Novel Structures in Scattering Amplitudes

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Brandhuber, Spence, GT	hep-th/0407214 and follow ups
Brandhuber, Heslop, GT	0707.1153 [hep-th]
Brandhuber, Heslop, Nasti, Spence, GT	0805.2763 [hep-th]
Brandhuber, Heslop, GT	0807.4097 [hep-th]

Strings & Strong Interactions, LNF, September 2008

Outline

- Amplitudes vs Feynman diagrams
- New structures in scattering amplitudes
  - MHV diagrams (and twistor space)
  - on-shell recursion relations
- More structure in planar N=4 SYM

I. iterative relations in the higher-loop expansion of MHV amplitudes

II. new duality: amplitudes/Wilson loops

III. new symmetry of the planar theory: dual superconformal symmetry

Motivations

- Scattering amplitudes in gauge theory and gravity are surprisingly simple quantities
  - geometry in Twistor Space (Witten)
  - recursive structures in the perturbative S-matrix of YM and GR
  - much of the structure emerges by studying singularities of S-matrix
  - uncovering hidden structures leads to new techniques for calculating
- Simplicity hidden by Feynman diagrams
  - high-multiplicity processes
  - loops

☞ Number of Feynman diagrams for $gg \longrightarrow n\,g$ scattering:

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

(tree level)

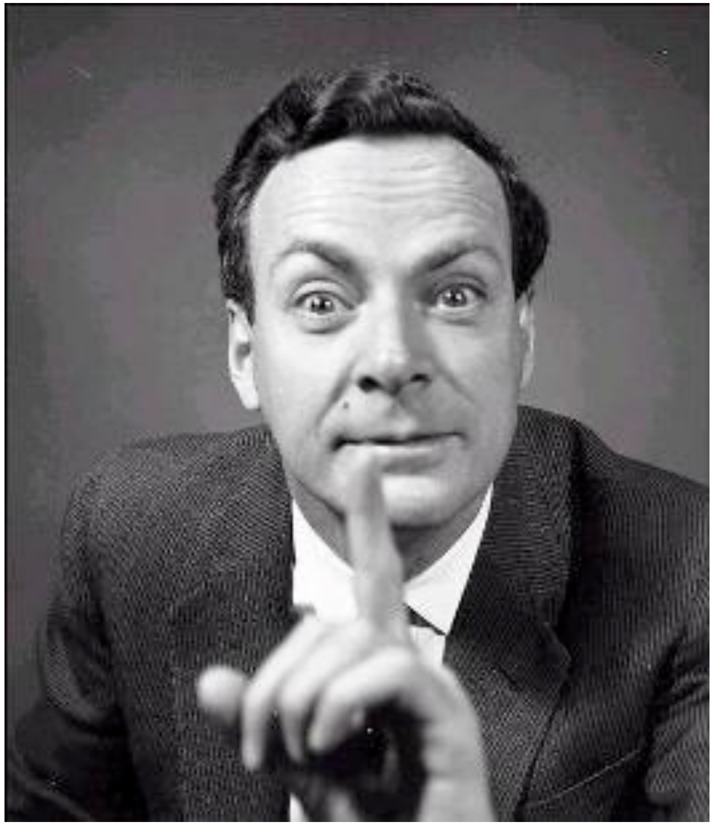
☞ Result is: $\mathcal{A}(1^{\pm}, 2^{+}, \ldots n^{+}) = 0$ at tree level

$$\mathcal{A}_{\mathrm{MHV}}(1^{+} \ldots i^{-} \ldots j^{-} \ldots n^{+}) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

(Parke & Taylor)

Large numbers of Feynman diagrams combine into mysteriously simple expressions

What's "wrong" with Feynman diagrams ?



Nothing! but ...

- Diagrams are not separately gauge invariant
  - vertices and propagators are off shell, leads to vast cancellations
- Huge redundancy from field redefinitions
- Solution: on-shell methods
  - calculate on-shell amplitudes rather than off-shell Green's functions
  - ♻ amplitudes with fewer legs/fewer loops ♻
- Unitarity-based & twistor-inspired methods:
  - gauge-invariant, on-shell data at each intermediate step of calculation
  - also in non-supersymmetric theories

Even more simplicity

- Amplitudes in N=4 SYM
  - All one-loop amplitudes expressed in terms of box functions (Bern, Dixon, Dunbar, Kosower)
  - Iterative structures in splitting amplitudes and in planar MHV amplitudes (Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)
    - planar: leading in $1/N$
  - More structure: N=4 MHV amplitudes from a polygonal Wilson loop calculation
    - strong coupling (Alday & Maldacena), after fermionic T-duality (Berkovits & Maldacena; Beisert, Ricci, Tseytlin, Wolf)
    - weak coupling (Drummond, Korchemsky, Sokatchev + Henn; Brandhuber, Heslop, GT)
  - Dual superconformal symmetry (Drummond, Henn, Korchemsky, Sokatchev)

- Amplitudes in N=8 supergravity increasingly similar to those of N=4 SYM
  - Absence of triangle and bubble subgraphs in amplitudes ("no-triangle hypothesis") (Bern, Dixon, Perelstein, Rozowsky; Bern, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove)
  - N=8 conjectured to be perturbatively finite (Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Chalmers; Bern, Dixon, Roiban; Green, Russo, Vanhove; Bern, Carrasco, Dixon, Johansson, Kosower, Roiban)
  - Iterative structures in the IR divergences (beyond those of Weinberg 1965) (Naculich, Nastase, Schnitzer; Brandhuber, Heslop, Nasti, Spence, GT)
  - Novel tree-level relations (Arkany-Hamed, Cachazo, Kaplan)

- Amplitudes in Gravity
  - KLT relations (Kawai, Lewellen, Tye)
  - UV behaviour of tree amplitudes under (complex) shifts much softer than expected (Bedford, Brandhuber, Spence, GT; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkany-Hamed, Kaplan)

New methods

- MHV diagrams (Cachazo, Svrcek, Witten)
  - Loop MHV diagrams (Brandhuber, Spence, GT)
- Tree-level recursion relations (Britto, Cachazo, Feng + Witten)
  - gravity (Bedford, Brandhuber, Spence, GT; Cachazo, Svrcek)
  - massive particles (Badger, Glover, Khoze, Svrcek)
  - rational one-loop amplitudes in QCD (Bern, Dixon, Kosower) and pure gravity (Brandhuber, McNamara, Spence, GT)

Specifically for loops: (not today, sorry)

- **Unitarity** (Bern, Dixon, Kosower 1994)
  - glue on-shell amplitudes to form loops
  - amplitudes reconstructed from its discontinuities across two-particle cuts
- **Generalised unitarity** (Britto, Cachazo, Feng 2005)
  - use multiple cuts $\longrightarrow$ loops from trees !
  - D-dimensional multiple cuts for complete amplitudes in non-supersymmetric YM (Brandhuber, McNamara, Spence, GT)

- Spinor integration (Britto, Buchbinder, Cachazo, Feng; Britto, Feng, Mastrolia)
    - D-dimensional applications for non-supersymmetric amplitudes

      (Anastasiou, Britto, Feng, Kunszt, Mastrolia)
- Maximal cuts (Bern, Carrasco, Johansson, Kosower 2008)
- Leading singularity (Cachazo 2008)

Spinor helicity formalism

- Massless particles: p_μ null vector
- Define $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$ where $\sigma^\mu = (1, \vec{\sigma})$
- If $p^2 = 0$ then $\det p = 0$
- Hence $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$
  - λ ($\tilde{\lambda}$) positive (negative) helicity spinors
  - Inner products $\langle 12 \rangle := \varepsilon_{ab} \lambda_1^a \lambda_2^b$

$$[12] := \varepsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$$

$$2(p_1 \cdot p_2) = \langle 12 \rangle [12]$$

- Momenta and wavefunctions re-expressed in terms of λ and $\tilde{\lambda}$

- E.g. for gluons, $\varepsilon^{(+)}_{a\dot{a}} = \frac{\tilde{\lambda}_{\dot{a}}\eta_a}{\langle\lambda\eta\rangle}$ $\qquad \varepsilon^{(-)}_{a\dot{a}} = \frac{\lambda_a\tilde{\eta}_{\dot{a}}}{[\tilde{\lambda}\tilde{\eta}]}$

  - η and $\tilde{\eta}$ are reference spinors, and $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}$

  - independent of η and $\tilde{\eta}$ (up to gauge transformation)

Scattering Amplitudes

$$\mathcal{A} \;=\; \mathcal{A}(\{\lambda_i, \tilde{\lambda}_i, ; h_i\})$$

- momenta and polarisation vectors expressed in terms of spinors and helicities
- Yang-Mills: use colour ordering
- Tree-level YM: single-trace structure is stripped off

- First examples:

$$\mathcal{A}(1^+, 2^+, \ldots n^+) = 0\,,$$

$$\mathcal{A}(1^-, 2^+, \ldots n^+) = 0$$

tree-level YM and GR

Maximally Helicity Violating amplitude, or MHV amplitude

- gluon helicities are a permutation of −−++....+

$$\mathcal{A}_{\mathrm{MHV}}(1^+ \ldots i^- \ldots j^- \ldots n^+) = \frac{\langle i\,j\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

Parke-Taylor formula

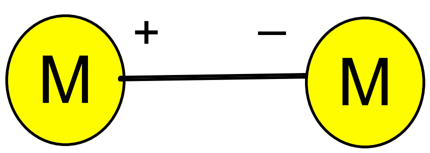
- Holomorphic function of the positive helicity spinors λ
  - geometry in twistor space (Witten, 2003)
- Covariant under dual superconformal symmetry [split-helicity case] (Drummond, Henn, Korchemsky, Sokatchev, 2008)

MHV diagrams

- MHV amplitudes ➡ localise on complex lines in Penrose's Twistor Space (Witten, 2003)
- Line in twistor space ➡ point in Minkowski space (Penrose)
- MHV amplitude ➡ local interaction in spacetime ! (Cachazo, Svrček, Witten, 2004)
  - now understood as a change of variables in YM path integral which maintains locality in lightcone time (Mansfield; Gorsky & Rosly, 2005)

Diagrammatic Rules

- MHV amplitude ➡ MHV vertex
- Off-shell continuation for internal momenta
  - Same as in lightcone Yang-Mills



- Internal momentum is off-shell
- Need to define spinor λ for an off-shell vector!

- Scalar propagators connect MHV vertices

Key points

- Off-shell prescription as in lightcone YM

$$L_{a\dot{a}} = l_{a\dot{a}} + z\eta_{a\dot{a}}$$

  - $l_{a\dot{a}} := l_a \tilde{l}_{\dot{a}}$ is the off-shell continuation

- Scalar propagators $\dfrac{i}{P^2 + i\varepsilon}$

  - At loop level, the $i\varepsilon$ prescription is crucial in correctly determining the integration range

- Draw all diagrams obtained by sewing $q - 1 + l$ MHV vertices

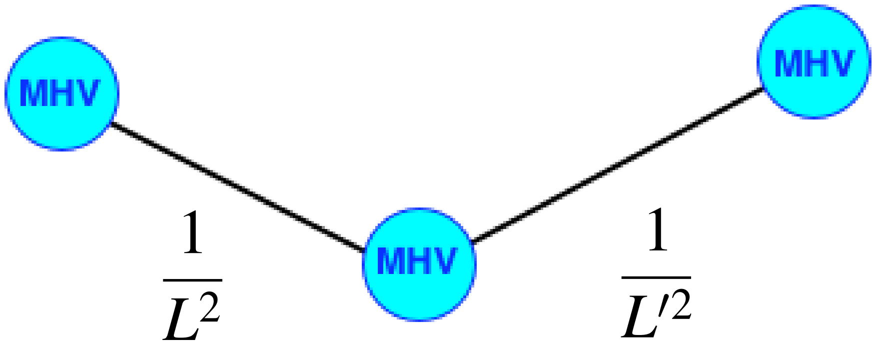
  q = # negative helicity gluons,

  l = # loops

  - each vertex provides one external negative helicity
  - $+$ and $-$ helicities treated asymmetrically

- Examples: (tree level)

  - Next-to-MHV (NMHV): $q=3, \quad l=0 \qquad d=2$



An MHV diagram contributing to a Next-to-Next-to-MHV amplitude

$q=4, \quad l=0 \qquad d=3$

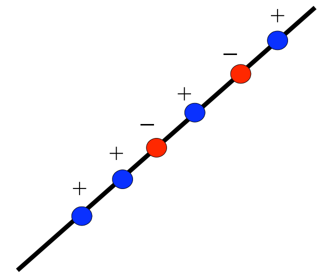
- Sum over all diagrams obtained by distributing external gluons among the vertices

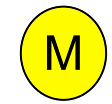
- Covariance achieved in the sum

- Result identical to Feynman diagram calculation

Amplitude | Twistor space structure | MHV diagrams

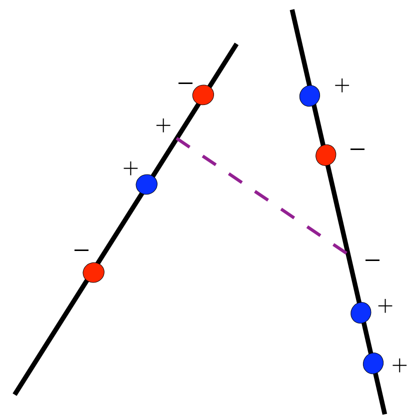
MHV





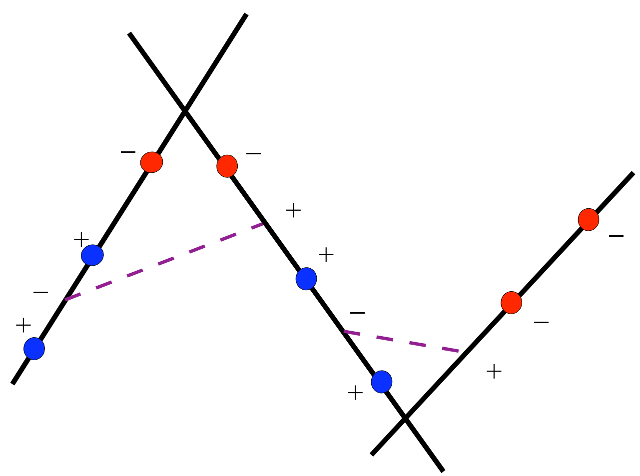
NMHV

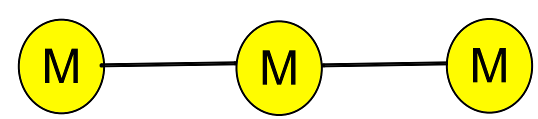
(Next-to-MHV)



NNMHV

(Next-to-Next-to-MHV)

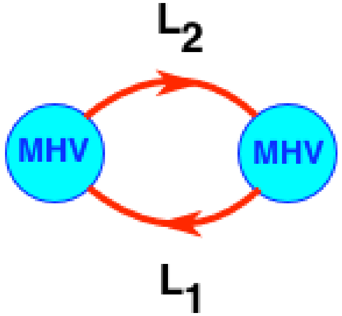




Loop MHV diagrams

(Brandhuber, Spence, GT, 2004)

- First application: MHV amplitudes in N=4

$\sum \int d\mathcal{M}$ 

  - sum over all possible ways to distribute the external −ve helicities among the vertices
  - sum over internal particle species (g, f, s)
  - measure turns out to be of the form:
    phase space X dispersive
  - applications to N=1,2 SYM and to pure Yang-Mills

MHV Lagrangian

- Mansfield's procedure: (in a nutshell)
  - Start from lightcone quantisation of YM, $A^{-} = 0$
  - integrate out A^{+} (no derivatives wrt lightcone time x^{-})
  - $A_z, A_{\bar{z}}$ correspond to physical polarisations

- Action is $$S = S^{-+} + S^{--+} + S^{++-} + S^{--++}$$

(Scherk, Schwarz)

anti MHV

- Change variables in path integral: $A_z, A_{\bar{z}} \rightarrow B_+, B_-$

$$(S^{-+} + S^{-++})[A_z, A_{\bar{z}}] = S^{-+}[B_+, B_-]$$

· LHS is SDYM action

· Bäcklund transformation

- Further require:
  - Transformation is canonical, with $A_z = A_z[B_+]$
  - Canonicality ➡ Jacobian equal to 1 (classically)
    - Subtleties related to $\det \partial_+$?

- Plug $A_z \sim B_+ + B_+^2 + B_+^3 + \cdots$ in

  $A_{\bar{z}} \sim B_-(1 + B_+ + B_+^2 + B_+^3 + \cdots)$

  $(S^{--+} + S^{--++})[A_z, A_{\bar{z}}]$

- Result is $S[B_+, B_-] = S^{-+} + S^{--+} + S^{--++} + S^{--+++} + \cdots$

  (Mansfield)

- New vertices have MHV helicity configuration

- Nontrivial derivation of rational loop amplitudes in pure YM

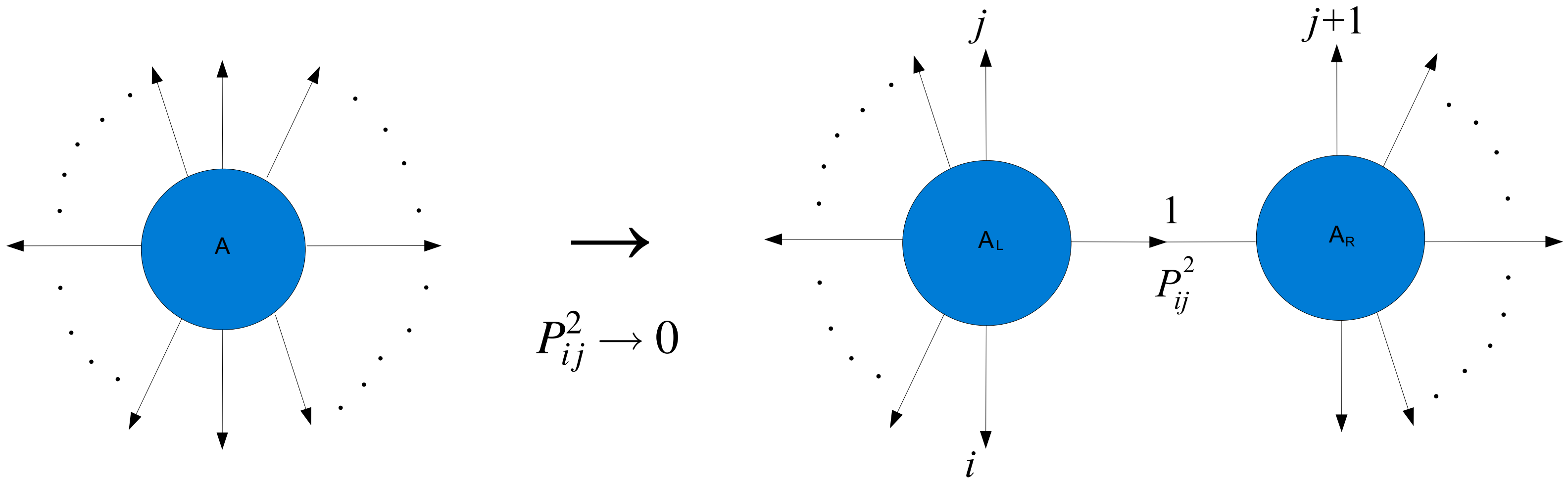
  (Ettle, Fu, Fudger, Mansfield, Morris; Brandhuber, Spence, Zoubos, GT)

On-shell recursion relations

(Britto, Cachazo, Feng; BCF + Witten, 2005)

- Exploit analytic structure of tree amplitudes

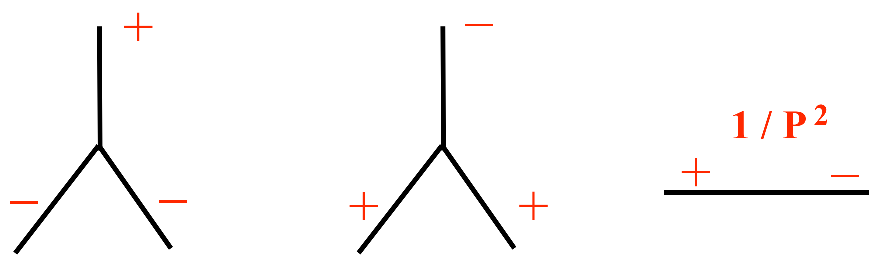
  - Input 1: factorisation of tree-level amplitudes



- Factorisation on multiparticle poles (simple poles, tree level)

- Bilinear structure

- Input 2: three-point amplitudes



- Elementary building blocks. Nonvanishing in complexified Minkowski

- Can be calculated without using Feynman rules, in fact without even knowing the Lagrangian

- Input 3: behaviour of amplitudes for large (complex deformations of) momenta

Concretely:

- Select two (adjacent) legs, and shift momenta:
  $\hat{p}_1 = p_1 + z\,\eta \qquad \hat{p}_2 = p_2 - z\,\eta \qquad \hat{p}_1 + \hat{p}_2 = p_1 + p_2$

- Impose $\hat{p}_1^2 = \hat{p}_2^2 = 0$ for all z

  - Solution is complex: $\eta = \lambda_1 \tilde{\lambda}_2 \;, \quad \eta = \lambda_2 \tilde{\lambda}_1$

- $\mathcal{A}(z) := \mathcal{A}(\hat{p}_1, \hat{p}_2, p_3, \ldots, p_n)$ is a one complex parameter family of amplitudes

- If $\mathcal{A}(z \to \infty) \to 0$, $\frac{1}{2\pi i} \oint_{C_\infty} \frac{dz}{z} \mathcal{A}(z) = 0 = \mathcal{A}(0) + \sum{}' \text{Res} \frac{\mathcal{A}(z)}{z}$

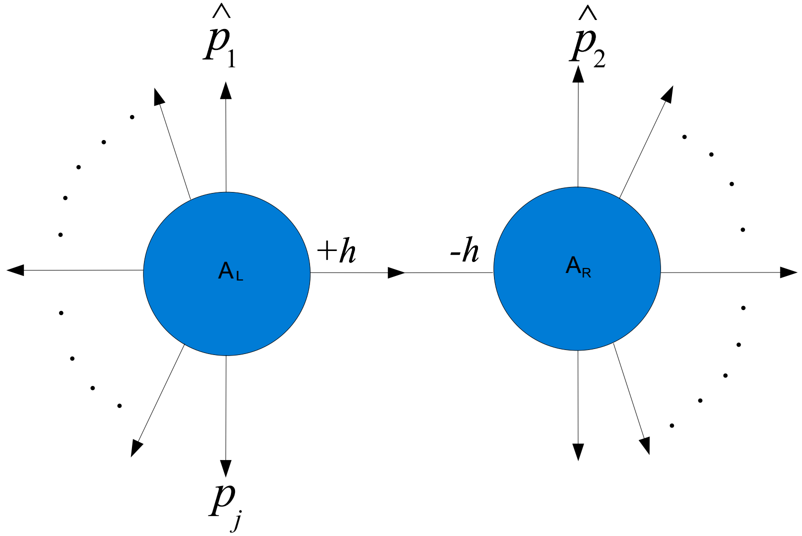
UV behaviour

This is the amplitude we want

Residues from multiparticle factorisation

- Final result:

$$\mathcal{A} = \sum_{j,h}$$



  - Amplitudes with fewer legs
  - Shifted momenta (complex!)
  - Proof has very general applicability, but need to check large-z behaviour (depends on the theory)

- Recursion relation works also in Gravity
  (Bedford, Brandhuber, Spence, GT; Cachazo, Svrcek)

  - Relation to possible finiteness of N=8 supergravity (Bern, Dixon, Roiban; Bern, Carrasco, Dixon, Johansson, Kosower, Roiban; Green, Russo, Vanhove; Bern, Carrasco, Forde, Ita, Johansson)

  - New relations for N=8 amplitudes, consequence of $1/z^2$ behaviour (Arkani-Hamed, Cachazo, Kaplan)

- Calculation of tree-level gravity amplitudes enormously simplified

  - Only need

$$\mathcal{A}_{\rm GR}(1^+2^+3^-) = [\mathcal{A}_{\rm YM}(1^+2^+3^-)]^2$$

$$\mathcal{A}_{\rm GR}(1^-2^-3^+) = [\mathcal{A}_{\rm YM}(1^-2^-3^+)]^2$$

KLT relations

to jump-start the recursion relation

➡ EH Lagrangian (and its Feynman rules) not needed

Bryce S. DeWitt , Phys. Rev. 162:1239-1256,1967.

corrections. For the Yang-Mills field it takes the form

$$V_{(\alpha\gamma''\sigma'')\beta'} \to -ic_{\alpha\beta\gamma}p'^{\sigma} = -ic_{\alpha\gamma\beta}(p^{\sigma}+p''^{\sigma}). \quad (2.3)$$

The propagators for the normal and fictitious quanta are, respectively,

$$G \to \gamma^{\alpha\beta}\eta_{\mu\nu}/p^2, \quad (2.4)$$

$$\hat{G} \to \gamma^{\alpha\beta}/p^2, \quad (2.5)$$

with p^2 being understood to have the usual small negative imaginary part.

The corresponding quantities for the gravitational field are much more complicated. In this case we shall employ the momentum-index combinations $p\mu\nu$, $p'\sigma'\tau'$, $p''\rho''\lambda''$, $p'''\iota'''\kappa'''$. The vertices must not only be symmetric in each index pair but must also remain unchanged under arbitrary permutations of the momentum-index triplets. At least 171 separate terms are required in the complete expression for S_3 in order to exhibit this full symmetry, and for S_4 the number is 2850. However, these numbers can be greatly reduced by counting only the combinatorially distinct terms[2] and leaving it understood that the appropriate symmetrizations are to be carried out. In this way S_3 is reduced to 11 terms and S_4 to 28 terms, as follows:

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \to$$

$$\begin{aligned}\mathrm{Sym}[&-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^{\sigma}p^{\lambda}\eta^{\mu\nu}\eta^{\tau\rho})\\ &-\tfrac{1}{2}P_3(p^{\tau}p'^{\mu}\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^{\rho}p^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^{\sigma}p'^{\lambda}\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu})\\ &-P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})], \end{aligned} \quad (2.6)$$

← 3-point vertex: 171 terms

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \to$$

$$\begin{aligned}\mathrm{Sym}[&-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\ &+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\ &+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^{\sigma}p^{\tau}\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\ &-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\ &-P_{12}(p^{\sigma}p^{\tau}\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^{\rho}p'^{\lambda}\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_{\sigma}p'^{\rho}\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^{\rho}p'^{\iota}\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\ &+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^{\sigma}p^{\rho}\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\ &-P_6(p^{\rho}p'^{\iota}\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})]. \end{aligned} \quad (2.7)$$

← 4-point vertex: 2850 terms

The "Sym" standing in front of these expressions indicates that a symmetrization is to be performed on each index pair $\mu\nu$, $\sigma\tau$, etc. The symbol P indicates that a summation is to be carried out over all distinct permutations of the momentum-index triplets, and the subscript gives the number of permutations required in each case.

Expressions (2.6) and (2.7) can be obtained in a straightforward manner by repeated functional differentiation of the Einstein action. This procedure, however, is exceedingly laborious. A more efficient (but still lengthy) method is to make use of the hierarchy of identities (II, 17.31). It is a remarkable fact that once $S_2{}^0$ is known all the higher vertex functions, and hence the complete action functional itself, are determined by the general coordinate invariance of the theory. It is convenient, in the actual computation of the vertices via (II, 17.31), to invent diagrammatic schemes for displaying the combinatorics of indices. Since each reader will devise the scheme which suits him best we shall not shackle him by describing one here. We also make no attempt to display S_5 or any higher vertices.

The vertex $V_{(\alpha i)\beta}$ has the following form for the gravitational field:

$$V_{(\mu{}^{\sigma''\tau''})\nu'} \to \tfrac{1}{2}\mathrm{Sym}[2p''_{\mu}p'^{\sigma}\delta_{\nu}{}^{\tau}-p''_{\mu}p'_{\nu}\eta^{\sigma\tau} +(p_{\nu}p'^{\sigma}-p'_{\nu}p^{\sigma})\delta_{\mu}{}^{\tau}+p\cdot p'\delta_{\mu}{}^{\sigma}\delta_{\nu}{}^{\tau}], \quad (2.8)$$

where the momentum-index combinations are $p\mu$, $p'\nu'$, $p''\sigma''\tau''$, and the symmetrization is to be performed on the index pair $\sigma\tau$. The propagators for the normal and fictitious quanta are given by

$$G \to (\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\sigma\tau})/p^2, \quad (2.9)$$

$$\hat{G} \to \eta^{\mu\nu}/p^2. \quad (2.10)$$

[2] The choice of terms is not completely unique since momentum conservation may be used to replace a given term by other terms. We give here what we believe (but have not proved) to be the expressions containing the smallest number of terms.

Gravity (and YM) amplitudes are much simpler than what one would expect from Feynman rules !

- Similarly, for Yang-Mills theory, need to know only three-point amplitudes
    - Four-point vertex not needed

- Three-point amplitudes of massless particles can be calculated using Witten's "auxiliary relation"

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \tilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}}\right) \mathcal{A} = -2 h_i \mathcal{A}$$

- ...together with the fact that three-point amplitudes are holomorphic or anti-holomorphic (depend either on λ or on $\tilde{\lambda}$)

  - Encodes the scaling of wavefunctions with the spinors

  - Valid for massless particles of any spin

Spin s three-point amplitudes

$$\mathcal{A}(1^{-s}, 2^{-s}, 3^{s}) \sim \left(\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,1\rangle} \right)^s$$

$$\mathcal{A}(1^{s}, 2^{s}, 3^{-s}) \sim \left(\frac{[1\,2]^3}{[2\,3]\,[3\,1]} \right)^s$$

Recent developments

- Supersymmetric recursion relations in N=4 SYM/N=8 sugra
  - Three-particle shifts (MHV type) (Bianchi, Elvang, Freedman)
  - Two particle shifts (BCF type)
    (Brandhuber, Heslop, GT; Arkani-Hamed, Cachazo, Kaplan)
- Advantages:
  - Amplitudes in N=4/N=8 efficiently generated (Bianchi, Elvang, Freedman)
  - Dual superconformal symmetry of N=4 manifest (Brandhuber, Heslop, GT)
  - large-z behaviour in N=4/N=8 is manifest (Arkani-Hamed, Cachazo, Kaplan)

Novel structures

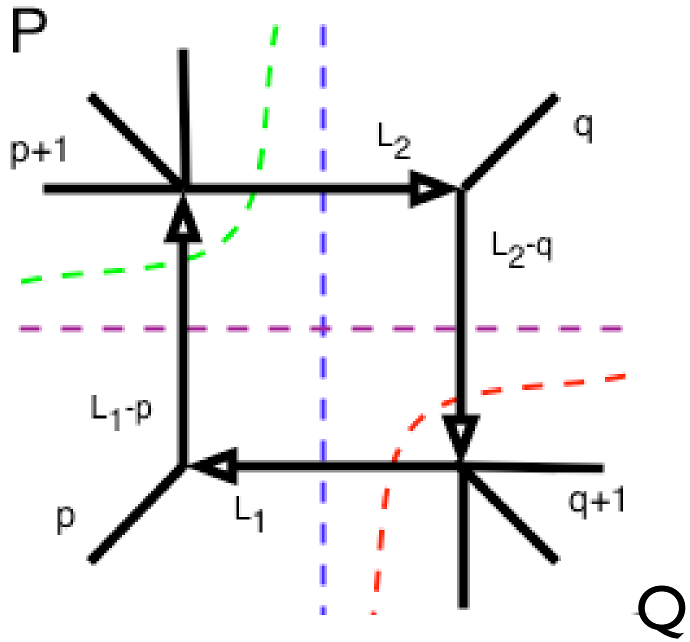
(planar N=4 SYM)

Simplest one-loop amplitude

- n-point MHV amplitude in N=4 SYM

$$\mathcal{A}_{\text{MHV}}^{1-\text{loop}} = \mathcal{A}_{\text{MHV}}^{\text{tree}} \sum$$

(Bern, Dixon, Dunbar, Kosower, 1994)



- Colour-ordered partial amplitude, leading term in $1/N$
- Sum of two-mass easy box functions, all with coefficient 1

Diagrammatic interpretation

I. Iterative structures at higher loops

(Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)

- $\mathcal{A}_{n,\mathrm{MHV}} = \mathcal{A}^{\mathrm{tree}}_{n,\mathrm{MHV}} \mathcal{M}_n$ $\mathcal{M}_n$ helicity-blind function

- BDS ansatz: (Bern, Dixon, Smirnov; earlier work of Anastasiou, Bern, Dixon, Kosower)

$$\mathcal{M}_n := 1 + \sum_{L=1}^{\infty} a^L \mathcal{M}_n^{(L)} \overset{\text{BDS}}{=} \exp\left[\sum_{L=1}^{\infty} a^L \Big(f^{(L)}(\epsilon) \mathcal{M}_n^{(1)}(L\epsilon) + C^{(L)} + \mathcal{O}(\epsilon) \Big)\right]$$

$$a \sim g^2 N/(8\pi^2)$$

- $\mathcal{M}_n^{(1)}(\epsilon)$ is the all-orders in ϵ one-loop amplitude, $D = 4 - 2\epsilon$

  ☝ regulates infrared divergences

- $f^{(L)}(\epsilon) = f_0^{(L)} + \epsilon f_1^{(L)} + \epsilon^2 f_2^{(L)}$

  ☝ anomalous dimension of twist-two operators at large spin, $\gamma_K^{(L)}/4$

☞ Higher-loop amplitudes expressed in terms of lower-loop amplitudes

Comments

- BDS Ansatz suggested by universal resummation of IR divergences (Catani; Magnea, Sterman; Sterman, Tejeda-Yeomans)

- BDS: exponentiation of finite parts

  - IR and finite parts entangled. Exponentiated finite remainders approach constants (independent of kinematics and # of particles)

- Signature of two-loop iteration: (take Log of the Ansatz)

$$\mathcal{M}_n^{(2)} - \frac{1}{2}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^2 = f^{(2)}(\epsilon)\,\mathcal{M}_n^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

  - Requires knowledge of lower-loop amplitude to higher orders in ϵ, hence go up by one loop only

Checks of BDS conjecture

- Two and three loops at four points (Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov). Confirmed result for three-loop cusp anomalous dimension obtained assuming maximal transcendentality (Kotikov, Lipatov, Onishcenko, Velizhanin)

- Two loops at five points (Bern, Czakon, Kosower, Roiban, Smirnov)

  - Parity odd terms cancel in the iteration

- Problems begin at two loops, six points (Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich)

- Exponent requires an additional finite remainder (discrepancy function)

II. Amplitude / Wilson Loop duality

(Alday, Maldacena; Drummond, Korchemsky, Sokatchev+Henn; Brandhuber, Heslop, GT)

- MHV amplitudes in planar N=4 super Yang-Mills appear in a completely different context:

$$< W[C] >$$

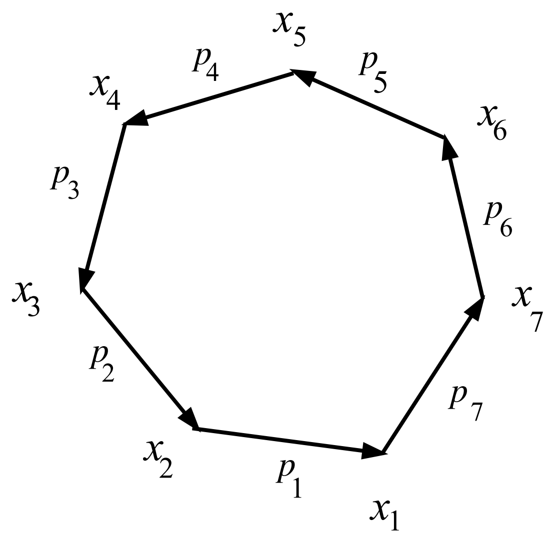
- Contour C is determined by the momenta of the scattered particles
- Strong coupling (Alday & Maldacena)
- Weak coupling (Drummond, Korchemsky, Sokatchev+Henn; Brandhuber, Heslop, GT)

- The contour of the Wilson loop:

  - at strong coupling, amplitude calculation identical to Wilson loop calculation with a particular polygonal contour
  - boundary of worldsheet tends to boundary of T-dual AdS space as IR cutoff is removed
  - colour ordering $\mathrm{Tr}(T^1 T^2 \cdots T^7)$

    $p_i = x_i - x_{i+1}$ lightlike momenta

    x's are the T-dual (region) momenta



  - momentum conservation $\sum_{i=1}^{n} p_i = 0$ ➡ closed contour
  - dual conformal symmetry acts on the T-dual momenta

- Try at weak coupling: $< W[C] >$ is the n-point MHV amplitude in N=4 SYM

  - modulo tree-level prefactor. Wilson loop reproduces the helicity-blind function $\mathcal{M}_n$
  - Unexpected! eikonal approximation usually reproduces IR behaviour only; we also get finite parts

- Conjecture: (Log) $< W[C] >$ = (Log) $\mathcal{M}$

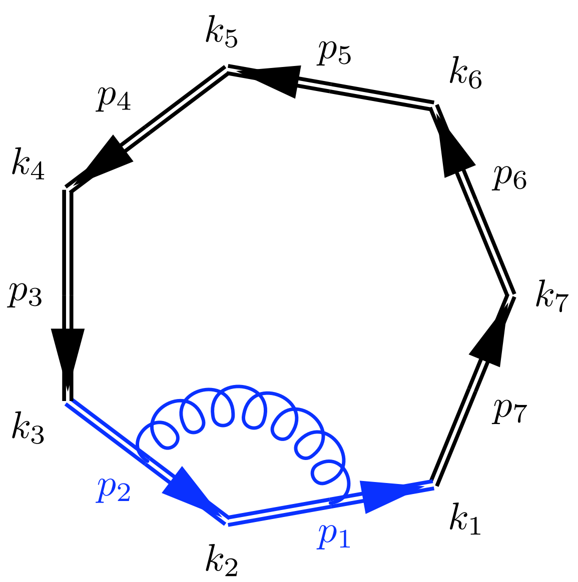
  (Drummond, Korchemsky, Sokatchev+Henn; Brandhuber, Heslop, GT)

  - Natural exponentiation for Wilson loops: nonabelian exponentiation theorem (Gatheral; Frenkel & Taylor)
  - Conjecture recently checked at two loops by Drummond, Henn, Korchemsky, Sokatchev for the four-, five-, and six-point case

$< W[C] >$ at one loop, n points

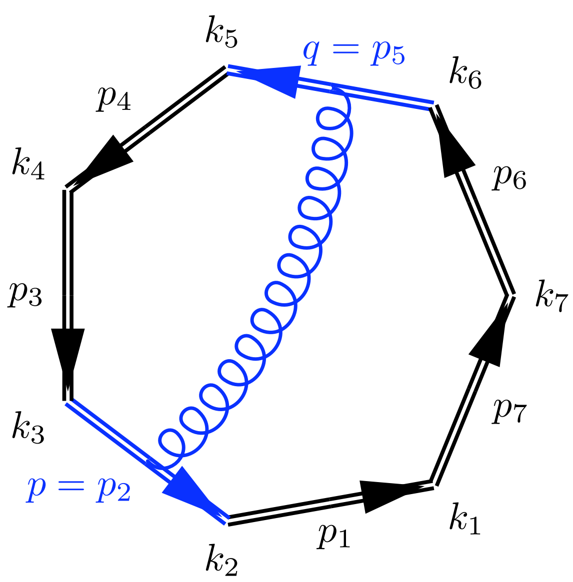
(Brandhuber, Heslop, GT)

- Calculation done (almost) instantly.
  Two classes of diagrams: [self-energy diagrams = 0]



Gluon stretched between two segments meeting at a cusp

A. Infrared divergent



Gluon stretched between two non-adjacent segments

B. Infrared finite

- Clean separation between infrared-divergent and infrared-finite terms
  - Important advantage, as ε can be set to zero in the finite parts from the start
- From diagrams in class A :

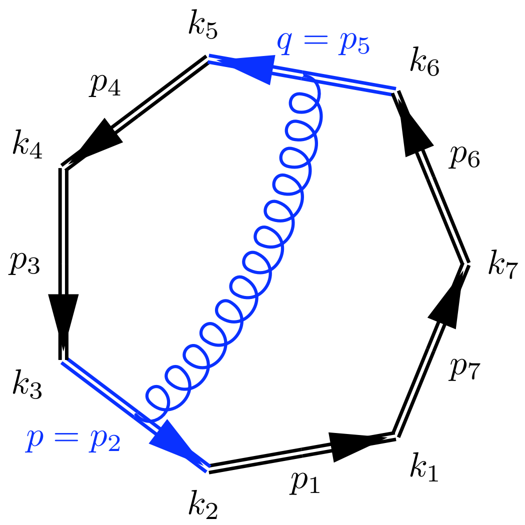
$$\mathcal{M}_n^{(1)}\big|_{IR} = -\frac{1}{\varepsilon^2}\sum_{i=1}^{n}\left(\frac{-s_{i,i+1}}{\mu^2}\right)^{-\varepsilon}$$

  - $s_{i,i+1} = (p_i + p_{i+1})^2$ is the invariant formed with the momenta meeting at the cusp

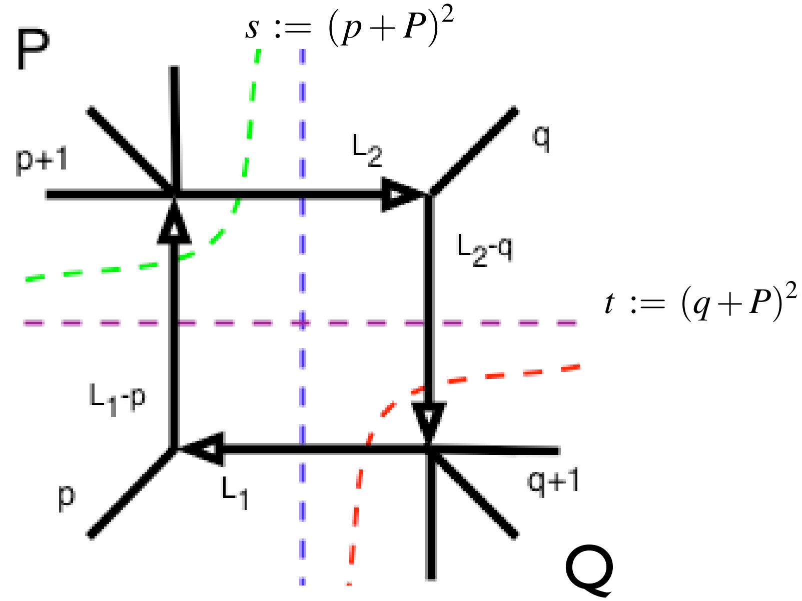
- Diagram in class B, with gluon stretched between p and q gives a result proportional to

$$\mathcal{F}_\varepsilon(s,t,P,Q) = \int_0^1 d\tau_p d\tau_q \, \frac{P^2+Q^2-s-t}{\left[-\left(P^2+(s-P^2)\tau_p+(t-P^2)\tau_q+(-s-t+P^2+Q^2)\tau_p\tau_q\right)\right]^{1+\varepsilon}}$$

- Explicit evaluation shows that this is the finite part of a two-mass easy box function
  - Two-dimensional representation of two-mass easy box function



$\leftrightarrow$



- In the example: $p = p_2 \quad q = p_5$

  $P = p_3 + p_4 \, , \quad Q = p_6 + p_7 + p_1$

- One-to-one correspondence between Wilson loop diagrams and finite parts of two-mass easy box functions

- Explains why each box function appears with coefficient equal to one in the expression of the one-loop N=4 MHV amplitude

- Explicit calculation gives:

$$a := \frac{2(pq)}{P^2Q^2 - st}$$

$$\mathcal{F}_\varepsilon = -\frac{1}{\varepsilon^2}$$

$$\left[\left(\frac{a}{1-aP^2}\right)^{\varepsilon} {}_2F_1\left(\varepsilon,\varepsilon,1+\varepsilon,\frac{1}{1-aP^2}\right) + \left(\frac{a}{1-aQ^2}\right)^{\varepsilon} {}_2F_1\left(\varepsilon,\varepsilon,1+\varepsilon,\frac{1}{1-aQ^2}\right)\right.$$

$$\left. - \left(\frac{a}{1-as}\right)^{\varepsilon} {}_2F_1\left(\varepsilon,\varepsilon,1+\varepsilon,\frac{1}{1-as}\right) - \left(\frac{a}{1-at}\right)^{\varepsilon} {}_2F_1\left(\varepsilon,\varepsilon,1+\varepsilon,\frac{1}{1-at}\right)\right]$$

- At $\varepsilon \longrightarrow 0$: $\mathcal{F}_{\varepsilon=0} = -\mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at) + \mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2)$

  - Box function in the same compact form derived from dispersion integrals using one-loop MHV diagrams

    (Brandhuber, Spence, GT)

- At 4 points, Wilson loop = all-orders in ε amplitude:

$$\mathcal{M}_4^{(1)}(\varepsilon) = -\frac{2}{\varepsilon^2}\left[\left(\frac{-s}{\mu^2}\right)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon,1-\varepsilon,1+\frac{s}{t}\right) + \left(\frac{-t}{\mu^2}\right)^{-\varepsilon} {}_2F_1\left(1,-\varepsilon,1-\varepsilon,1+\frac{t}{s}\right)\right]$$

  - Agrees with result of Green, Schwarz and Brink

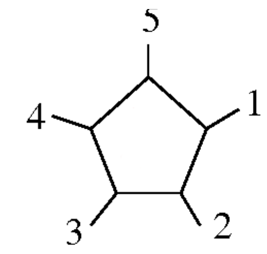
- For $n > 4$ points: missing topologies

  - New diagrams appear at $O(\varepsilon)$
  - Result correct only up to and including $O(1)$

- E.g. for $n = 5$, pentagons

$$\mathcal{M}_5^{(1)} \sim \sum_{p,q} F^{2\text{me}}(p,q) + \varepsilon\, \varepsilon(1,2,3,4)\, F_5^{6-2\varepsilon}$$

$$\varepsilon(1,2,3,4) = 4i\varepsilon_{\mu\nu\rho\lambda} p_1^\mu p_2^\nu p_3^\rho p_4^\lambda$$

$F_5^{6-2\varepsilon} =$ 

(finite in 6 dimensions)

- Coefficient is parity odd, vanishes under collinear limits (and for $\varepsilon \to 0$)

- At one loop, Wilson loop gives parity-even part to all orders in ε

    - Q: what part of amplitude does the Wilson loop calculate in general ? (higher loops, higher points, higher orders in ε)

III. Dual (super)conformal symmetry

(Drummond, Henn, Korchemsky, Sokatchev)

- On the Wilson loop side, dual conformal symmetry:
  - is the standard conformal symmetry acting on the dual variables x's
  - is anomalous due to ultraviolet divergences (UV for the Wilson loop = IR for the amplitude)
- BDS ansatz solves dual conformal Ward identities
  - Unique solution up to five points (modulo constants)
  - For $n \geq 6$: nontrivial conformally invariant ratios, e.g. $\frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{36}^2}$
  - Dual conformal symmetry leaves room for an arbitrary function of cross ratios (discrepancy function)

- Dual superconformal symmetry (Drummond, Henn, Korchemsky, Sokatchev)
    - nontrivial action on spinor variables
    - requires formulation in dual N=4 superspace in order to become manifest
    - Amplitudes lifted superamplitudes

- Conjecture: dual superconformal symmetry is a symmetry of the planar N=4 S-matrix
(Drummond, Henn, Korchemsky, Sokatchev)

  - planarity, maximal supersymmetry, on-shellness

- Evidence:

Trees

  - MHV tree superamplitude (Drummond, Henn, Korchemsky, Sokatchev)
  - tree-level S-matrix, using a new super-recursion relation which is manifestly dual SC invariant (Brandhuber, Heslop, GT)

Loops

  - MHV, Next-to-MHV amplitudes at one loop (DHKS)
  - Coefficients of the expansion of the superamplitudes in a basis of box functions (BHT)

DSC invariance of tree-level S-matrix

(Brandhuber, Heslop, GT)

- Idea: write down an N=4 supersymmetric recursion relation $\mathcal{A} = \sum_P \int d^4\eta_{\hat{P}} \, \mathcal{A}_L(z_P) \frac{i}{P^2} \mathcal{A}_R(z_P)$
    - building blocks are superamplitudes
    - three-point superamplitudes are covariant
        - MHV is manifestly covariant
        - anti-MHV amplitude (Brandhuber, Heslop, GT; Arkani-Hamed, Cachazo, Kaplan)
    - propagator + delta function + fermionic integration
    - proof of DSC obtained by induction
    - similar proof for the expansion coefficients of one-loop amplitudes in a basis of box functions

Summary

- Novel structures uncovered in gauge theory and gravity, new methods to calculate amplitudes
    - twistor space geometry
    - MHV diagrams
    - On-shell recursion relations
- Planar N=4
    - Iterative structures
    - Amplitude/Wilson loop duality
    - Dual superconformal symmetry

Open questions

- Can we understand why MHV amplitudes and Wilson loops are related ?
- Can we extend this to non-MHV amplitudes ?
- Higher-loop analytic calculations of Wilson loops?
- Use integrability of worldsheet theory to find differential equations which determine the discrepancy function
- Strong coupling: beyond 4 points, beyond MHV
- Origin of dual superconformal symmetry in field theory?
- Extensions of dual superconformal symmetry ?
- N=8 supergravity

and many more...

Superamplitudes

- $$\mathcal{A} \;=\; \delta^{(4)}\Big(\sum_i \lambda_i \tilde{\lambda}_i\Big)\; \delta^{(8)}\Big(\sum_i \eta_i \lambda_i\Big)\; A$$ (Nair)
  - $\eta^A\,,\; A = 1, \ldots 4$ fermionic variables, A is an SU(4) index
  - expansion in η generates all component amplitudes
  - p powers of η_i corresponds to helicity $h_i = 1 - p/2$
    (Georgiou, Khoze)
  - MHV: $A \;=\; \dfrac{1}{\langle 1\,2\rangle\langle 2\,3\rangle \cdots \langle n\,1\rangle}$
  - gluons i^-, j^- : get factor of $(\eta_i)^4 (\eta_j)^4 \,\langle i\, j\rangle^4$