Mohammad Bahrami, Department of Physics, University of Trieste, Italy

Nonlinear term in SN equation

$$U(\mathbf{x}-\mathbf{x}') = -G \int \frac{d\mathbf{r}\, d\mathbf{r}'\, \varrho(\mathbf{r})\, \varrho(\mathbf{r}')}{|\mathbf{x}-\mathbf{x}'+\mathbf{r}-\mathbf{r}'|} = -\frac{G}{2\pi^2} \int d\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \frac{|\tilde{\varrho}(\mathbf{k})|^2}{k^2},$$

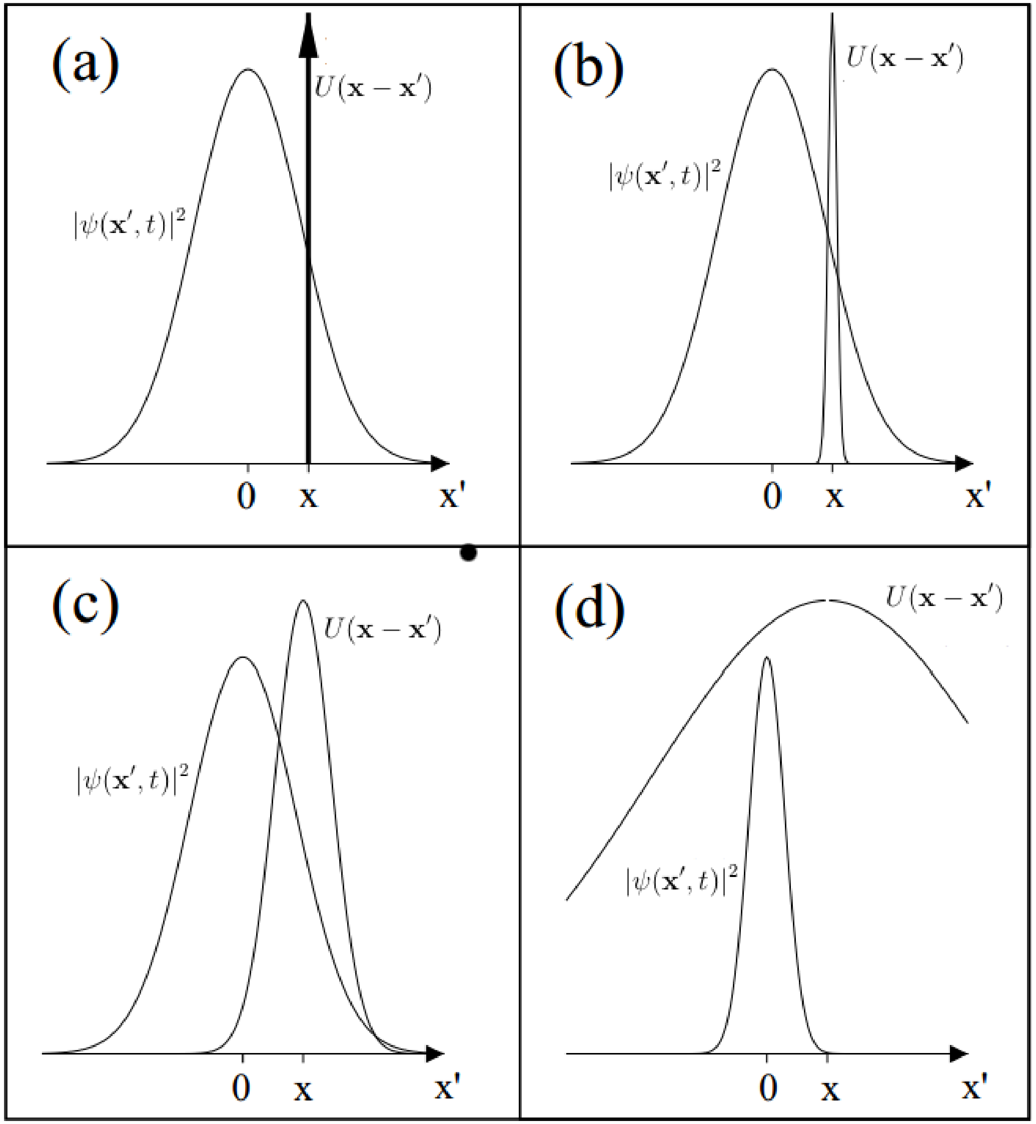
where $\varrho(\mathbf{r})$ is the internal density of the system and $\tilde{\varrho}(\mathbf{k}) = \int d\mathbf{r}\, \varrho(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}$. For a system of size R and mass M, one gets $\tilde{\varrho}(\mathbf{k}) \approx M e^{-k^2 R^2/8}$, then the Newtonian potential $U(\mathbf{x}-\mathbf{x}')$ can be approximated as follows

$$U(\mathbf{x}-\mathbf{x}') \approx -GM^2 \frac{\operatorname{erf}(|\mathbf{x}-\mathbf{x}'|/R)}{|\mathbf{x}-\mathbf{x}'|}, \qquad \text{with “erf” the error function.}$$

- In fact, *U(**x**-**x’**)* acts like a distribution with the width *R* around the position ***x***

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

The self-gravity term *U*(x-x')



SN equation in two limits

- Accordingly, |**x**-**x**'| is effectively the delocalization of the wave function, e.g., in the case of an individual wavepacket it is the width of the wavepacket, and in the case of a superposition of two separated wavepackets, it is the separation between the centers of wavepackets.

SN equation for Matter-Wave Experiments

- For example, in matter-wave interferemetry experiments, the size *R* is smaller than the localization distance

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) \approx \left(-\frac{\hbar^2}{2M}\nabla^2_{\mathbf{x}} + V(\mathbf{x}) - GM^2\int \frac{d\mathbf{x}'\,|\psi(\mathbf{x}',t)|^2}{|\mathbf{x}-\mathbf{x}'|}\right)\psi(\mathbf{x},t).$$

- Here, SN equation is like one-particle SN equation with mass *M*

SN equation for Optomechanical Systems

- For example, in optomechanical systems, the size *R* is larger than the localization distance

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) \approx \Big(-\frac{\hbar^2}{2M}\nabla^2_{\mathbf{x}} + V(\mathbf{x}) + \frac{1}{2}\,M\,\omega_G^2\,x^2 \\ - M\,\omega_G^2\,\Big(\mathbf{x}\cdot\langle\mathbf{x}\rangle_t - \frac{1}{2}\langle\mathbf{x}^2\rangle_t\Big) + \text{const.}\Big)\,\psi(\mathbf{x},t),$$

with $\omega_G = \sqrt{GM/R^3}$ and $\langle\mathbf{x}\rangle_t = \langle\psi(\mathbf{x},t)|\hat{\mathbf{x}}|\psi(\mathbf{x},t)\rangle$

This is similar to equation 2 obtained in
H. Yang et. al, Phys. Rev. Lett. 110, 170401, (2013)

The limit where the Newtonian nonlinearity is dominant compared to Schrodinger term

$$\int d\mathbf{x}\, \psi^*(\mathbf{x},t) \left(-\frac{\hbar^2}{2M}\nabla^2_{\mathbf{x}} + V(\mathbf{x})\right) \psi(\mathbf{x},t) \sim \int d\mathbf{x}\, d\mathbf{x}'\, |\psi(\mathbf{x}',t)|^2\, |\psi(\mathbf{x},t)|^2\, |U(\mathbf{x}-\mathbf{x}')|.$$

The term $|U(\mathbf{x}-\mathbf{x}')|$ is bounded from above:

$$|U(\mathbf{x}-\mathbf{x}')| \leq \frac{2GM^2}{R\sqrt{\pi}}$$

Introducing this bound into above equation, we obtain:

$$\int d\mathbf{x}\, \psi^*(\mathbf{x},t) \left(-\frac{\hbar^2}{2M}\nabla^2_{\mathbf{x}} + V(\mathbf{x})\right) \psi(\mathbf{x},t) \sim \frac{GM^2}{R},$$

which means that for the system where the average of Hamiltonian is comparable with GM^2/R then the nonlinear Newtonian effects become observable.

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

The limit where the Newtonian nonlinearity is dominant compared to Schrodinger term

- Accordingly, for a wavepackets of width σ (or for a superposition of two wavepackets of widths σ and separation d where d>>σ), one finds:

$$\frac{G\,M^2}{R} \sim \frac{\hbar^2}{M\,\sigma^2},$$

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

Estimations for matter-wave experiments

- Let us consider the state-of-the-art of matter-wave interferometery experiments:

$$\sigma \sim 50 - 500\,\mathrm{nm} \quad , R \sim 1 - 10\,\mathrm{nm}\ \text{with mass}\ M \sim 10^4 - 10^5\,\mathrm{amu}$$

- Then one finds:

$$\frac{G\,M^2}{R} \sim 10^{-13} - 10^{-11}\,\mathrm{Hz}\ \text{and}\ \frac{\hbar^2}{M\,\sigma^2} \sim 1 - 10^4\,\mathrm{Hz}$$

meaning the SN effects (if any) are negligible for these ranges of variables.

- To observe the nolinear effect as predicted by SN equation, one needs:

$$M \sim 10^9\mathrm{amu},\ \sigma \sim 500\mathrm{nm}\ \text{and}\ R \sim 10\mathrm{nm},$$

Estimations for optomechanical experiments

- Let us consider the case where σ is the zero-width-point of a harmonic oscillator, as in optomechanics. Then, one easily get:

$$GM^2/R \sim \hbar\omega.$$

For $R = 50\,\mathrm{nm}$ and $\varrho_0 = 2 \times 10^3\,\mathrm{kg/m^3}$ (i.e. $M \approx 10^9\,\mathrm{amu}$) with $\omega/2\pi = 5 \times 10^5\,\mathrm{s^{-1}}$, one gets:

$$GM^2/R \sim 10^{-5}\mathrm{Hz}$$

- To reach the regime for observation of SN effects, one needs

***M* ~ *10^15^* amu**

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

Effect of nonlinearity in SN equation

Accordingly, the nonlinearity practically induces:

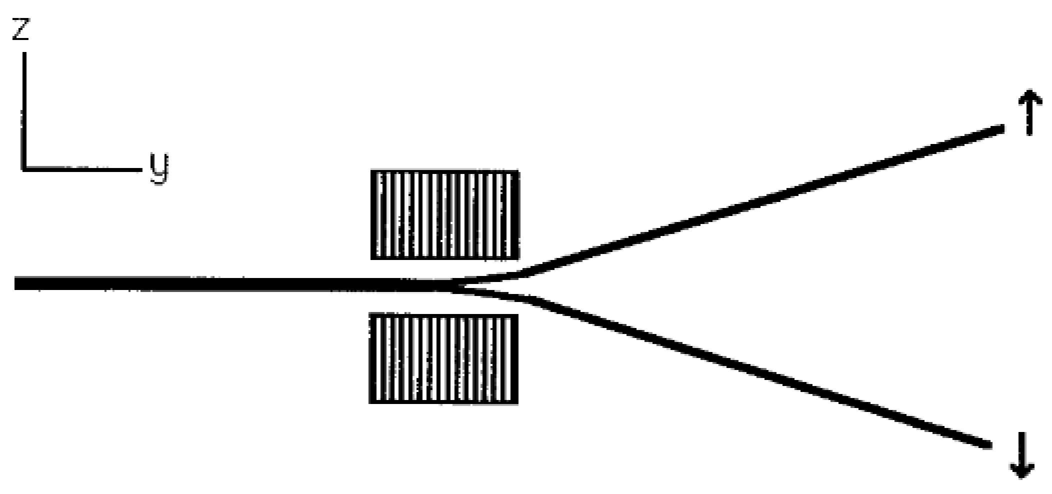
- Self-focusing -> Shrinking
- Gravitational lensing -> attraction

Anyhow, we know that the deterministic nonlinear modification of Schrodinger equation is fundamentally problematic.

N. Gisin, Hel. Phys. Acta 62, 363 (1989); Phys. Lett. A 143, 1 (1990).
N. Gisin, and M. Rigo, J. Phys. A 28, 7375 (1995).
J. Polcinski, Phys. Rev. Lett. 66, 397 (1991).

SN equation and Stern-Gerlach Experiment

Consider the case of a spin-1/2 particle in a Stern-Gerlach apparatus with a magnetic force in z-direction here the position of particle along z-axis is finally observed at the detector



SN equation and Stern-Gerlach Experiment

For the initial state $\phi(z)|\pm\rangle$

the total state $\Psi(z,t) = \psi_{\pm}(z,t)|\pm\rangle$ evolves as:

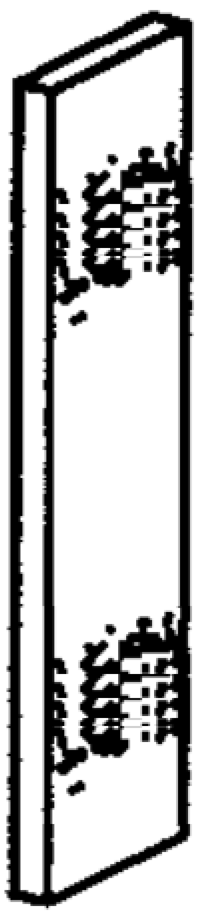
$$i\hbar\frac{\partial}{\partial t}\psi_{\pm}(z,t) = \left(-\frac{\hbar^2}{2M}\nabla_z^2 - GM^2\int dz'\,\frac{|\psi_{\pm}(z',t)|^2}{|z-z'|}\right)\psi_{\pm}(z,t).$$

with the initial conditions: $\psi_{\pm}(z,0) = e^{\pm ik_z z}\phi(z)$ where k_z is the momentum induced by the Stern-Gerlach field and $\phi(z)$ is the initial wavepacket.

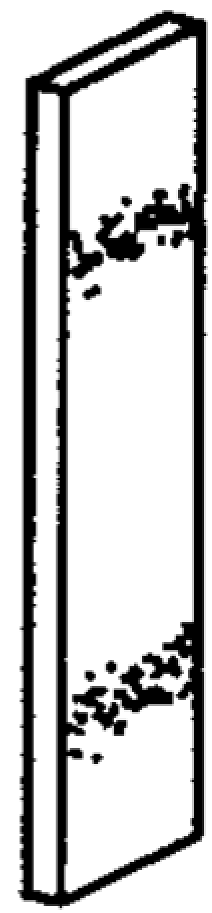
<u>Here the nonlinear Newtonian Term acts as a self-focusing</u>

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

SN equation and Stern-Gerlach Experiment



SN nonlinearity acts as
self-focusing (shrinking)



Mohammad Bahrami, Department of Physics, University of Trieste, Italy

SN equation and Stern-Gerlach Experiment

Now consider the case with initial states $|x_{\pm}\rangle$. Then the final state $\Psi(z,t) = \frac{1}{\sqrt{2}}\left(\psi_+(z,t)|+\rangle + \psi_-(z,t)|-\rangle\right)$ evolves as:

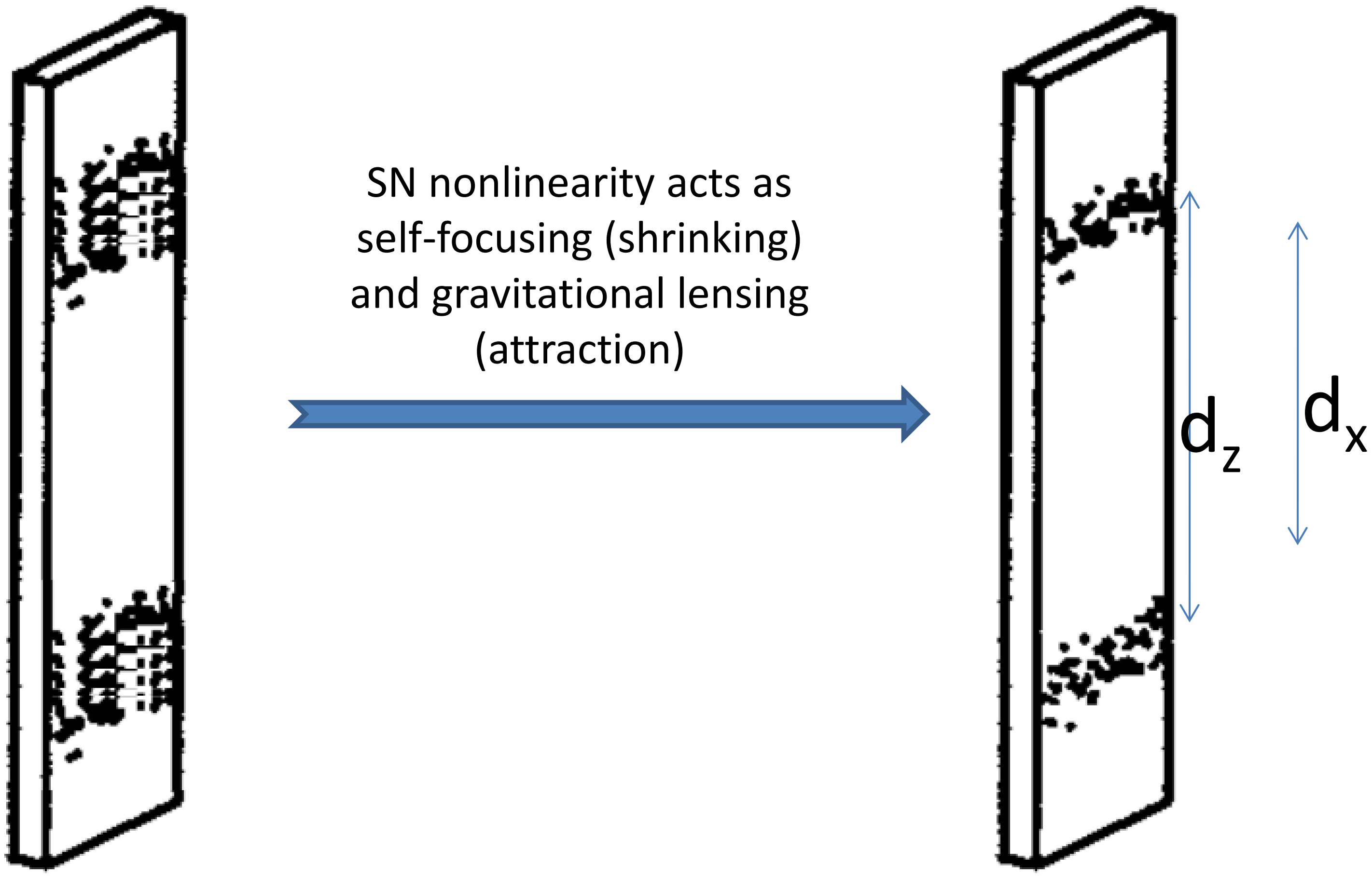
$$i\hbar\frac{\partial}{\partial t}\psi_+(z,t) = \left(-\frac{\hbar^2}{2M}\nabla_z^2 - \frac{GM^2}{2}\int dz'\,\frac{|\psi_+(z',t)|^2}{|z-z'|} - \frac{GM^2}{2}\int dz'\,\frac{|\psi_-(z',t)|^2}{|z-z'|}\right)\psi_+(z,t)$$

$$i\hbar\frac{\partial}{\partial t}\psi_-(z,t) = \left(-\frac{\hbar^2}{2M}\nabla_z^2 - \frac{GM^2}{2}\int dz'\,\frac{|\psi_-(z',t)|^2}{|z-z'|} - \frac{GM^2}{2}\int dz'\,\frac{|\psi_+(z',t)|^2}{|z-z'|}\right)\psi_-(z,t)$$

- Like before, the 2nd terms on the right-side of above equations are the self-focusing force while the 3rd erms correspond to focusing forces (attraction) between two wavepackets

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

SN equation and Stern-Gerlach Experiment



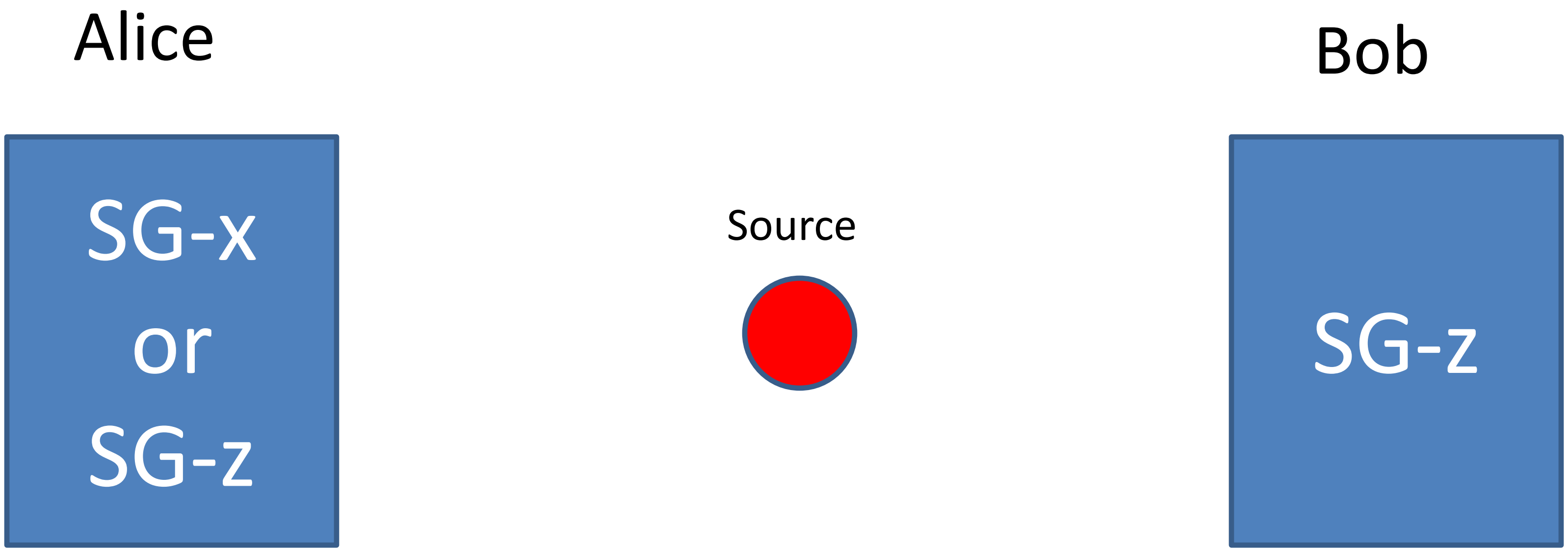
Mohammad Bahrami, Department of Physics, University of Trieste, Italy

SN equation and Stern-Gerlach Experiment

- The aforementioned gravitational lensing can, at least in principle, be exploited experimentally to distinguish standard quantum theory from SN equation.
- Now we shall show that this effect can be used in order to send a faster-than-light signalling because of the nonlocal nature of the SN equation.

Mohammad Bahrami, Department of Physics, University of Trieste, Italy

Superluminality in SN equation



$$\Psi(z_A, z_B, t=0) = \frac{1}{\sqrt{2}}\left(|+\rangle_A|-\rangle_B - |-\rangle_A|+\rangle_B\right)\phi_1(z_A)\phi_2(z_B)$$

Superluminality in SN equation

- If Alice measures the spin along the z axis, then Bob's particle is prepared in one the state |±> , thus finally Bob will get spots separated by d_z.
- If Alice measures the spin along x-axis, Bob's particle will be prepared in states |x±> , meaning Bob shall will get spots separated by d_x
- With this scenario, Bob can understand which measurement Alice performed by only looking at the position where he observes the particle.
- Since the distance between Alice and Bob can be arbitrary large, this implies faster than light signaling.

Conclusion

- SN equation is the weak-field non-relativistic limit of semi-classical gravity.
- The Newotonian self-interaction is introduced as an attractive nonlinearity, induces the self-focusing (shrinking) and gravitational lensing (attraction).
- However, SN equation gives rise to superluminal effects, as expected from any deterministic nonlinear Schrodinger equation.

Thanks (SEPAAS)