Hadronization of QCD

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Collaborations

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Hadronization of QCD is the reformulation of its action in terms of hadronic fields and fermionic fields with the quark quantum numbers. These fermionic fields cannot be the bare quark fields (Pauli principle, double counting).

The problem of hadronization is a special case of a general problem which appears in many fields of Physics:

Given a Lagrangian which admits bound states, how to construct an equivalent Lagrangian in which bound states and constituent fields appear on the same footing.

This problem has been considered by many authors (Weinberg, Salam,...). The result of their investigations is a conjecture of a

compositeness condition: the wave function renormalization constants of the bound states fields in the Lehmann spectral representation must vanish.

I will present a constructive procedure in which a compositeness condition is exactly implemented.

Examples of fermionic composites

There are many finite and infinite fermion systems whose partition function at low energy is dominated by bosonic modes. This is always the case when, due to spontaneous breaking of a global symmetry, there are Goldstone bosons.

• Many-body systems : phonons, Cooper pairs in metals and atomic nuclei, Cooper and molecular pairs in ultracold fermi systems in magnetic traps

• QCD : mesons and baryons, and Cooper pairs (diquarks) in the color superconducting and color locking phases

The structure of composites and condensates changes with temperature, fermion number and other control parameters

The constituents can live in equilibrium with the composites.

Mixing of composite and elementary bosons

Composites can mix with elementary bosons with the same quantum numbers

- polari(zation pho)tons :
  particle-hole states and photons in condensed matter
- vector dominance in QCD :
  quark-antiquark vector resonances and photons
- tensor dominance in quantum gravity :
  particle-hole states and gravitons
  (matter stress tensor and metric tensor) in quantum gravity

Outline of the present approach

I perform an independent Bogoliubov transformation at each time in the partition function. The time-dependent parameters of the transformation are then associated to dynamical composite fields while the transformed fermionic fields (quasiparticles) represent fermionic states in the presence of the composites.

Quasiparticles exactly satisfy by construction a compositeness condition which is exactly implemented.

The resulting action of composites and quasiparticles is exactly equivalent to the original one, and as such can be used in numerical simulations.

Analytic approximations are also possible, based on an expansion in the inverse of the index of nilpotency of the composites, which is the number of fermionic states in their structure functions.

Operator form of partition function in relativistic field theories

$$\mathcal{Z} = \int [d\sigma] \exp\left[-S_{(}\sigma)\right] \mathrm{Tr}^{\mathsf{F}} \left\{ \prod_{t=0}^{L_0-1} \hat{\mathcal{T}}_t \right\}$$

σ the elementary bosonic fields coupled to the fermions

Tr^{F} trace on the fermion Fock space, $\hat{\mathcal{T}}_t$ transfer matrix

$$\hat{\mathcal{T}}_t = \hat{T}_t^{\dagger} \, \hat{T}_{t+1}$$

$$\hat{T}_t = \exp\left[-\hat{u}^{\dagger} M(\sigma_t)\, \hat{u} - \hat{v}^{\dagger} M(\sigma_t)^T \hat{v}\right] \exp\left[\, \hat{v}\, N(\sigma_t)\, \hat{u}\right]$$

$\hat{u}, \hat{v}$ = fermion-antifermion canonical annihilation operators

The matrices M, N depend on time only through the bosonic fields. Their form depends on the regularization adopted, but what follows is not affected by their explicit expression. One can adopt a lattice regularization to be able to use some nonperturbative numerical results.

Operator form of partition function in QCD

In the case of gauge theories the matrices M_t, N_t are functions of the spatial gauge fields at Euclidean time t. The temporal gauge fields appear in the partition function in the following form

$$\mathcal{Z} = \int [dU] \exp\left[-S_G(U)\right] \mathrm{Tr}^{\mathsf{F}} \left\{ \prod_{t=0}^{L_0-1} \left(\hat{T}_t^\dagger \, \hat{V}_t \exp(\mu \, \hat{n}_B) \hat{T}_{t+1} \right) \right\}$$

$U_{\mu,t}$ = link variables

S_G = gluon action

μ = chemical potential

$\hat{n}_B$ = baryon number

$$\hat{V}_t = \exp\left(\hat{u}^\dagger \ln U_{0,t} \, \hat{u} + \hat{v}^\dagger U^*_{0,t} \, \hat{v} \right)$$

The matrices M, N with Kogut-Susskind fermions in the flavor basis

Kogut-Susskind fermions in the flavor basis have Dirac and taste indices on which the matrices γ_μ, t_μ act

$$
\begin{aligned}
M &= 0 \\
N &= -2\,\gamma_0 \otimes \mathbb{1} \left\{ m + \sigma + \sum_{j=1}^{3} \gamma_j \otimes \mathbb{1} \left[P_j^{(-)} \nabla_j^{(+)} + P_j^{(+)} \nabla_j^{(-)} \right] \right\} \\
&= \text{Dirac Hamiltonian}
\end{aligned}
$$

$$P_\mu^{(\pm)} = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \pm \gamma_\mu \gamma_5 \otimes t_5 t_\mu)$$

$$\nabla_j^{(+)} = \frac{1}{2}\left(U_j\, T_j^{(+)} - 1\right), \quad \nabla_j^{(-)} = \frac{1}{2}\left(1 - T_j^{(-)} U_j^\dagger\right) = \text{covariant derivatives}$$

$T_\mu^{(\pm)}$ = forward / backward translation operators of one block

Global Bogoliubov transformations

$$\hat{\alpha}_i = \left[\mathsf{R}^{\frac{1}{2}} \left(\hat{u} - \mathcal{F}^\dagger \, \hat{v}^\dagger \right) \right]_i \qquad \hat{\beta}_i = \left[\left(\hat{v} + \hat{u}^\dagger \, \mathcal{F}^\dagger \right) \mathring{\mathsf{R}}^{\frac{1}{2}} \right]_i$$

$$\mathsf{R} = (1 + \mathcal{F}^\dagger \mathcal{F})^{-1} \qquad \mathring{\mathsf{R}} = (1 + \mathcal{F} \mathcal{F}^\dagger)^{-1}$$

Bogoliubov transformations are unitary for arbitrary matrices $\mathcal{F}$

Quasiparticle operators $\hat{\alpha}, \hat{\beta}$ have quark-antiquark quantum numbers

but in general do not have definite transformation properties with respect to the symmetries of the Lagrangian, which therefore are not respected term by term.

We can give the quasiparticle operators the correct transformation properties and restore term by term all symmetries by requiring that the matrices $\mathcal{F}$ transform in the appropriate way, which implies that they must be dynamical fields.

Time-dependent Bogoliubov transformations

Dynamical fields can be introduced by performing an independent Bogoliubov transformation at each time.

Since nothing depends on the $\mathcal{F}_t$, I can integrate over them in the partition function with an arbitray measure $d\mu(\mathcal{F}^\dagger, \mathcal{F})$.

The trace over the transformed states in the partition function can be performed exactly yielding its functional form

$$\mathcal{Z} = \int [dU] \exp\left[-S_G(U)\right] \int d\mu(\mathcal{F}^\dagger, \mathcal{F}) \exp\left[S_{eff}\right]$$

Physical interpretation

The effective action S_{eff} is a function of the quasiparticle (Grassmann) variables and of the matrices $\mathcal{F}_t$.

For a physical interpretation I expand these matrices on a time independent basis $\mathbf{\Phi}_K$ with time-dependent coefficients $\phi_{K,t}$

$$\mathcal{F}_t = \sum_K \phi_{K,t} \mathbf{\Phi}_K^\dagger$$

$S_{eff} = S_{eff}(\phi^*, \phi, \alpha^*, \alpha, \beta^*, \beta, \mathbf{\Phi}^\dagger, \mathbf{\Phi})$ depends on the holomorphic variables ϕ^*, ϕ, the Grassmann variables $\alpha^*, \alpha, \beta^*, \beta$ and the matrices $\mathbf{\Phi}^\dagger, \mathbf{\Phi}$.

I associate the fields ϕ_K with bosonic composites with quantum numbers K, and the Grassmann variables $\alpha^*, \alpha, \beta^*, \beta$ with fermionic quasiparticles

Index of nilpotency

index of nilpotency $= \Omega_K$ = largest integer such that $\left(\hat{\Phi}_K\right)^{\Omega_K} \neq 0$

$\hat{\Phi}_K = \hat{v}\,\Phi_K\,\hat{u}$ composite annihilation operators

do not satisfy canonical commutation relations (no problem)

Φ_K= structure functions of the composites

Ω_K = number of fermionic states in structure function Φ_K, much greater than the number of intrinsic degrees of freedom

Ω_K counts the number of composites we can put in the state K

Necessary condition for physical interpretation

$$\Omega_K >> 1$$

Compositeness condition

The trace in the partition function is evaluated over coherent states.

The Bogoliubov transformations transform coherent states of quarks-antiquarks according to

$$\mathcal{S} \exp\left(\hat{u}^{\dagger}\alpha + \hat{v}^{\dagger}\beta\right)|0\rangle = \langle\phi|\phi\rangle^{-\frac{1}{2}} \exp(\hat{\alpha}^{\dagger}\alpha + \hat{\beta}^{\dagger}\beta)|\phi\rangle$$

$$|\phi\rangle = \exp\left(\hat{u}^{\dagger}\mathcal{F}\hat{u}^{\dagger}\right)|0\rangle$$

$$\hat{\alpha}_i|\phi\rangle = \hat{\beta}_i|\phi\rangle = 0 .$$

Quasiparticle operators satisfy by construction the compositeness condition

$$\langle\phi|\,\hat{\alpha}\,\hat{\beta}\,\hat{\Phi}^{\dagger}_{K}\,|\phi\rangle = 0\,,$$ quasiparticle states orthogonal to composites

Meson effective action

Neglecting quasiparticles we get the meson action

$$\mathcal{Z}_{\text{mesons}} = \int [dU] \exp\left[S_G(U)\right] \int \left[\frac{d\phi_t d\phi_t^*}{2\pi i}\right] \exp\left[-S_{\text{mesons}}(\phi^*, \phi)\right]$$

$$S_{\text{mesons}} = \sum_t \mathrm{tr}_- \left[-\ln \mathrm{R}_t + \ln \mathcal{R}_t + M_t^\dagger \right]$$

$$\begin{aligned} \mathcal{R}_t \quad &= \quad \left[1 + \left(N_t + e^{-M_t}\, U^\dagger_{0,t-1}\, \mathcal{F}_{t-1}\, U_{0,t-1}\, e^{-M_t} \right)^\dagger \right. \\ &\left. \times \left(N_t + e^{-M_t}\, \mathcal{F}_t\, e^{-M_t} \right) \right]^{-1} e^{-M_t^\dagger} . \end{aligned}$$

Finite baryon density

In order to discuss the case of finite baryon density I should introduce baryonic composites constructed in terms of quasiquarks (not bare quarks). I will later show one way to introduce baryons, but as a first step it is sufficient to consider a space of mesons, quasiquarks and quasiantiquarks, which contains the space of mesons and baryons

$$
\begin{aligned}
S_{\text{mesons-quarks}} &= S_{\text{mesons}} - \sum_t \alpha_t^* \left(\nabla_t - \mathcal{H}_t\right) \alpha_{t+1} \\
&- \beta_{t+1} \left(\mathring{\nabla}_t - \mathring{\mathcal{H}}_t\right) \beta_t^* + \beta_t \mathcal{I}_t^{(2,1)} \alpha_t + \alpha_t^* \mathcal{I}_t^{(1,2)} \beta_t^*
\end{aligned}
$$

where

$$
\nabla_t = e^{\mu} U_{0,t} - T_0^{(-)}, \quad \mathring{\nabla}_t = e^{-\mu} U_{0,t}^{\dagger} - T_0^{(+)}
$$

$$
\mathcal{H} = e^{\mu} \left[U_{0,t} - \mathrm{R}_t^{-\frac{1}{2}} E_{t+1}^{-1} \mathrm{R}_{t+1}^{-\frac{1}{2}} \right], \quad \mathring{\mathcal{H}} = e^{-\mu} \left[U_{0,t}^{\dagger} - \mathring{\mathrm{R}}_t^{-\frac{1}{2}} \mathring{E}_{t+1}^{-1} \mathring{\mathrm{R}}_{t+1}^{-\frac{1}{2}} \right]
$$

The coefficients I, E, $\mathring{E}$

The Ω^{-1} expansion

The $\mathbf{\Omega^{-1}}$ expansion is a saddle point expansion in which the asymptotic parameter is the index of nilpotency. Saddle point equations are obtained by performing the variation of the mesonic Lagrangian with respect to the structure function of the condensed meson.

It is remarkable that these equations coincide with the equations

$$\mathcal{I}^{(1,2)} = \mathcal{I}^{(2,1)} = 0$$

which eliminate the direct mixing of quasiparticles and quasiantiparticles in the effective action.

The Bogoliubov transformation at the saddle point is therefore equivalent to the Foldy-Wouthuysen transformation which separates particles from antiparticles in the Dirac Hamiltonian.

We are presently studying the saddle point equations for QCD in the normal and color superconducting phases

Application to a four-fermion model

Model with N_f degenerate fermions with quartic interaction in 3+1 dimensions

$$\mathcal{S} = \sum_x \sum_y \bar{\psi}(x)\ [m\,\mathbb{1} \otimes \mathbb{1}\ + Q]_{x,y}\,\psi(y) + \frac{1}{2}\,\frac{g^2}{4N_f} \sum_x (\bar{\psi}(x)\psi(x))^2$$

m = the mass parameter, g^2 = coupling constant and

$$Q = \sum_\mu \gamma_\mu \otimes \mathbb{1} \left[P_\mu^{(-)}\nabla_\mu^{(+)} + P_\mu^{(+)}\nabla_\mu^{(-)}\right]$$

The model has a discrete chiral symmetry at $m = 0$:

$$\psi \ \rightarrow\ -\gamma_5 \otimes \mathbb{1}\,\psi\,, \qquad \bar{\psi}\ \rightarrow\ \bar{\psi}\,\gamma_5 \otimes \mathbb{1}$$

I linearize the action introducing an auxiliary field $\sigma(x)$

$$\mathcal{S}' = \sum_x \sum_y \bar{\psi}(x)\,(m + \sigma + Q)_{xy}\psi(y) + \frac{4N_f}{2g^2} \sum_x \sigma^2(x)\,.$$

Saddle-point approximation at zero fermion density

$$S_{\text{mesons}} = \sum_t \mathrm{tr}_- \left\{ \ln(1+\mathcal{F}^\dagger\mathcal{F}) - \ln\left[1+(N+\mathcal{F}\ )^\dagger (N+\mathcal{F}\ )\right]\right\}$$

$$\frac{\partial}{\partial\sigma} S_{\text{mesons}} = 0 \to \text{ standard gap equation}$$

$$\frac{\partial}{\partial\mathcal{F}} S_{\text{mesons}} = 0 \to \bar{\mathcal{F}}$$

$$\bar{\mathcal{F}} = \left(\frac{N}{2H}\right)\left(\sqrt{1+H^2}-H\right), \quad H_p = p^2 + \bar{\sigma}^2$$

$$\bar{\mathcal{B}} = \sigma_2\left(\sqrt{1+\epsilon^2}-\epsilon\right), \quad \epsilon_p = \frac{1}{2m\,\triangle}(p^2 - p_F^2) \quad \triangle = \text{gap energy}$$

$$\frac{N}{2H}, \quad \sigma_2 = \text{Pauli matrix} = \text{unitary factors}$$

The fluctuations around the minimum give the action of a scalar field with mass = $2\bar{\sigma}$

Saddle point approximation at finite fermion density

Integrating over the quasifermions and summing on Euclidean time at constant fields

$$\bar{S}_{\text{effective}} = -\frac{1}{2} L_0 \, \mathrm{tr}_- \left\{ \mu \, \theta \left(e^{\mu} - \bar{\mathcal{R}}^{-1} \bar{\mathrm{R}} \right) + \ln(\bar{\mathrm{R}} \, \bar{\mathcal{R}}^{-1}) \, \theta \left(\bar{\mathcal{R}}^{-1} \, \bar{R} - e^{\mu} \right) \right\} ,$$

θ = the step function defining the Fermi surface which depends on the chemical potential.

Variation with respect to $\bar{\mathcal{F}}$ gives

$$\bar{\mathcal{F}} = \frac{N}{2H} (\sqrt{1 + H^2} - H), \quad \bar{\mathcal{H}} > e^{\mu} - 1$$

The components of the boson composite for which the above condition is satisfied (which only enter in the effective action) remain unaltered

The condition on fermion number gives

$$-\frac{2}{L_0}\frac{\partial}{\partial\mu}\bar{S}_{\text{effective}} = \mathrm{tr}_-\,\theta\left[\exp\mu - 1 - \bar{\mathcal{H}}\right] = n_F\,.$$

For $\mu < \bar{\sigma}$, $n_F = 0$. *For* $\mu > \bar{\sigma}$, *quasifermions occupy the states from zero energy up to a maximum energy* E_{n_F} *depending on the fermion number* n_F.

At the minimum

$$\bar{S}_{\text{effective}} = -L_0\,\mathrm{tr}_-\left\{\ln\left(\sqrt{1+H^2}+H\right)^2\theta\left(\bar{\mathcal{H}}+1-\exp\mu\right)\right\}.$$

Stationarity with respect to $\bar{\sigma}$ yields the gap equation

$$\frac{4L_0N_f}{g^2}\,\bar{\sigma} = 2\,L_0\,\bar{\sigma}\,\mathrm{tr}_-\left\{\frac{1}{H\sqrt{1+H^2}}\;\theta\left(\bar{\mathcal{H}}+1-\exp\mu\right)\right\}.$$

Increasing the fermion density, quasifermions occupy higher and higher energy states depleting the condensate, until only the solution $\bar{\sigma} = 0$ remains and chiral invariance is restored.

A variational application application to QCD at finite baryon density

In the case of QCD the solution of the variational equations for the structure functions is more difficult because the structure functions of the chiral mesons must depend on the gauge fields, unless they are point-like.

As a first application of the present method I exploit its variational feature assuming a point-like structure for the chiral σ-meson

$$\begin{aligned} \Phi_\sigma &= \gamma_0 \otimes \mathbb{1} \otimes \mathbb{1}_c \otimes \mathbb{1}_s \\ \mathbb{1}_c, \mathbb{1}_s &= \text{identity in color and spatial coordinates} \end{aligned}$$

I assume that at the saddle point the σ-field, but not the gauge field, is constant

$$\bar{S}_{\text{effective}} = -\text{Tr}_{-}\left[\ln(1+\bar{\phi}_\sigma^2) - \ln\left(1 - e^{\mu'} U_0 T_0^{(+)} + NN^\dagger + \bar{\phi}_\sigma^2\right)\right]$$

$$\mu' = \mu + \ln(1+\bar{\phi}_\sigma^2)\,.$$

First term = contribution of σ-meson condensate

Second term = contribution of quarks of mass $\bar{\phi}_\sigma$ (due to the condensate) interacting with gauge fields

One must determine the minimum of $\bar{S}_{\text{effective}}$ with respect to $\bar{\phi}_\sigma$ under the condition on baryon number

$$-\frac{2}{L_0}\frac{\partial}{\partial\mu'}\bar{S}_{\text{effective}} = \frac{1}{1+\bar{\phi}_\sigma^2}\; n_F\,.$$

One expects that numerical simulations be more stable due to the effective mass $\bar{\phi}_\sigma$ of the quarks even with zero bare mass

Analitical investigation under way

Diquarks and Baryons

Diquarks and baryons are constructed satisfying the compositeness condition

Diquark coherent states

$$|d\rangle = \exp\left(\sum_{K} d_K \hat{D}_K^{\dagger}\right)|0\rangle\,, \quad \hat{D}_K = \frac{1}{2}\hat{\alpha}\, D_K\, \hat{\alpha}$$

D_K = diquark structure function with quantum number K

d_K = holomorphic variables associated to diquark fields

Baryon coherent states

$$|b\rangle = \exp\left(\sum_{K,H,i} d_K^{*}\,(B_H)_{K,i}\; b_H\,\hat{\alpha}_i^{\dagger}\right)|0\rangle$$

$(B_H)_{K,i}$ = baryonic structure function with quantum number H

b_H = Grassmann variables associated to baryon fields

$\hat{\alpha}_i$ = creation operator of quasiquark

Effective actions derived but not reported here

Summary and outlook

- Many-body theory : The method reproduces exact known results and goes beyond the Random Phase approximation

- Relativistic field theory : The method reproduces the results of a four fermion interaction model and in addition yields the explicit form of the composite boson structure function and its dependence on the chemical potential.

- QCD : Effective actions have been derived for QCD in different phases. They can be used in numerical simulations which should be more stable because of the presence of the condensate even for bare quark masses exactly zero. At the moment the structure functions can be parametrized in the spirit of a variational calculation, but

- Study of the Ω^{-1} expansion is in progress. If we get in this way analytical results for the structure functions, like in the four-fermion interaction model, we can used them as inputs in numerical simulations.